Some variations on connected layer families

Subset partition graphs

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Background

Combinatorial abstractions of polytope graphs

Considered by:

- Adler, Dantzig, and Murty (abstract polytopes)
- Kalai (ultraconnected set systems)
- Eisenbrand, Hähnle, Razborov, and Rothvoß (connected layer families)

What happens if we modify the axioms/conditions to our combinatorial abstraction? Need a slightly more general setting...

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Fix a set S, $|S| = n$.

Graph $G = (A, E)$ with $A \subseteq \binom{S}{d}$ such that:

1. for each $u, v \in A$, there is a path connecting u and v whose intermediate vertices all contain $u \cap v$.
2. $(u, v) \in E$ iff $|u \cap v| = d - 1$.
3. if $f \subseteq S$ and $|f| = d - 1$, then $|\{v \in V : f \subseteq v\}| \in \{0, 2\}$.

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A d-dimensional connected layer family on the symbol set S ($n := |S|$) is a collection $\{\mathcal{F}_0, \ldots, \mathcal{F}_\delta\}$ of non-empty sets such that:

- the elements of $\mathcal{F}_i$ are d-subsets of S
- if $i \neq j$, then $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$
- if $i < j < k$ and $u \in \mathcal{F}_i$ and $w \in \mathcal{F}_k$, then there is a $v \in \mathcal{F}_j$ such that $u \cap w \subseteq v$

The CLF has diameter δ.

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Upper bounds

Kalai-Kleitman [1994], Eisenbrand-Hähnle-Razborov-Rothvoß [2010]:

Theorem

The diameter of CLFs (and thus the diameters of all polyhedra graphs) is bounded above by $n^{1+\log d}$.

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Lower bounds

Eisenbrand, Hähnle, Razborov, Rothvoß construct a lower bound on the diameter of CLFs when n is even:

Theorem: Eisenbrand-Hähnle-Razborov-Rothvoß

Let $D(n)$ be the maximal diameter of d-dimensional CLFs on $n = 2d$ symbols. Then,

$$\limsup_{n\to\infty} \frac{D(n)}{n^2/\log n} \geq \text{constant}.$$

A new abstraction

What if CLFs satisfied properties 2 and 3 also?

Let

$$A := \bigcup_{i=0}^{\delta} \mathcal{F}_i \subseteq \binom{S}{d}.$$

What if a CLF was defined as before, but satisfying properties 1, 2, and 3?

- **adjacency:** If $a, a' \in A$ and $|a \cap a'| = d - 1$, then a and a' are in the same or adjacent layers.
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Definition: Subset partition graphs

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The graph G is a subset partition graph of A on the symbol set S.

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Main conditions

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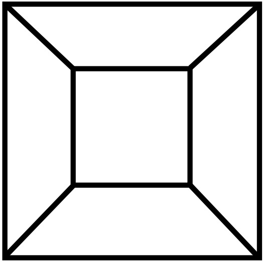

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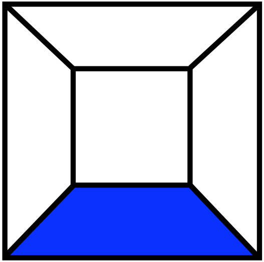

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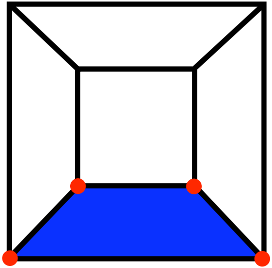

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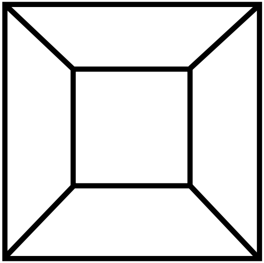

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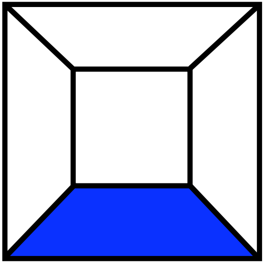

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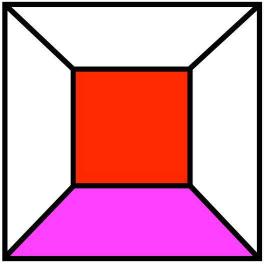

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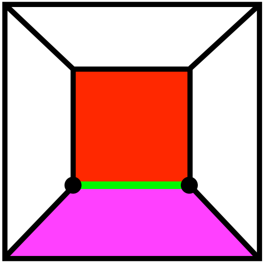

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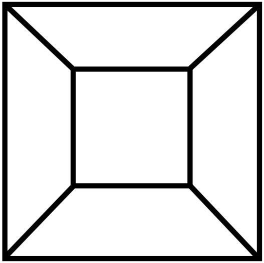

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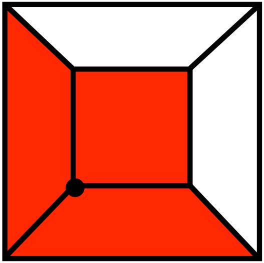

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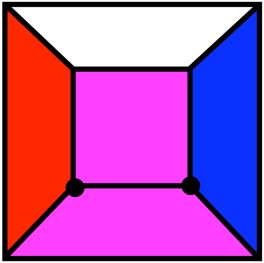

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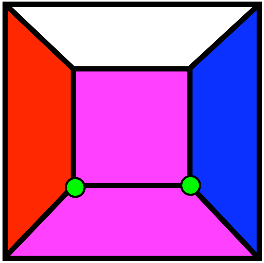

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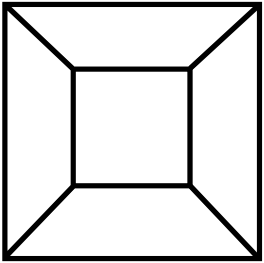

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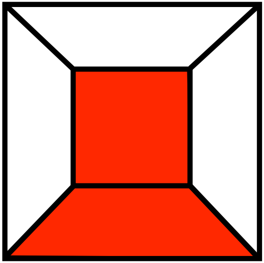

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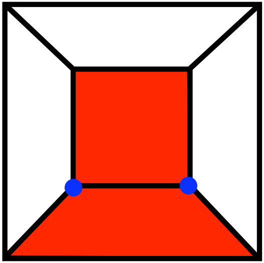

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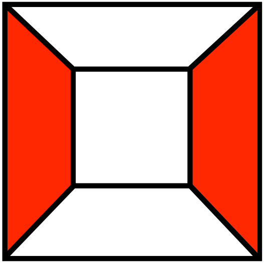

Summary of main conditions

The main properties we want to consider for SPGs:

- **dimension reduction:** if $f \subset S$ and $|f| \leq d$, the induced subgraph F of G whose vertices contain a set $a \supseteq f$ is connected.
- **convexity:** if $f \subseteq f' \subseteq S$ and $|f'| \leq d$, then $F \supseteq F'$
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G path + dimension reduction + convexity $\leftrightsquigarrow$ CLF

Three operations on SPGs

1 contraction, **2** edge addition, **3** induction

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Other conditions on SPGs

Additional combinatorial properties:

- If $a, a' \in A$, then a is not contained in a'.
- The graph G is d-connected.
- The graph G is d-regular.
- For every $a \in A$, $|\{a' \in A \setminus a : |a \cap a'| = d - 1\}| = d$.
- $|v_i| = 1$ for each i.

Diameter, dimension...

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The subset partition graph G satisfies the DR, C, A, or EC conditions (respectively) if ....

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Upper bound

Upper bound for convex, dimension-reduction SPGs

Let G be a subset partition graph of A on the symbol set S which satisfies the convexity and dimension-reduction properties.

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Analogous to Kalai-Kleitman, Eisenbrand-Hähnle-Razborov-Rothvoß:

Theorem

The maximal diameter among d-dimensional subset partition graph on n symbols which satisfies the convexity and dimension-reduction properties is at most $n^{1+\log d}$.

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Induction on CLFs

Inspired by Eisenbrand-Hähnle-Razborov-Rothvoß...

Induction on a symbol $s \in S$

1. Let $A' = \{a \in A : s \in a\}$ and $S' = S \setminus \{s\}$.
2. Let G' be the induced subgraph of G consisting of vertices which contain a subset $a \ni s$.
3. Remove s from each subset in A'.

Lemma

Given a d-dimensional subset partition graph on n symbols which satisfies the convexity and dimension-reduction properties, induction on any symbol $s \in S$ produces a $(d-1)$-dimensional subset partition graph on $n-1$ symbols.

Induction on CLFs

Inspired by Eisenbrand-Hähnle-Razborov-Rothvoß...

Induction on a symbol $s \in S$

1. Let $A' = \{a \in A : s \in a\}$ and $S' = S \setminus \{s\}$.
2. Let G' be the induced subgraph of G consisting of vertices which contain a subset $a \ni s$.
3. Remove s from each subset in A'.

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Upper bound for convex, dimension-reduction SPGs

Let $J(d, n)$ denote the maximal diameter among d-dimensional subset partition graph on n symbols which satisfies the convexity and dimension-reduction properties.

Analogous to
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Theorem

$$J(d, n) \leq n^{1+\log d}.$$

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- Let $a_1 \in v_1$ and $a_2 \in v_2$ be two arbitrary subsets in A.
- Let k_i be maximal such that at most $\frac{n}{2}$ symbols are in $\{a \in v : \text{dist}(v, v_i) \leq k_i\}$.
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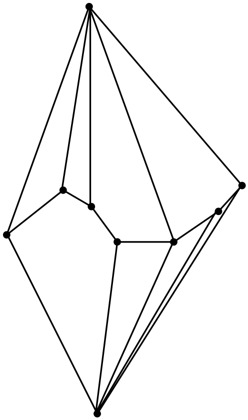
Lower bounds

Spindles



Santos introduced spindles in the construction of a counter-example to the Hirsch Conjecture:

Definition

A spindle is a polytope with two distinguished vertices a_1 and a_2 (called the apices) such that every facet contains exactly one of the apices.

- The length of the spindle is the distance between a_1 and a_2.
- A d-spindle is long if its length exceeds d.

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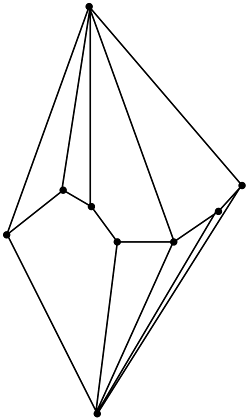

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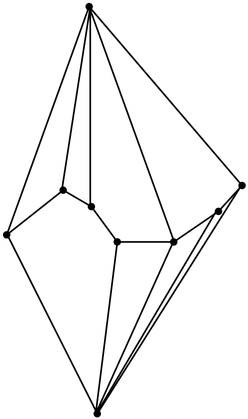

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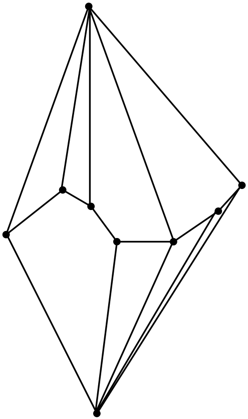
Spindles

Theorem: Santos [2010]

- Long 5-dimensional spindles exist.
- Moreover, existence of long spindles implies existence of a long spindle P with $n = 2d$ facets. Thus, P is non-Hirsch.

A subset partition graph satisfies the **spindle property** if there are two distinguished subsets a_1 and a_2 (called the apices) in A such that a_1 and a_2 is a partition of S.

We examine some spindle-SPGs with $n = 2d$ symbols.



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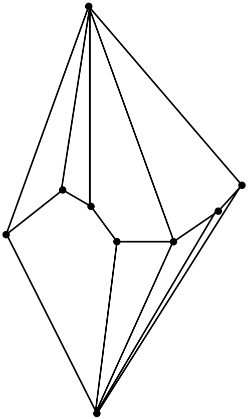

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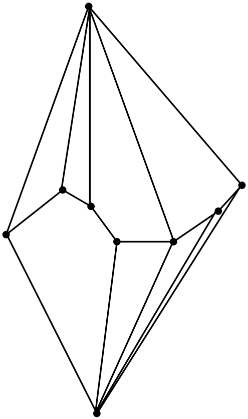

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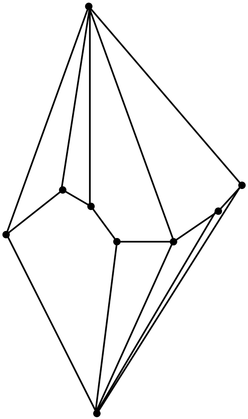

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Superlinear diameter: abstract spindles

Theorem: K. [2010]

Let $T(n)$ be the maximal length of d-dimensional SPG spindles on $n = 2d$ symbols satisfying the adjacency and edge-count properties. Then,

$$\limsup_{n\to\infty} \frac{T(n)}{n^{3/2}} \geq \text{constant.} \tag{1}$$

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Corollary

The maximal diameter $T'(n)$ also satisfies (1).

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Proof sketch

- Fix $m \in \mathbb{N}$. Let $S = [2m] \times [2m]$. Let $d = \frac{n}{2} = (2m)^2$.
- The apices of the abstract spindle G are

$$\{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq 2m\}$$

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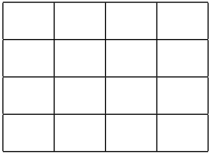
Example: $m = 2$

$S = \{1, \ldots, 4\}^2$.



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- **Number of symbols:**

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- **Number of elements in each subset:**

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$$dm = 2m^3 = \Theta(n^{3/2}).$$

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Transportation SPGs

A SPG of A on the symbol set $S = [p] \times [q] \times [r]$ satisfies the **3-way axial transportation property** if: for every $a \in A$, the set $S \setminus a$ is the support set of the vertex of some $p \times q \times r$ axial 3-way transportation polytope.

Proposition: K.

Let $U(m)$ be the maximal diameter among SPGs of the symbol set $[m]^3$ satisfying the adjacency, edge-count, and 3-way axial transportation properties. Then, $U(m) \geq (m-1)^3$.

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- Challenge: Satisfy the four main properties (convexity, dimension reduction, adjacency, edge-count) and construct a superlinear ($n^{3/2}$? $n \log n$?) family.
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  2. Gain the other main properties with contraction and edge addition.
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- "Please do not hesitate to repeat an idea raised by yourself or by other if you think it can be useful."
- "...ideas, no matter how desparate, for counterexamples?"

– Gil Kalai

Challenges/Invitations

- Try to disprove the Linear Diameter Conjecture:
  1. Start with a family of SPGs satisfying at least the edge-count property with superlinear growth.
  2. Gain the other main properties with contraction and edge addition.
  3. Realize the new family of graphs as polytopes.
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Thank you! Any questions?

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