Central Swaths
(a generalization of the central path)

Jim Renegar
- $p : \mathbb{R}^d \rightarrow \mathbb{R}$ homogeneous polynomial of degree $n$
- $p(e) > 0$

**Defn:** The polynomial $p$ is

“hyperbolic in direction $e$”

if for all $x \in \mathbb{R}^d$, the univariate polynomial $t \mapsto p(te + x)$ has only real roots.

Roots: $\lambda_{1,e}(x) \leq \lambda_{2,e}(x) \leq \cdots \leq \lambda_{n,e}(x)$

“eigenvalues of $x$ (in direction $e$)”
LP:

1. $p(x) = x_1 \cdots x_n$
2. $e > 0$

\[
\lambda \mapsto p(\lambda e - x) = (\lambda e_1 - x_1) \cdots (\lambda e_n - x_n)
\]

Eigenvalues of $x$ in direction $e$: $\frac{x_1}{e_1}, \ldots, \frac{x_n}{e_n}$

SDP:

1. $p(x) = \det(x)$
2. $e \succ 0$

\[
\lambda \mapsto \det(\lambda e - x) = \det(e) \det(\lambda I - e^{-1/2}xe^{-1/2})
\]

Eigenvalues of $x$ in direction $e$

= traditional eigenvalues of $e^{-1/2}xe^{-1/2}$
\begin{align*}
\lambda_1,e(x) & \leq \lambda_2,e(x) \leq \cdots \leq \lambda_n,e(x) \quad \text{roots of } \lambda \mapsto p(x - \lambda e) \\
\text{Hyperbolicity Cone:} & \\
\Lambda_{++} & := \{x : \lambda_1,e(x) > 0\} \\
& = \text{connected component of } \{x : p(x) > 0\} \text{ containing } e
\end{align*}

\textbf{Gårding (1959):} \quad p \text{ is hyperbolic in direction } e \text{ for all } e \in \Lambda_{++} \\
\text{Corollary:} \quad \Lambda_{++} \text{ is a convex cone} \\
\text{Corollary:} \quad x \mapsto \lambda_n,e(x) \text{ is a convex function}
Bauschke, Güler, Lewis & Sendov:

If $f : \mathbb{R}^n \to \mathbb{R}$ is a convex and permutation-invariant
then $x \mapsto f(\bar{\lambda}_e(x))$ is convex.

Helton-Vinnikov Theorem:

Assume $e \in \Lambda_{++}$ and $x, y \in \mathbb{R}^d$.
There exist $n \times n$ symmetric matrices $X$ and $Y$ satisfying

$$(r, s, t) \mapsto p(rx + sy + te) = p(e) \det(rX + sY + tl).$$

Is every hyperbolicity cone a slice of a PSD cone?

Chua: Every homogeneous cone is a slice of a PSD cone.
The literature on hyperbolic polynomials is relatively small, but its growth is accelerating, its quality in general is distinctly impressive, and its reach is surprisingly broad.


\( \phi \) a univariate polynomial

If \( \phi \) has only real roots then:
- \( \phi' \) has only real roots.
- Roots are interlaced: \( \lambda_1 \leq \lambda'_1 \leq \lambda_2 \leq \ldots \leq \lambda'_{n-1} \leq \lambda_n \)

\[ p \] a multivariate polynomial

\[ p'_e(x) := \langle \nabla p(x), e \rangle \] (directional derivative)

If \( p \) is hyperbolic in direction \( e \) then:
- \( p'_e \) is hyperbolic in direction \( e \).
- \( \Lambda_+ \subseteq \Lambda'_{e,+} \)
Inductively: \( p_e^{(i+1)}(x) = \langle \nabla p_e^{(i)}(x), e \rangle \)

LP: \( p(x) := x_1 x_2 \cdots x_n, \quad e > 0 \)

\[
p_e^{(i)}(x) = i! \ p(e) \ \sum_{j_1 < j_2 < \cdots < j_{n-i}} \left( \frac{x_{j_1}}{e_{j_1}} \right) \left( \frac{x_{j_2}}{e_{j_2}} \right) \cdots \left( \frac{x_{j_{n-i}}}{e_{j_{n-i}}} \right)
\]

In general: \( p_e^{(i)}(x) = i! \ p(e) \ E_{n-i}(\bar{x}_e(x)) \)

where \( E_k = \text{elementary symmetric polynomial of degree } k \)
\[ \Lambda_+ = \Lambda_{e,+}^{(0)} \subseteq \Lambda_{e,+}^{(1)} \subseteq \cdots \subseteq \Lambda_{e,+}^{(n-1)} = \text{a halfspace} \]

\[ \Lambda_{e,++}^{(i)} = \text{connected component of} \]
\[ \{ x : p_e^{(i)}(x) > 0 \} \text{ containing } e \]
\[ = \{ x : p_e^{(k)}(x) > 0 \text{ for } k = i, \ldots, n \} \]
Computing $p_e^{(i)}(x), \ \nabla p_e^{(i)}(x), \ \nabla^2 p_e^{(i)}(x)$

Univariate interpolation using a primitive root of unity $\omega$:

If $\phi(t) := \sum_{i=0}^{n} a_{n-i} t^i$ then $a_i = \frac{1}{n} \sum_{j=1}^{n} \omega^{ij} \phi(\omega^j)$

Letting $\phi(t) := p(x + te)$ gives

$$p_e^{(i)}(x) = \frac{i!}{n} \sum_{j=1}^{n} \omega^{ij} p(x + \omega^j e)$$

Differentiating both sides wrt $x$:

$$\nabla p_e^{(i)}(x) = \frac{i!}{n} \sum_{j=1}^{n} \omega^{ij} \nabla p(x + \omega^j e)$$

$$\nabla^2 p_e^{(i)}(x) = \frac{i!}{n} \sum_{j=1}^{n} \omega^{ij} \nabla^2 p(x + \omega^j e)$$
Hyperbolic Program (HP):

\[
\begin{aligned}
& \text{min } \langle c, x \rangle \\
& \text{s.t. } Ax = b \\
& x \in \Lambda_+
\end{aligned}
\]

Introduced by Güler (mid-90’s) in context of ipm’s.

“Central Path” = \{x(\eta) : \eta > 0\}

where \(x(\eta)\) solves

\[
\begin{aligned}
& \text{min } \eta \langle c, x \rangle - \ln p(x) \\
& \text{s.t. } Ax = b
\end{aligned}
\]

\(O(\sqrt{n}) \log(1/\epsilon)\) iterations suffice

to reduce \(\alpha := \langle c, x \rangle - \langle b, y \rangle\) to \(\epsilon \alpha\)
Hyperbolic Program relaxation:

\[
\begin{align*}
&\min \quad \langle c, x \rangle \\
&\text{s.t.} \quad Ax = b \\
&\quad x \in \Lambda_+ \\
\end{align*}
\]

\[\text{HP}\]

\[\begin{align*}
&\min \quad \langle c, x \rangle \\
&\text{s.t.} \quad Ax = b \\
&\quad x \in \Lambda^{(i)}_e,++ \\
\end{align*}
\]

\[\text{HP}^{(i)}_e\]

Defn: The “\(i^{th}\) central swath” is the set of directions \(e\) satisfying

- \(Ae = b, \quad e \in \Lambda^{(i)}_e,++\) \quad (strict feasibility)
- \(\text{HP}^{(i)}_e\) has an optimal solution

Thm: central path = \((n - 1)^{th}\) central swath
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<td>$\text{min } \langle c, x \rangle$</td>
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<td>s.t. $Ax = b$</td>
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<tr>
<td>$x \in \Lambda_+$</td>
<td>$x \in \Lambda_{+,e}^{(i)}$</td>
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\[\begin{align*}
\text{min } \langle c, x \rangle \\
\text{s.t. } Ax = b \\
x \in \Lambda_+
\end{align*}\]
Hyperbolic Program relaxation:

\[
\begin{align*}
\min & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \Lambda_+ \\
\end{align*}
\]

\(\overset{\text{relax}}{\Rightarrow}\)

\[
\begin{align*}
\min & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \Lambda_{e,+}^{(i)} \\
\end{align*}
\]

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- \(Ae = b, \quad e \in \Lambda_{e,+}^{(i)}\) (strict feasibility)
- \(\text{HP}_{e}^{(i)}\) has an optimal solution

**Thm:** central path = \((n - 1)^{th}\) central swath
How might solving $\text{HP}^{(i)}_e$ help in solving $\text{HP}$?

$z(e)$: optimal solution for $\text{HP}^{(i)}_e$

- assume we know $z(e)$ exactly

Obtaining a derivative direction with better objective value is easy:

Just move $e$ towards $z(e)$

Dynamics on $e$: $\dot{e} = z(e) - e$

Of course $z(e)$ then changes dynamically, too.

Are the dynamics well-defined?
If so, where do they lead?
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\text{s.t.} & \quad Ax = b \\
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\end{align*}
\]
\[
\begin{align*}
\min & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \Lambda_e^{(i)}_+ \\
\end{align*}
\]

Recall: \[ \text{Swath}(i) = \{ e \in \Lambda_{++} : Ae = b \text{ and } \text{Opt}_e^{(i)} \neq \emptyset \} \]

Define \[ \text{Core}(i) = \{ e \in \text{Swath}(i) : \text{Opt}_e^{(i)} = \text{Opt} \} \]

**Thm:** Assume \( 1 \leq i \leq n - 2 \).

For \( e \in \text{Swath}(i) \setminus \text{Core}(i) \),
\[ \text{Opt}_e^{(i)} = \{ z(e) \} \quad \text{(a single point)} \]

and \( e \mapsto z(e) \) is analytic.

In other words, the dynamics are well-defined.
Another reason for appropriateness of the terminology “central swaths”:

Thm: If $i = n - 2$ and $e(0)$ is on the central path then $e(t)$ traces the central path.
\[ e(0) \in Swath(i) \setminus Core(i) \]

\[ \{ e(t) : 0 \leq t < T \} \quad \text{maximal trajectory} \]

\[ T \text{ finite means trajectory leaves } Swath(i) \setminus Core(i) \text{ at time } T \]

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**Main Thm (Part I):**

Assume \( 1 \leq i \leq n - 2 \) and \( \text{Opt} \) is nonempty and bounded.

A) If \( T = \infty \) then all limit points of \( \{ e(t) \} \) are in \( \text{Opt} \).

B) If \( T < \infty \) then \( \{ e(t) \} \) has a unique limit point, and the point is in \( \text{Core}(i) \).

Moreover, the path \( \{ z(e(t)) \} \) is then bounded, and each of its limit points lies in \( \text{Opt} \).
\[
\begin{align*}
\min & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \Lambda_+ \\
\end{align*}
\]

\[\text{HP} \]

\[
\begin{align*}
\min & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \Lambda_{e,+}^{(i)} \\
\end{align*}
\]

\[\text{HP}^{(i)}_e \]

\[z(e) \quad \text{optimal solution of } HP^{(i)}_e\]

\[\text{Thm: } z(e) \text{ also is the unique optimal solution for}\]

\[
\begin{align*}
\min_x & \quad - \ln \langle c, e - x \rangle - \frac{p_e^{(i)}(x)}{p_e^{(i+1)}(x)} \\
\text{s.t.} & \quad Ax = b \\
\end{align*}
\]

a linearly-constrained optimization problem with convex objective function.
More generally . . .

\[
\begin{align*}
\min & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \Lambda_+
\end{align*}
\]

\[\text{HP}\]

**Thm:** The optimal solutions of HP satisfying \(z \notin \partial \Lambda_{e,+}'\) are precisely the optimal solutions for

\[
\begin{align*}
\min_x & \quad -\ln \langle c, e - x \rangle - \frac{p(x)}{p'_e(x)} \\
\text{s.t.} & \quad Ax = b
\end{align*}
\]

a linearly-constrained optimization problem with convex objective function.
**Thm:** If $p$ is hyperbolic in direction $e$

then $p/p'_e$ is a concave function on $\Lambda'_{e,++}$

**Pf:**

- $P(x, t) := tp(x)$ is hyperbolic in direction $(e, 1)$
- Hence, $P'_{(e,1)}$ is hyperbolic in direction $(e, 1)$
- Hyperbolicity cone of $P'_{(e,1)}$ is epigraph of

  $$x \mapsto -p(x)/p'_e(x)$$
\begin{align*}
\text{min} & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \Lambda_+ \\
\end{align*}

\begin{align*}
\text{min} & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \Lambda_+^{(i),e} \\
\end{align*}
\[
\begin{align*}
\min & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \Lambda_+ \\
\quad \text{HP} \\
\max & \quad \langle b, y \rangle \\
\text{s.t.} & \quad A^*y + s = c \\
& \quad x \in \Lambda^*_+ \\
& \quad \text{HP}^*
\end{align*}
\]

Let \( s(e) := \frac{\langle c, e-z \rangle}{p^{(i+1)}_e(z)} \nabla p^{(i)}_e(z) \) where \( z := z(e) \)

and let \( y(e) \) solve \( A^*y + s(e) = c \)

**Main Thm (Part II):**

The path \( t \mapsto (y(e(t)), s(e(t))) \)

is strictly feasible for \( \text{HP}^* \) and bounded.

Moreover, \( t \mapsto \langle b, y(e(t)) \rangle \) converges monotonically to the optimal value for \( \text{HP}^* \).