Integral Geometry and the Polynomial Hirsch Conjecture

Jonathan Kelner, MIT

Partially based on joint work with Daniel Spielman

Introduction

- A lot of recent work on Polynomial Hirsch Conjecture has focused on combinatorial abstractions
- Goal of this talk is to propose an alternative geometric program for the problem
- Motivated by integral geometry techniques used by Spielman and me in our polynomial time simplex method
- Also draws from some ideas from smoothed analysis literature
- But don’t need to worry about being algorithmic

- Polynomial Hirsch conjecture should be easier for random polytopes than for general polytopes
  - So should be able to resolve this case if we ever hope to do general case
- Something stronger should be true
  - And pretty approachable
  - In fact, maybe this talk plus one really good paragraph…
- Will show that if counterexamples to polynomial Hirsch conjecture exist, they are very fragile
  - But won't quite get a diameter bound
- **Alternative titles:**
  - An Un-Abstract View on the Polynomial Hirsch Conjecture
  - How to Almost Prove that the Hirsch Conjecture is Almost Always Almost True

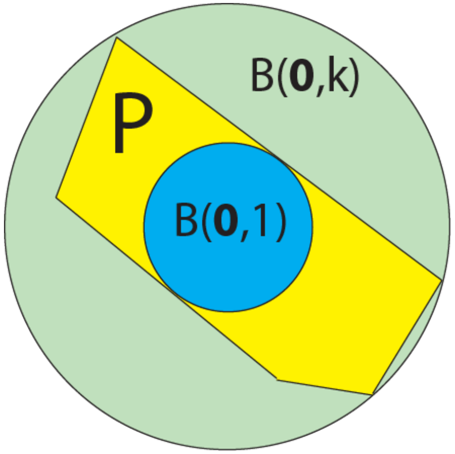
Goals for this talk

- Set forth a geometric program for the polynomial Hirsch Conecture
- Advocate for the following conjecture as a good next step:
- **Smoothed Hirsch Conjecture:** Slightly perturbing **any** polytope (randomly) results in one with polynomial diameter
- Show how to almost prove it
  - And give a general geometric picture for these sorts of problems

Notation

- Polytope $P = \{x \mid Ax \leq b\}$
  - $x \in \mathbf{R}^d$
  - $b \in \mathbf{R}^n$, positive
  - $A = n \times d$ matrix
- A has rows $a_1, \ldots, a_n$
- Can rescale s.t. $b = \mathbf{1}$ WLOG

Well-Rounded Polytopes



- Will be useful to have notion of coordinate system in which a convex body is "round"
- ***Definition:*** We say that a convex body is *k-well-rounded* if
$$B(\boldsymbol{0},1) \subseteq P \subseteq B(\boldsymbol{0},k),$$
where B(**0**,r)=radius r ball centered at origin
- **Fritz John's Theorem:** For any bounded convex body, exists coordinate system in which it's d-well-rounded

Our Perturbation Model

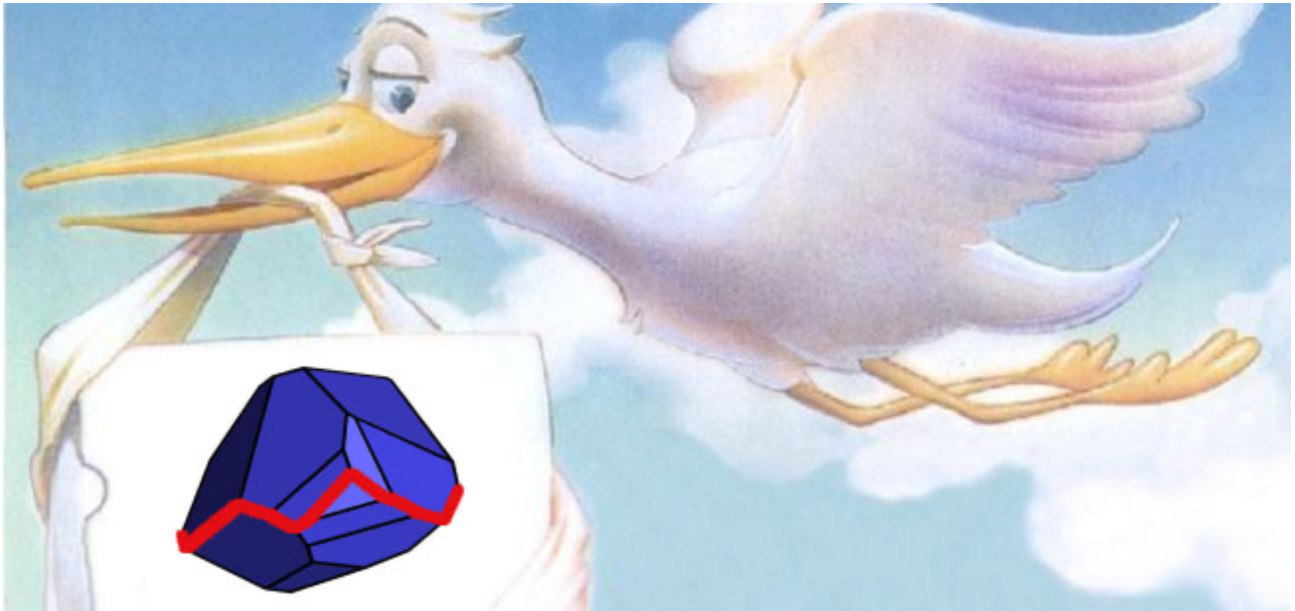
- Start with d-well-rounded $P = \{x \mid \bar{a}_i^T x \leq 1\}$
- Perturb LHS:
  - Let a_i be n-d normal centered at $\bar{a}_i$ with variance = 1/poly
- Perturb RHS to have entries $1+r_i$, where r_i are indep. exp. random variables with expectation λ:

  $$\Pr\left[r_i \geq t\right] = e^{-t/\lambda}$$

- Get perturbed polytope

$$Q = \left\{\boldsymbol{x} : \forall i,\ \boldsymbol{a}_i^T \boldsymbol{x} \leq 1 + r_i\right\}$$

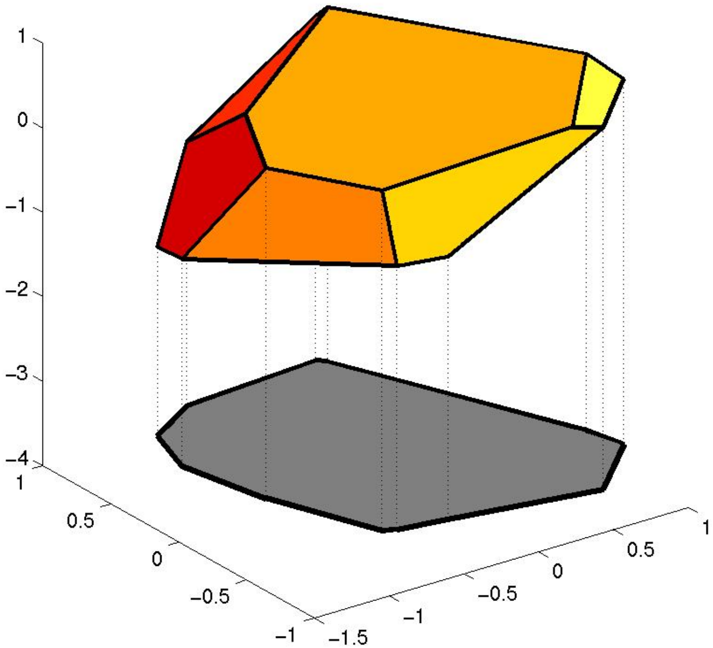
Where do short paths on polytopes come from?



- We'll find paths by looking at 2-d projections
  - Based on "Shadow vertex pivot rule" for simplex method

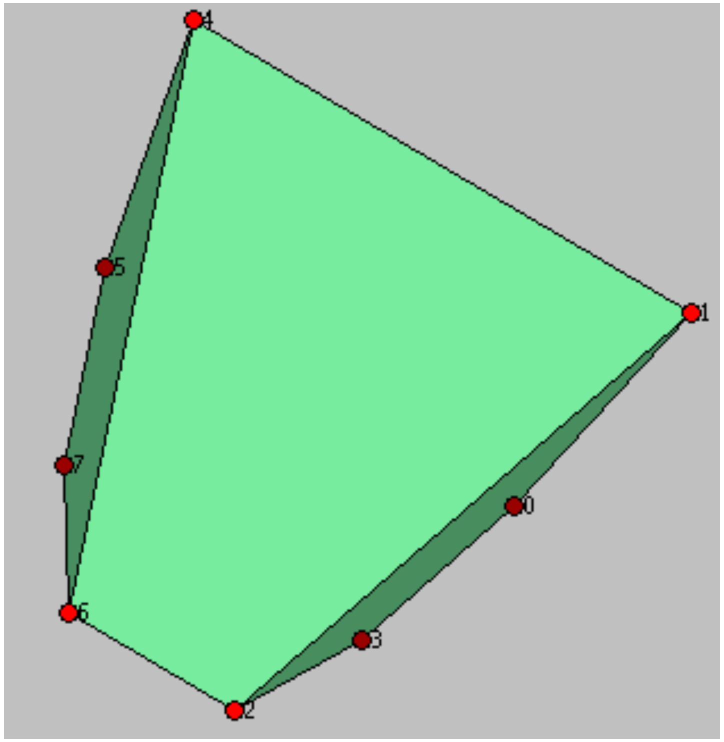
The Shadow Vertex Pivot Rule

- Project onto 2-d subspace V (called **shadow plane**)
  - Projection called **shadow**
  - Vertices project to vertices
  - Edges project to edges
- Walk along edges that appear in projection
- A vertex appears on boundary of shadow iff it maximizes some objective function c∈V



The Shadow Vertex Pivot Rule

- To get path between verts s and t:
  - Let c be an obj. fn. maximized at s
  - Let d be an obj. fn. maximized at t
  - Choose shadow plane to be span(c,d)
- Has well-known examples when it takes exponentially many steps



- Will argue that projection onto uniformly random subspace won’t have a lot of edges
  - This will only need perturbation of RHS(!)
- Argument will go through unchanged as long as subspace has reasonable amount of randomness
  - E.g. span(c,d), where c, d fixed vects plus 1/poly noise

Getting Randomness in the 2-plane

- If constraints from $a_1,\ldots,a_d$ tight at s, then s maximizes any c in pos($a_1,\ldots,a_d$)
- Because of perturbation, cone of such c very unlikely to be really small
  - I.e., will contain ball of radius 1/poly w.h.p.
  - More on this later
- Similarly for d
- So can think of c, d as being fixed vects plus 1/ poly noise
  - This is only place we use perturbation of the a_i

- ***Theorem:*** Let V be a random 2-plane and Q k-well-rounded with RHS perturbation described above. Then the expectation over V and $\{r_i\}$ of the number of edges of the projection of Q onto V is at most

$$\frac{12\pi k(1+\lambda\ln(ne))\sqrt{dn}}{\lambda}.$$

  - And similar (up to poly factors) for any reasonably random V
- So:
  - B/c of RHS perturbation, any reasonably random projection has polynomially many edges in expectation
  - B/c of LHS perturbation, vertices tend to optimize a reasonably large cone of objective functions
    - So given c,d, exist a bunch of planes containing them, and thus a short path, w.h.p.
- **Question:** Why doesn't this give a diameter bound???

Proof of Shadow Bound

- **Proof will proceed by analyzing expected lengths of edges on boundary**
- Let $r=\max_i r_i$.
- Q is $k(1+r)$-well-rounded
- Implies shadow of Q on V is contained in ball of radius $k(1+r)$, so perimeter of shadow is at most $2\pi k(1+r)$.
- Can show $E[r]\leq\lambda\ln(ne)$, so $E[\text{perimeter}]\leq 2\pi k(1+\lambda\ln(ne))$.
- So expected total length of edges of shadow is bounded above
- Get our bound by showing each edge is unlikely to be very short, so can't have too many of them

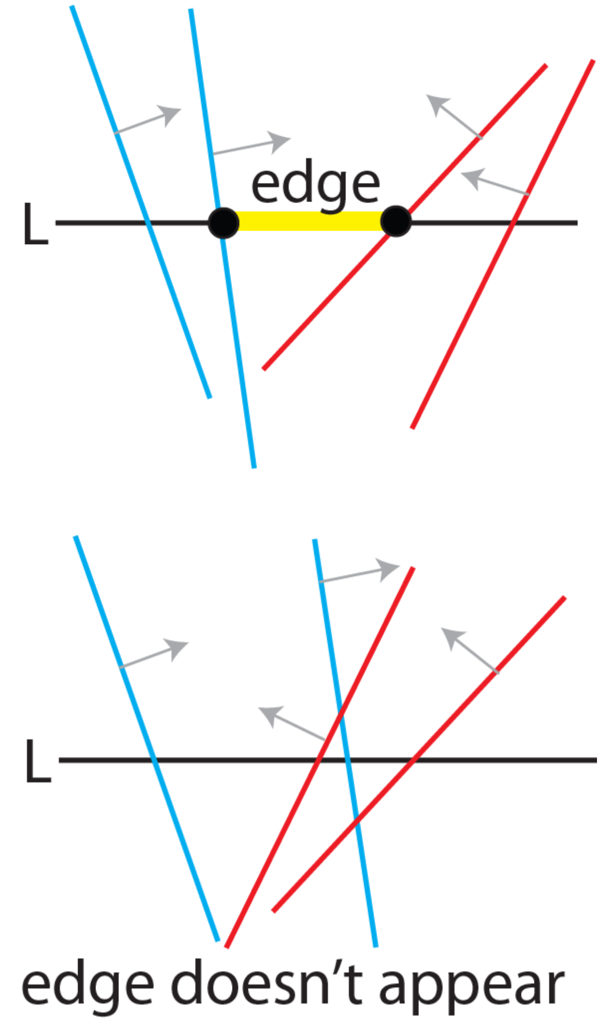
Proof of Well-Rounded Shadow Bound (cont.)

- **Step 2: Length of a given edge of Q expected to be long, if it appears**
- An edge is determined by the d-1 constraints that are tight on it
- For each I $\in \binom{[n]}{d-1}$, let A(I) be event that it appears on the convex hull of Q.
- If I appears, let δ(I) be its length
- ***Lemma:***

$$\Pr\left[\delta(I) < \epsilon \middle| A(I)\right] \leq \frac{n\epsilon}{2\lambda}.$$

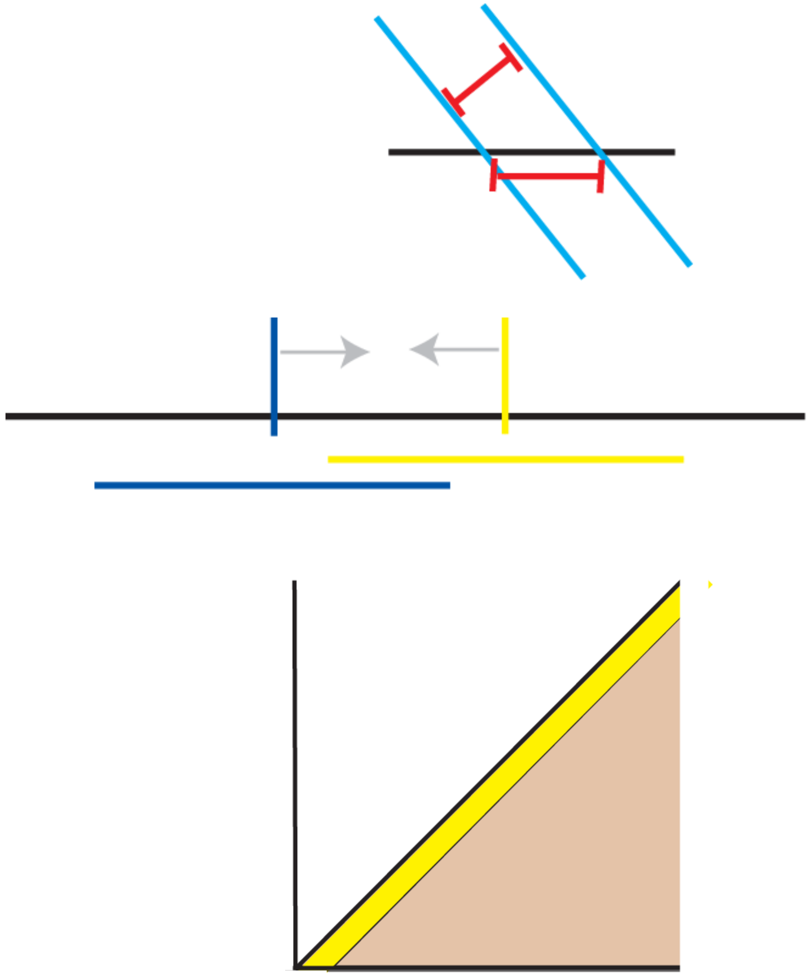
Intuition for Proof of Lemma

- Arbitrarily set r_i for all $i \in I$
- Consider line L of points satisfying $a_i^T x = 1 + r_i$ for all $i \in I$
- Every other constraint intersects this line either positively or negatively
- Edge length is distance between intersection points of max neg. constraint and min pos. constraint



Intuition for Proof of Lemma (cont.)

- If perturb RHS, intersection point moves by at least size of perturbation
- Now consider range of possible locations of constraints
- Contingent upon edge appearing, likely to be fairly long



Proof of Well-Rounded Shadow Bound (cont.)

- **<u>Step 3:</u> Projection is unlikely to decrease edge length too much.**
- Let $S_I(V)$ be event that I appears on shadow
- If I in shadow, let $\ell(I)$ be its projected length
  - $\ell(I) = \delta(I)\cos(\theta(I))$, where θ is angle I makes with V
- **Lemma:**

$$\mathbf{E}_{V,r_1,\ldots,r_n}\left[\ell(I)|S_I(V)\right] \geq \frac{\lambda}{6\sqrt{d}n}$$

Intuition for Proof of Lemma

- Suppose just projected a unit vector v onto a uniformly random unit vector u
  - Dot product probably not too small: $\Pr\left[u \cdot v \leq \varepsilon\right] \leq \sqrt{d}\,\varepsilon$
- We have to condition on $S_I(V)$, but get to project onto a *2-d* subspace V
  - Write V as span(v,w), v objective function optimized by our edge, w orthogonal to it
  - For any w, edge will appear in projection, because of v, so should be almost the same as projecting onto uniformly random w, which would give us desired bound
  - Full proof a little harder, needs some Jacobians and such, in order to actually get probabilities right.

Putting the Steps Together

$$2\pi k(1+\lambda\ln(ne)) \geq \sum_{I\in\binom{[n]}{d-1}} \mathbf{E}\left[\ell(I)\right]$$

$$= \sum_{I\in\binom{[n]}{d-1}} \mathbf{E}\left[\ell(I)|S_I(V)\right]\Pr\left[S_I(V)\right]$$

$$\geq \sum_{I\in\binom{[n]}{d-1}} \frac{\lambda}{6\sqrt{d}n}\Pr\left[S_I(V)\right]$$

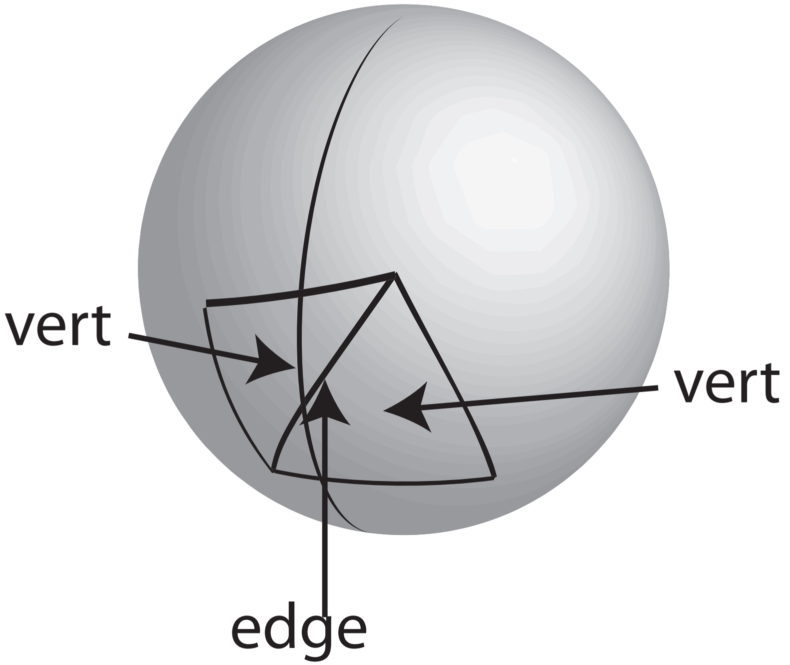
$$\Downarrow$$

$$\mathbf{E}\left[\text{number of edges}\right] = \sum_{I\in\binom{[n]}{d-1}} \Pr\left[S_I(V)\right] \leq \frac{12\pi k(1+\lambda\ln(ne))\sqrt{d}n}{\lambda}$$

A Polar View

- Can also get results by looking at polar (a.k.a. dual) polytope
- If $P = \{x \mid a_i^T x \leq 1\}$, **polar polytope** P* given by conv(a_i)
- Vertices become facets, edges become ridges (=dim d-2 faces), etc.
    - Given by conv(obj. fns. they optimize)
- Projection becomes intersection
    - Vertices appear in 2-d projection if plane intersects corresponding facets
    - Similarly for edges

- Verts give partition (usually triangulation) of sphere
- 2-planes intersect sphere in great circles
- Perturbation makes it very unlikely that a facet is ill-conditioned
  - RHS moves points in and out, which changes proj. onto sphere



- Claim that prob. a vertex appears on random 2-d proj. is proportional to surface area of corresp. spherical triangle
  - Other dims—spherical mixed vols

- Well-conditioned triangles: not too much surface area for given volume
  - And total vol fixed (=vol of sphere)
- Expectation proportional to total surface area, so isn't too big
- Also, most directions are contained in well-conditioned triangles
- Pretty robust: works well as long as everything "reasonably random"

What We Can/Can't Show

- **General plan:** To get short path from s to t, pick random 2-plane containing both
  - Almost all verts well-cond.
  - So almost always have enough randomness to show expected short path w.h.p.
    - Which guarantees existence
- **Nice trick:** Claim that enough to look at 2-planes span (c,r), where c opt at s, r unif. rand.
  - Why?
- So get that short paths b/w verts optimizing any given pair of directions w.h.p.
- **Only problem:** exponentially many verts., polynomially small probs., so can't union bound

Why Smoothed Diameter Bound Shouldn't Need Much More…

- How could this be true, but diameter still large?
  - Could have very small cones of directions where need very long paths
- But can we really?
- Call vertex "good" if can get to vert. optimizing unif. random obj. fn. in poly steps (in expectation), "bad" otherwise
- A vertex is good if any combinatorially nearby vertices are good
- May have some vertices where argument fails, can all nearby vertices fail too?
- Also, note that not really using all of our randomness
  - RHS perturbation by itself enough to get short paths between almost every pair of directions!
- Suggests approaches to un-smoothed problem

Conclusions

- Geometry can help
- Shouldn’t hope to do general case if we can’t even do random case
- If we can do random case, can probably do smoothed case
- If we can do smoothed case, gives ideas for general problem
- Can almost do smoothed case