Central Swaths
(a generalization of the central path)

Jim Renegar

- $p : \mathbb{R}^d \to \mathbb{R}$ homogeneous polynomial of degree n
- $p(e) > 0$

Defn: The polynomial p is

"hyperbolic in direction e"

if for all $x \in \mathbb{R}^d$, the univariate polynomial $t \mapsto p(te + x)$ has only real roots.

$\lambda \mapsto p(\lambda e - x)$

Roots: $\lambda_{1,e}(x) \leq \lambda_{2,e}(x) \leq \cdots \leq \lambda_{n,e}(x)$

"eigenvalues of x (in direction e)"

LP:

- $p(x) = x_1 \cdots x_n$
- $e > 0$

$$\lambda \mapsto p(\lambda e - x) = (\lambda e_1 - x_1) \cdots (\lambda e_n - x_n)$$

Eigenvalues of x in direction e: $\frac{x_1}{e_1}, \dots, \frac{x_n}{e_n}$

SDP:

- $p(x) = \det(x)$
- $e \succ 0$

$$\lambda \mapsto \det(\lambda e - x) = \det(e) \det(\lambda I - e^{-1/2} x e^{-1/2})$$

Eigenvalues of x in direction e

= traditional eigenvalues of $e^{-1/2} x e^{-1/2}$

$\lambda_{1,e}(x) \leq \lambda_{2,e}(x) \leq \cdots \leq \lambda_{n,e}(x)$ roots of $\lambda \mapsto p(x - \lambda e)$

Hyperbolicity Cone:

$$\begin{aligned} \Lambda_{++} &:= \{x : \lambda_{1,e}(x) > 0\} \\ &= \text{connected component of} \\ &\qquad \{x : p(x) > 0\} \text{ containing } e \end{aligned}$$

Gårding (1959): p is hyperbolic in direction e for all $e \in \Lambda_{++}$

Corollary: Λ_{++} is a convex cone

Corollary: $x \mapsto \lambda_{n,e}(x)$ is a convex function

Bauschke, Güler, Lewis & Sendov:

If $f : \mathbb{R}^n \to \mathbb{R}$ is a convex and permutation-invariant

then $x \mapsto f(\vec{\lambda}_e(x))$ is convex

Helton-Vinnikov Theorem:

Assume $e \in \Lambda_{++}$ and $x, y \in \mathbb{R}^d$.

There exist $n \times n$ symmetric matrices X and Y satisfying

$$(r, s, t) \mapsto p(rx + sy + te) = p(e)\,\det(rX + sY + tI)\ .$$

Is every hyperbolicity cone a slice of a PSD cone?

Chua: Every homogeneous cone is a slice of a PSD cone.

The literature on hyperbolic polynomials is relatively small,
but its growth is accelerating,
its quality in general is distinctly impressive,
and its reach is surprisingly broad.

L. Gurvits, Van der Waerden/Schrijver-Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials, Elec. Jour. of Comb. (2008)

F.R. Harvey and H.B. Lawson Jr., Hyperbolic polynomials and the Dirichlet problem, arXiv (2009)

J. Borcea and P. Brändén, Pólya-Schur master theorems for circular domains and their boundaries, Ann. of Math. (2009)

ϕ a univariate polynomial

If ϕ has only real roots then:

- ϕ' has only real roots.
- Roots are interlaced: $\lambda_1 \leq \lambda'_1 \leq \lambda_2 \leq \cdots \leq \lambda'_{n-1} \leq \lambda_n$

---

p a multivariate polynomial

$p'_e(x) := \langle \nabla p(x), e \rangle$ (directional derivative)

If p is hyperbolic in direction e then:

- p'_e is hyperbolic in direction e.
- $\Lambda_+ \subseteq \Lambda'_{e,+}$

Inductively: $p_e^{(i+1)}(x) = \langle \nabla p_e^{(i)}(x), e \rangle$

LP: $p(x) := x_1 x_2 \cdots x_n, \quad e > 0$

$$p_e^{(i)}(x) = i!\, p(e) \sum_{j_1 < j_2 < \cdots < j_{n-i}} \left(\frac{x_{j_1}}{e_{j_1}}\right)\left(\frac{x_{j_2}}{e_{j_2}}\right) \cdots \left(\frac{x_{j_{n-i}}}{e_{j_{n-i}}}\right)$$

In general: $p_e^{(i)}(x) = i!\, p(e)\, E_{n-i}(\vec{\lambda}_e(x))$

where E_k = elementary symmetric polynomial of degree k

$$\Lambda_+ = \Lambda_{e,+}^{(0)} \subseteq \Lambda_{e,+}^{(1)} \subseteq \cdots \subseteq \Lambda_{e,+}^{(n-1)} = \text{a halfspace}$$

$$\Lambda_{e,++}^{(i)} = \text{connected component of } \{x : p_e^{(i)}(x) > 0\} \text{ containing } e$$

$$= \{x : p_e^{(k)}(x) > 0 \text{ for } k = i, \ldots, n\}$$

Computing $p_e^{(i)}(x)$, $\nabla p_e^{(i)}(x)$, $\nabla^2 p_e^{(i)}(x)$

Univariate interpolation using a primitive root of unity ω:

If $\phi(t) := \sum_{i=0}^{n} a_{n-i} t^i$ then $a_i = \frac{1}{n} \sum_{j=1}^{n} \omega^{ij} \phi(\omega^j)$

Letting $\phi(t) := p(x + te)$ gives

$$p_e^{(i)}(x) = \frac{i!}{n} \sum_{j=1}^{n} \omega^{ij} p(x + \omega^j e)$$

Differentiating both sides wrt x:

$$\nabla p_e^{(i)}(x) = \frac{i!}{n} \sum_{j=1}^{n} \omega^{ij} \nabla p(x + \omega^j e)$$

$$\nabla^2 p_e^{(i)}(x) = \frac{i!}{n} \sum_{j=1}^{n} \omega^{ij} \nabla^2 p(x + \omega^j e)$$

Hyperbolic Program (HP):

$$\begin{array}{rl} \min & \langle c, x \rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_+ \end{array}$$

Introduced by Güler (mid-90's) in context of ipm's.

"Central Path" $= \{x(\eta) : \eta > 0\}$

where $x(\eta)$ solves

$$\begin{array}{rl} \min & \eta \langle c, x \rangle - \ln p(x) \\ \text{s.t.} & Ax = b \end{array}$$

$O(\sqrt{n}) \log(1/\epsilon)$ iterations suffice

to reduce $\alpha := \langle c, x \rangle - \langle b, y \rangle$ to $\epsilon \, \alpha$

Hyperbolic Program relaxation:

$$\underbrace{\begin{array}{rl} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_+ \end{array}}_{\text{HP}} \quad \xrightarrow{\text{relax}} \quad \underbrace{\begin{array}{rl} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_{e,+}^{(i)} \end{array}}_{\text{HP}_e^{(i)}}$$

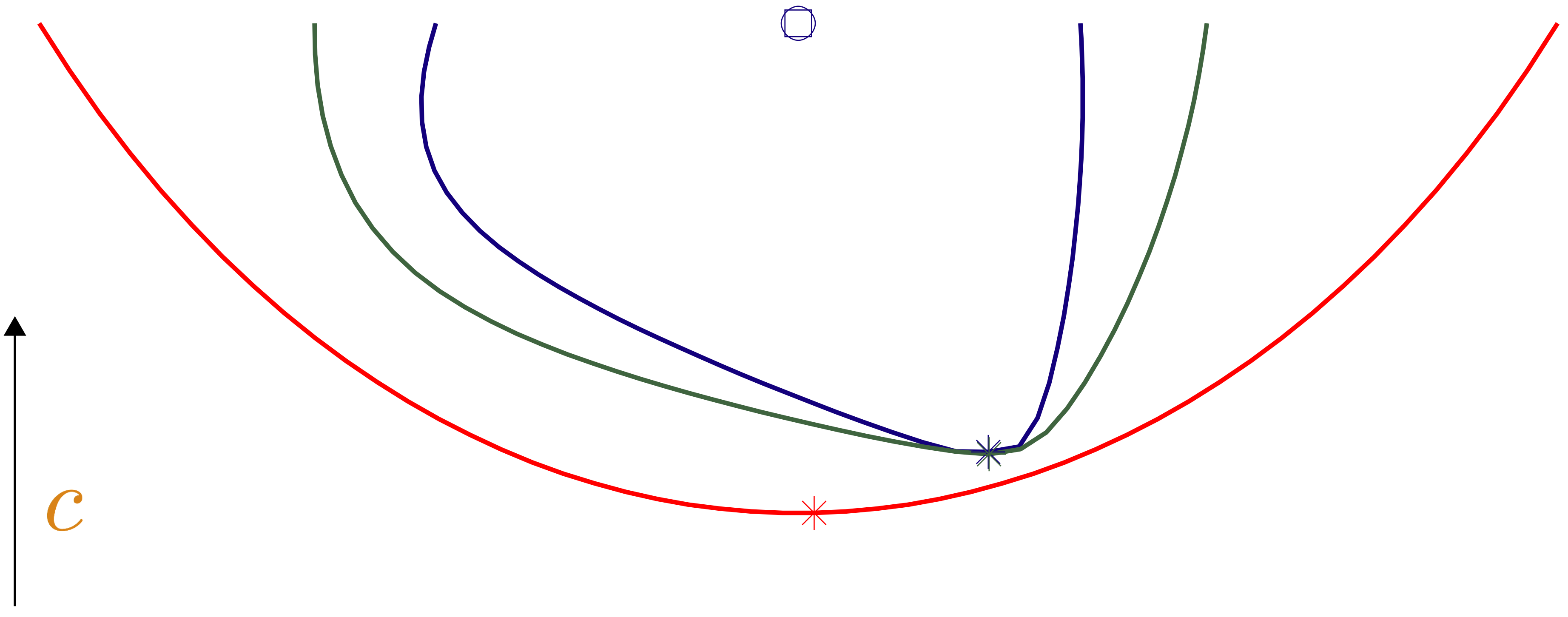
Defn: The "i^{th} central swath" is the set of directions e satisfying

- $Ae = b, \quad e \in \Lambda_{e,++}^{(i)}$ (strict feasibility)
- $\text{HP}_e^{(i)}$ has an optimal solution

Thm: central path = $(n-1)^{th}$ central swath

$$\begin{aligned} \min \quad & \langle c, x\rangle \\ \text{s.t.} \quad & Ax = b \\ & x \in \Lambda_+ \end{aligned}$$

$$\begin{aligned} \min \quad & \langle c, x\rangle \\ \text{s.t.} \quad & Ax = b \\ & x \in \Lambda_{+,e}^{(i)} \end{aligned}$$



Hyperbolic Program relaxation:

$$\underbrace{\begin{array}{rl} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_+ \end{array}}_{\text{HP}} \quad \xrightarrow{\text{relax}} \quad \underbrace{\begin{array}{rl} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_{e,+}^{(i)} \end{array}}_{\text{HP}_e^{(i)}}$$

Defn: The "i^{th} central swath" is the set of directions e satisfying

- $Ae = b, \quad e \in \Lambda_{e,++}^{(i)}$ (strict feasibility)
- $\text{HP}_e^{(i)}$ has an optimal solution

Thm: central path = $(n-1)^{th}$ central swath

How might solving $\mathrm{HP}_e^{(i)}$ help in solving HP?

$z(e)$: optimal solution for $\mathrm{HP}_e^{(i)}$

– assume we know $z(e)$ exactly

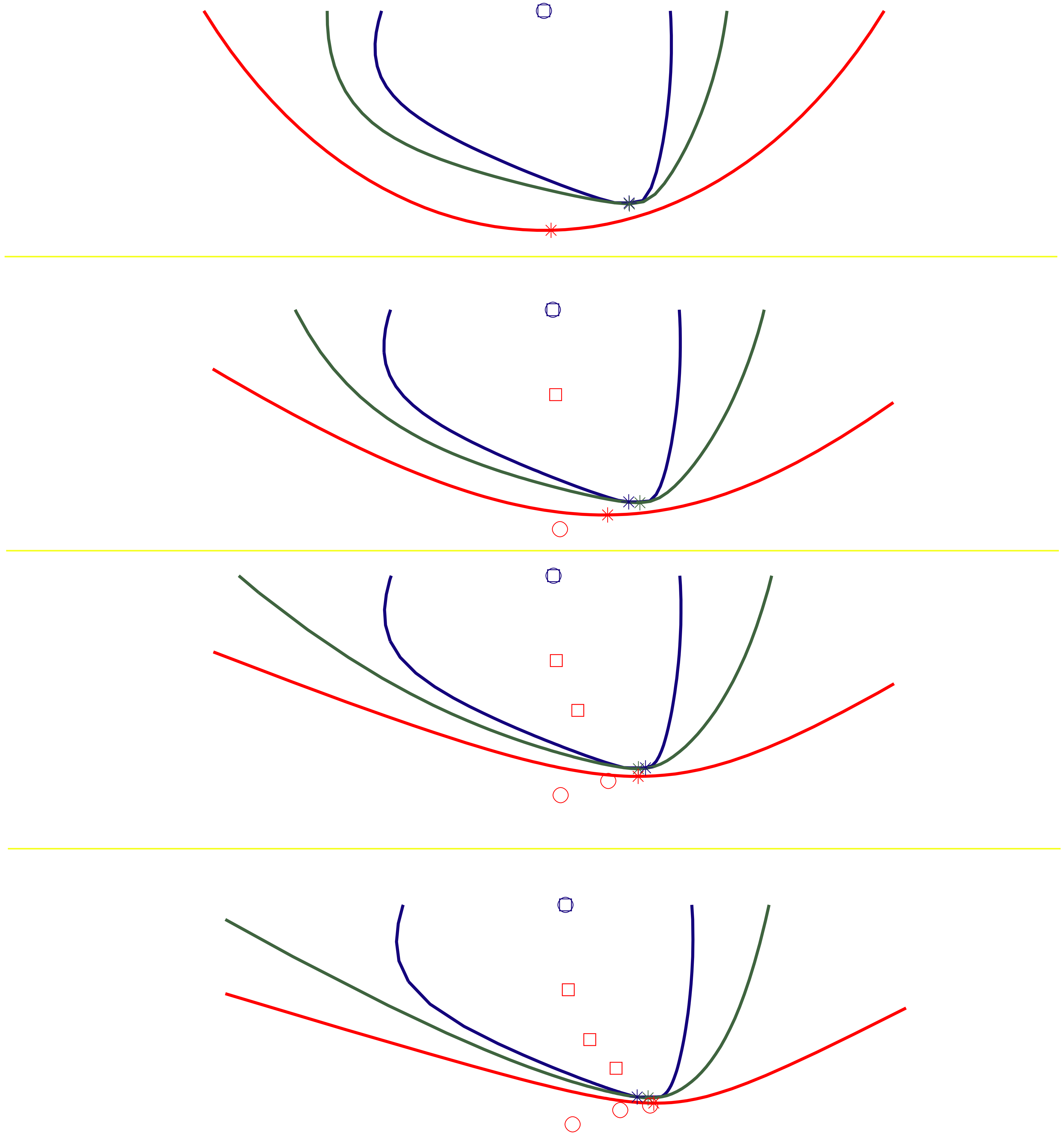
Obtaining a derivative direction with better objective value is easy:

Just move e towards $z(e)$

Dynamics on e : $\dot{e} = z(e) - e$

Of course $z(e)$ then changes dynamically, too.

Are the dynamics well-defined?
If so, where do they lead?



How might solving $\mathrm{HP}_e^{(i)}$ help in solving HP?

$z(e)$: optimal solution for $\mathrm{HP}_e^{(i)}$

– assume we know $z(e)$ exactly

Obtaining a derivative direction with better objective value is easy:

Just move e towards $z(e)$

Dynamics on e : $\dot{e} = z(e) - e$

Of course $z(e)$ then changes dynamically, too.

Are the dynamics well-defined?
If so, where do they lead?

$$\begin{array}{rl} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_+ \end{array} \quad \xrightarrow{\text{relax}} \quad \begin{array}{rl} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_{e,+}^{(i)} \end{array}$$

Recall: $\mathrm{Swath}(i) = \{e \in \Lambda_{++} : Ae = b \text{ and } \mathrm{Opt}_e^{(i)} \neq \emptyset\}$

Define $\mathrm{Core}(i) = \{e \in \mathrm{Swath}(i) : \mathrm{Opt}_e^{(i)} = \mathrm{Opt}\}$

Thm: Assume $1 \leq i \leq n - 2$.

For $e \in \mathrm{Swath}(i) \setminus \mathrm{Core}(i)$,

$\mathrm{Opt}_e^{(i)} = \{z(e)\}$ (a single point)

and $e \mapsto z(e)$ is analytic.

In other words, the dynamics are well-defined.

Another reason for appropriateness

of the terminology "central swaths":

Thm: If $i = n - 2$ and $e(0)$ is on the central path

then $e(t)$ traces the central path.

$e(0) \in \mathrm{Swath}(i) \setminus \mathrm{Core}(i)$

$\{e(t) : 0 \leq t < T\}$ maximal trajectory

T finite means trajectory leaves $\mathrm{Swath}(i) \setminus \mathrm{Core}(i)$ at time T

Main Thm (Part I):

Assume $1 \leq i \leq n-2$ and Opt is nonempty and bounded.

A) If $T = \infty$ then all limit points of $\{e(t)\}$ are in Opt.

B) If $T < \infty$ then $\{e(t)\}$ has a unique limit point,

and the point is in $\mathrm{Core}(i)$.

Moreover, the path $\{z(e(t))\}$ is then bounded,

and each of its limit points lies in Opt.

$$\underbrace{\begin{array}{rl} \min & \langle c, x \rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_+ \end{array}}_{\text{HP}} \quad \xrightarrow{\text{relax}} \quad \underbrace{\begin{array}{rl} \min & \langle c, x \rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_{e,+}^{(i)} \end{array}}_{\text{HP}_e^{(i)}}$$

$z(e)$ optimal solution of $\text{HP}_e^{(i)}$

Thm: $z(e)$ also is the unique optimal solution for

$$\begin{array}{rl} \min_x & -\ln\langle c, e - x \rangle - \frac{p_e^{(i)}(x)}{p_e^{(i+1)}(x)} \\ \text{s.t.} & Ax = b \ , \end{array}$$

a linearly-constrained optimization problem

with convex objective function.

More generally . . .

$$\underbrace{\begin{array}{rl} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_+ \end{array}}_{\text{HP}}$$

Thm: The optimal solutions of HP satisfying $z \notin \partial\Lambda'_{e,+}$ are precisely the optimal solutions for

$$\begin{array}{rl} \min_x & -\ln\langle c, e - x\rangle - \frac{p(x)}{p'_e(x)} \\ \text{s.t.} & Ax = b \ , \end{array}$$

a linearly-constrained optimization problem

with convex objective function.

Thm: If p is hyperbolic in direction e

then p/p'_e is a concave function on $\Lambda'_{e,++}$

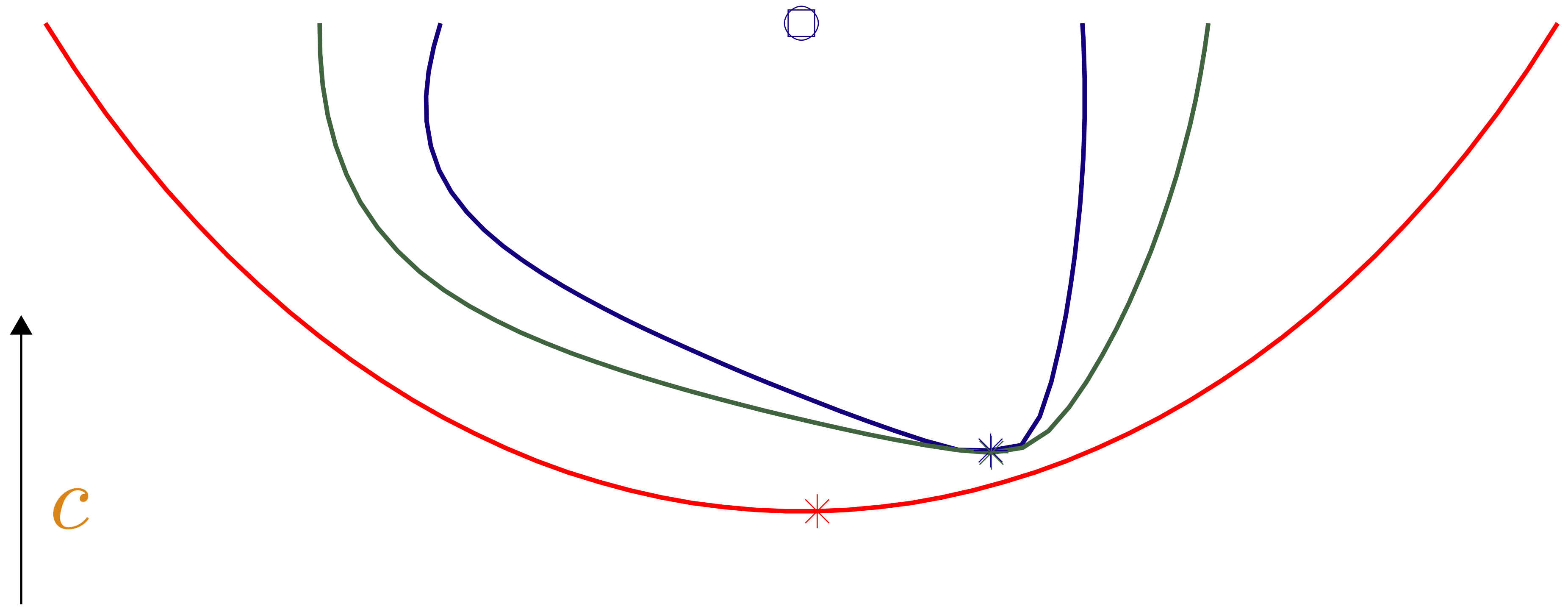
Pf:

- $P(x,t) := tp(x)$ is hyperbolic in direction $(e,1)$
- Hence, $P'_{(e,1)}$ is hyperbolic in direction $(e,1)$
- Hyperbolicity cone of $P'_{(e,1)}$ is epigraph of

$$x \mapsto -p(x)/p'_e(x)$$

$$\begin{aligned} \min \quad & \langle c, x\rangle \\ \text{s.t.} \quad & Ax = b \\ & x \in \Lambda_+ \end{aligned}$$

$$\begin{aligned} \min \quad & \langle c, x\rangle \\ \text{s.t.} \quad & Ax = b \\ & x \in \Lambda_{+,e}^{(i)} \end{aligned}$$



$$\underbrace{\begin{array}{rl} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_+ \end{array}}_{\text{HP}} \qquad \underbrace{\begin{array}{rl} \max & \langle b, y\rangle \\ \text{s.t.} & A^* y + s = c \\ & x \in \Lambda_+^* \end{array}}_{\text{HP}^*}$$

Let $s(e) := \frac{\langle c, e-z\rangle}{p_e^{(i+1)}(z)} \nabla p_e^{(i)}(z)$ where $z := z(e)$

and let $y(e)$ solve $A^* y + s(e) = c$

Main Thm (Part II):

The path $t \mapsto (y(e(t)), s(e(t)))$

is strictly feasible for HP^* and bounded.

Moreover, $t \mapsto \langle b, y(e(t))\rangle$ converges monotonically

to the optimal value for HP^*.