Geodesic Embeddings of Path Complexes

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work with Deza, Hua, Holt, Klee, Schewe

Motivation

- diameter of simple polytopes
- computational/exhaustive approach
- enumerate combinatorial types of paths instead of combinatorial types of polytopes.

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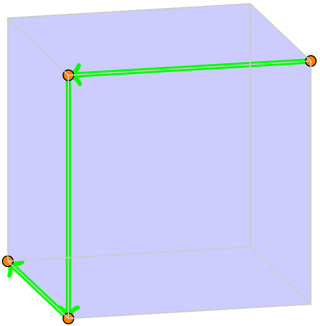

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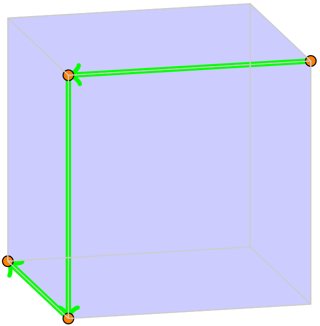
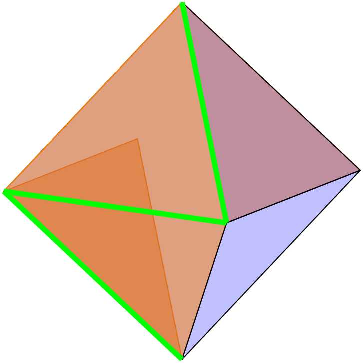
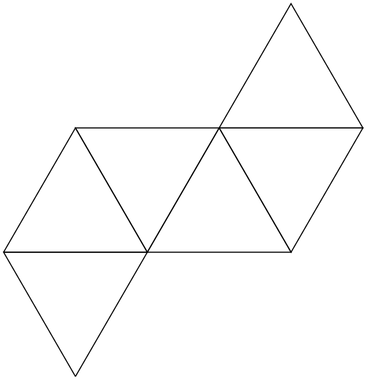




Simplicial complexes

- A *path complex* is a pure simplicial complex whose dual graph (with edges defined by ridges) is a path.
- A path complex is *end disjoint* if the two facets of dual-degree 1 are disjoint (complementary, prismatoid).



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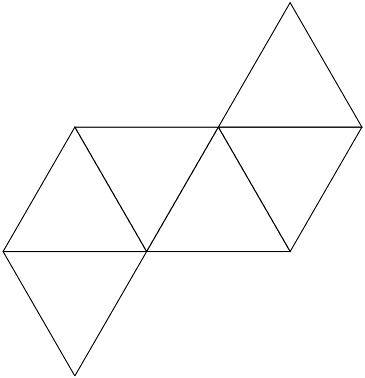
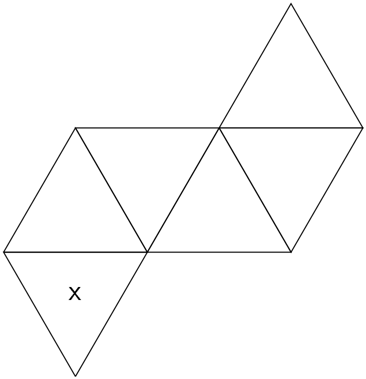


Directed Path Complexes

- A *directed* path complex is a path complex with distinguished *start facet*
- Every path complex corresponds to at most two non-isomorphic *orientations* (directed path complexes)



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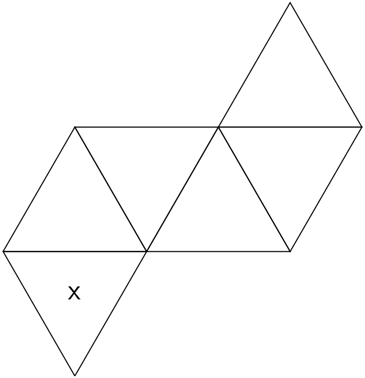
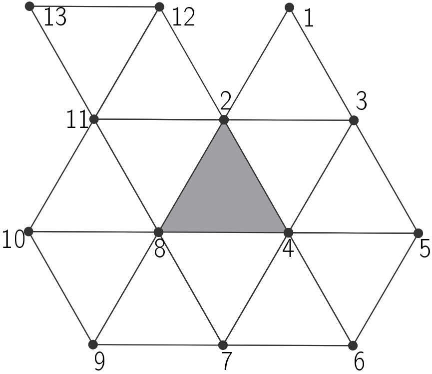


Revisits

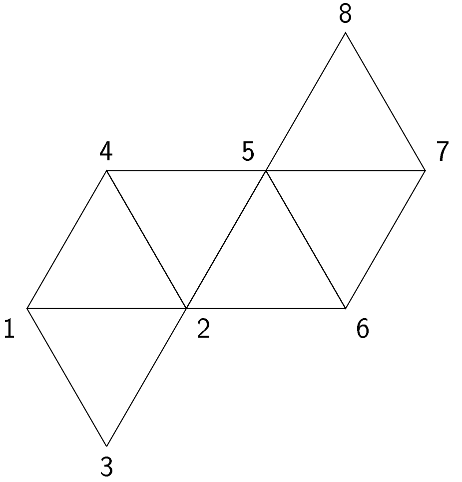


Directed Non-Revisiting Paths

- pivots and pivot sequences $(3,4)(1,5)(4,6)(2,7)(6,8)$
- Table representation

1	2	3
1	2	$\underline{4}$
$\underline{5}$	2	4
5	2	$\underline{6}$
5	$\underline{7}$	6
5	7	$\underline{8}$

- index sequence $\langle 3, 1, 3, 2, 3 \rangle$



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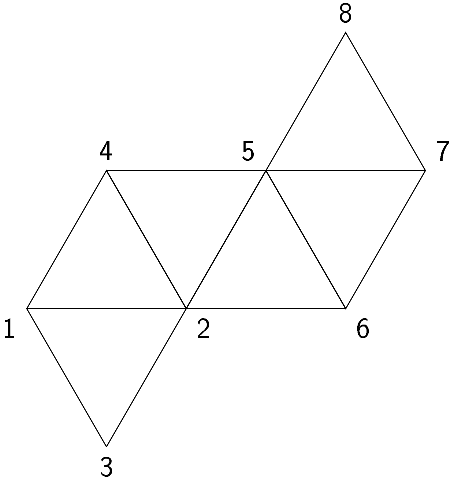

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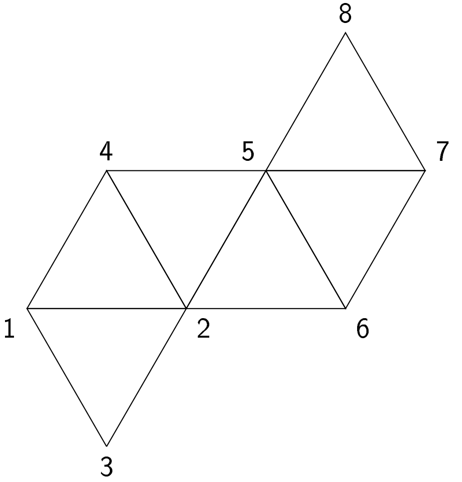
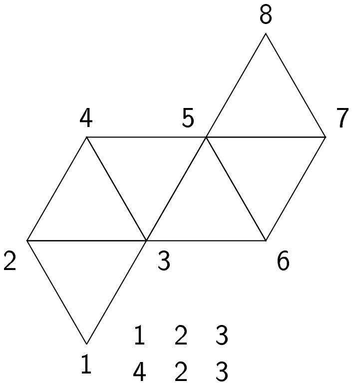


Canonical Index Sequences



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canonical index seq. indicies in order.

$$\langle 1, 2, 1, 3, 1 \rangle$$

recursion No index occurs twice in a row

$$t(d, l) = (d-1)t(d, l-1) + t(d-1, l-1).$$

restricted growth func. indices occur in order

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stirling numbers $t(d, l) = \left\{ {l-1 \atop d-1} \right\}$.

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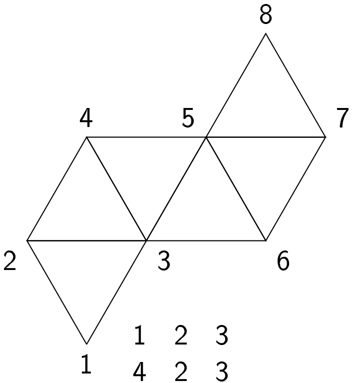

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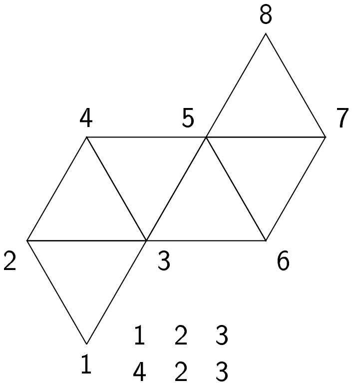

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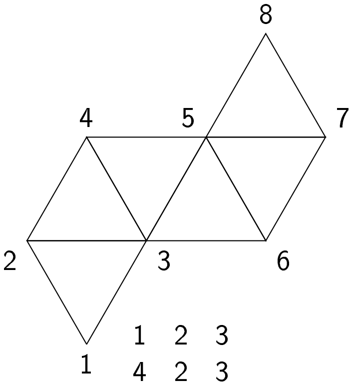

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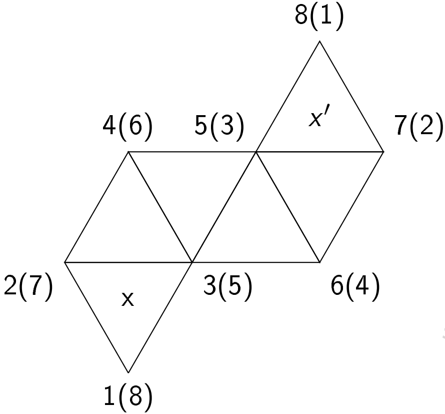
Counting Revisit Free Undirected Paths

A path complex is called *symmetric* if the two possible orientations of it are isomorphic.



$$\begin{aligned} s(d,l) = (d-1)s(d,l-2) &+ \\ s(d-1,l-2) &+ \\ s(d-2,l-2) & \end{aligned}$$

$$s(2,l) = s(d,d) = 1$$

$$s(d,d+1) = \left\lfloor \frac{d}{2} \right\rfloor = \begin{cases} \frac{d}{2} & \text{if } d \text{ even} \\ \frac{d-1}{2} & \text{if } d \text{ odd.} \end{cases}$$

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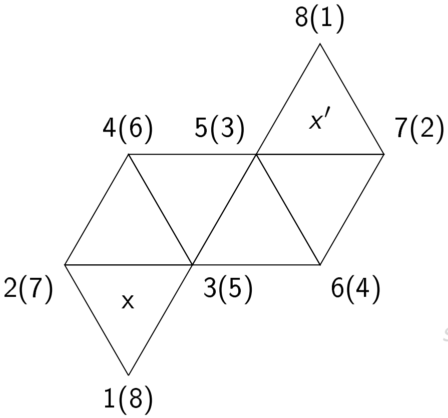

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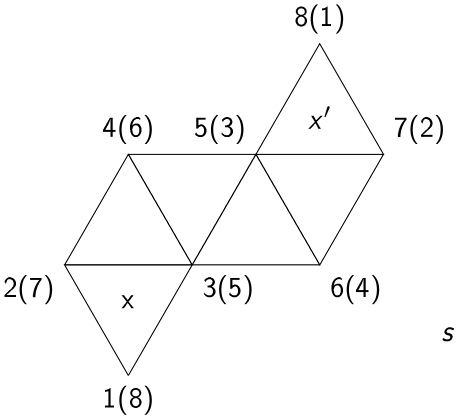

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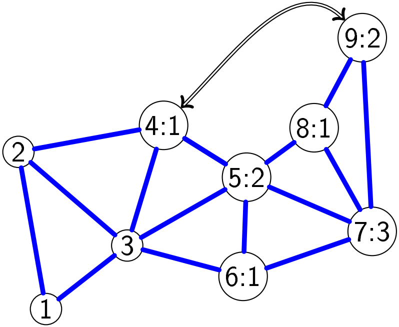
Necessary conditions for revisits

$\langle 1, 2, [1, 3, 1, 2] \rangle$



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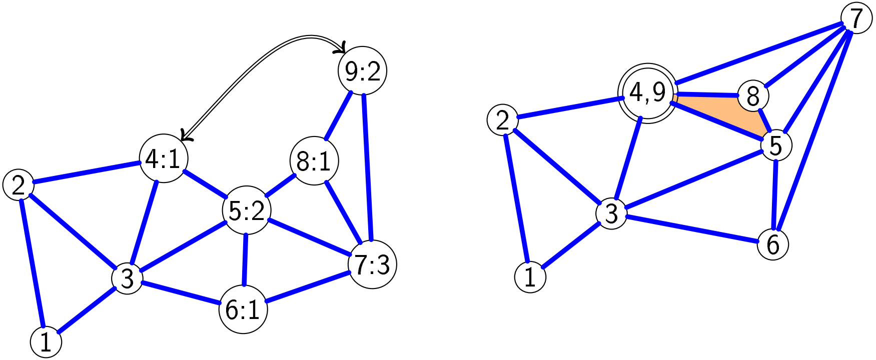

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Breaking Symmetry of Path Complexes

Generically Reduction to graph isomorphism, solve with partition backtracking

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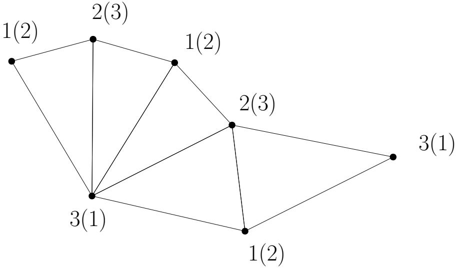

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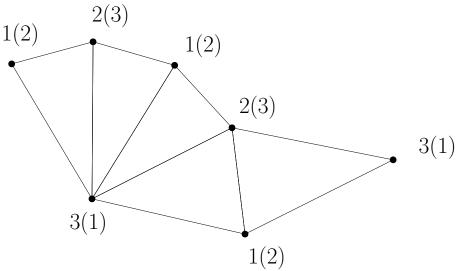

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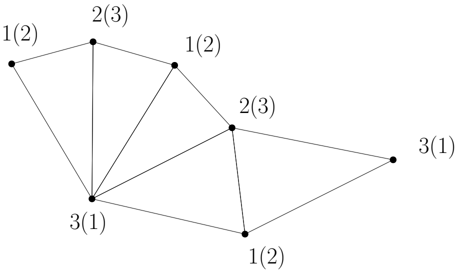

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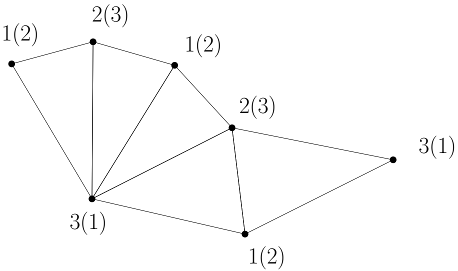

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Images of paths

Lemma

If $\Delta(d-1, n-1) < l-1$*, then a canonical index sequence of length* l *on* d *symbols in which the symbol* 1 *appears uniquely at the beginning or in which* d *appears uniquely at the end cannot correspond to a shortest path.*

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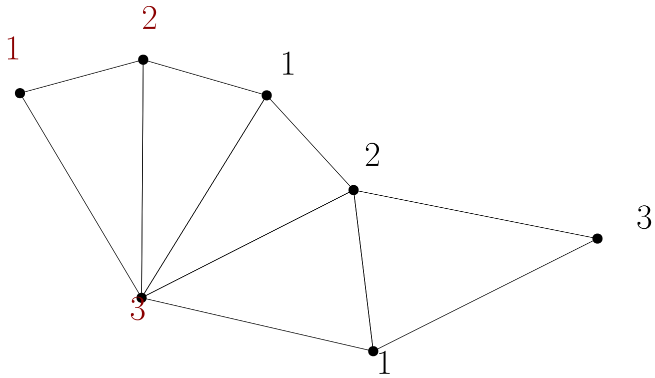
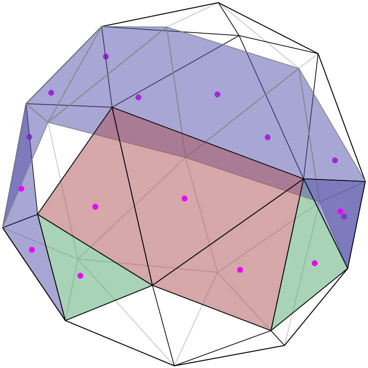


Shortcut free embedding.

- Q is a *shortcut* for P if it shares end facets with P and has a shorter dual graph.
- A *geodesic embedding* of a path complex P is a simplicial sphere S containing P as a sub-complex and not containing any shortcut for P as a subcomplex.



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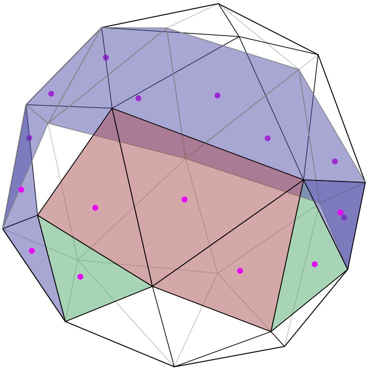
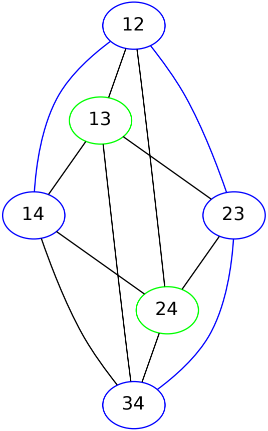


Exhaustive generation of shortcuts.

- Path $\pi = v_0, v_1 \ldots v_k$ is *inclusion-minimal* if no proper subset of $v_0 \ldots v_k$ is a path from v_0 to v_k.
- *Pivot graph* nodes are d-sets, edges are pivots.
- All inclusion-minimal paths from s to t in the pivot graph can be generated by backtracking.



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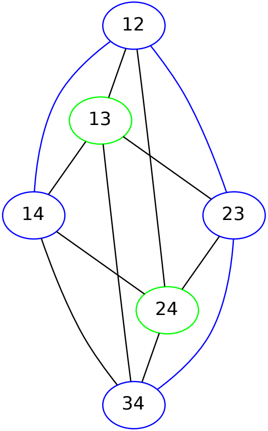

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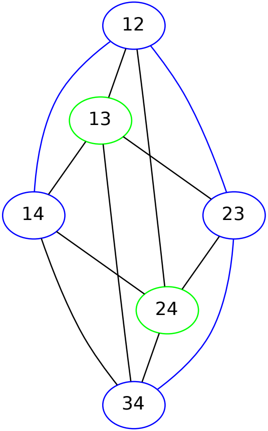
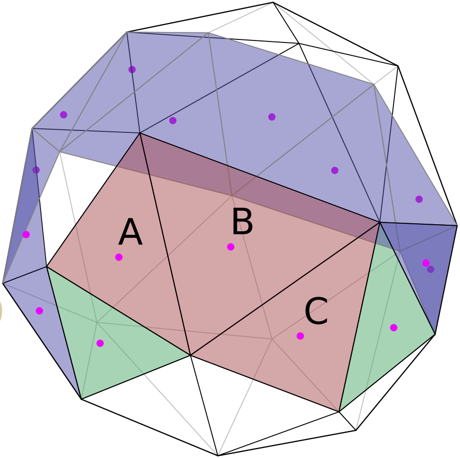


Incremental discovery of shortcuts.

Incremental Construction

1. Find candidate embedding
2. Find shortcuts (BFS)
3. Add constraint
   $\neg(\mathrm{facet}(A) \wedge \mathrm{facet}(B) \wedge \mathrm{facet}(B))$



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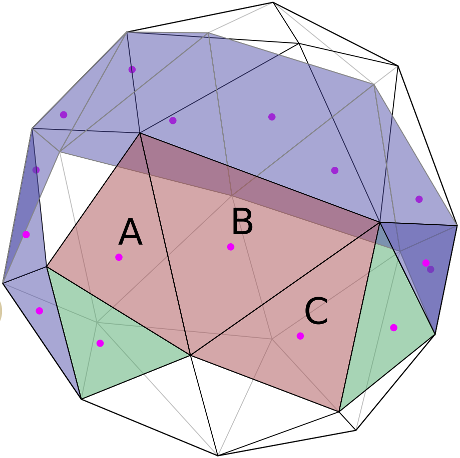

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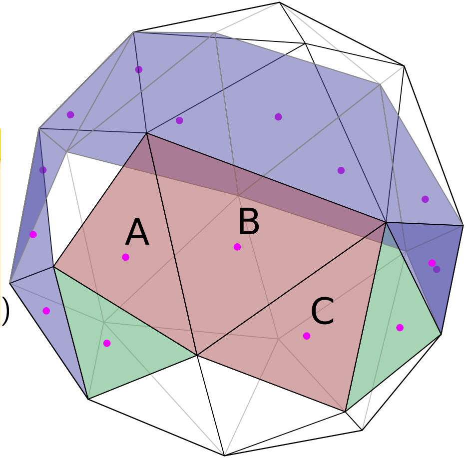

Grassmann-Plücker Relations

Given $\{ x_1 \dots x_{d-1}, y_1, y_2, y_3, y_4 \} \in \mathbb{R}^{d+1}$

$$\begin{aligned}0 &= \det(x_1 \dots x_{d-1}, y_1, y_2) \cdot \det(x_1 \dots x_{d-1}, y_3, y_4)\\ &+ \det(x_1 \dots x_{d-1}, y_1, y_4) \cdot \det(x_1 \dots x_{d-1}, y_2, y_3)\\ &- \det(x_1 \dots x_{d-1}, y_1, y_3) \cdot \det(x_1 \dots x_{d-1}, y_2, y_4)\end{aligned}$$

$$\chi(i_1, \dots i_{d+1}) := \operatorname{sign} \det x_{i_1}, \dots x_{i_{d+1}})$$

$$\begin{aligned}\chi(1,2,3)\chi(1,4,5) &= +1\\ \chi(1,2,4)\chi(1,3,5) &= -1\\ \chi(1,2,5)\chi(1,3,4) &= +1\end{aligned}$$

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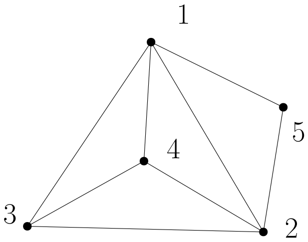

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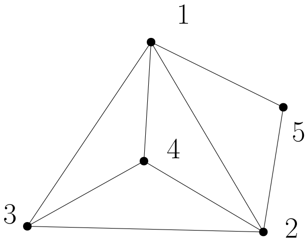

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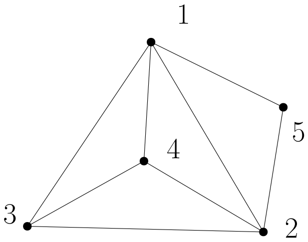
Grassmann-Plücker Relations

Given $\{ x_1 \ldots x_{d-1}, y_1, y_2, y_3, y_4 \} \in \mathbb{R}^{d+1}$

$$\begin{aligned}0 &= \det(x_1 \ldots x_{d-1}, y_1, y_2) \cdot \det(x_1 \ldots x_{d-1}, y_3, y_4)\\ &+ \det(x_1 \ldots x_{d-1}, y_1, y_4) \cdot \det(x_1 \ldots x_{d-1}, y_2, y_3)\\ &- \det(x_1 \ldots x_{d-1}, y_1, y_3) \cdot \det(x_1 \ldots x_{d-1}, y_2, y_4)\end{aligned}$$



$$\chi(i_1, \ldots i_{d+1}) := \text{sign} \det x_{i_1}, \ldots x_{i_{d+1}})$$

$$\begin{aligned}\chi(1,2,3)\chi(1,4,5) &= +1\\ \chi(1,2,4)\chi(1,3,5) &= -1\\ \chi(1,2,5)\chi(1,3,4) &= +1\end{aligned}$$

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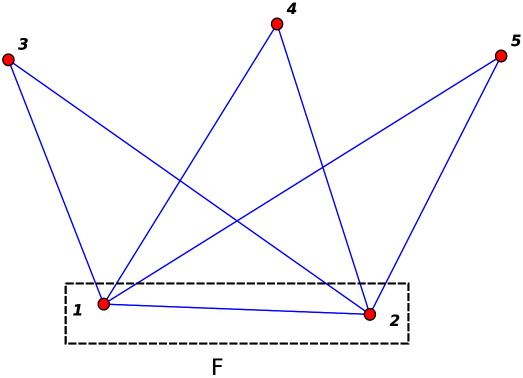
Determining facets and non-facets

F is a *facet* iff for all $x, y \notin F$

$$\chi(F\ x) = \chi(F\ y)$$

- no interior points
- Facet constraints are two variable equations.
- Non-facet constraints are "not-all-equal" constaints.



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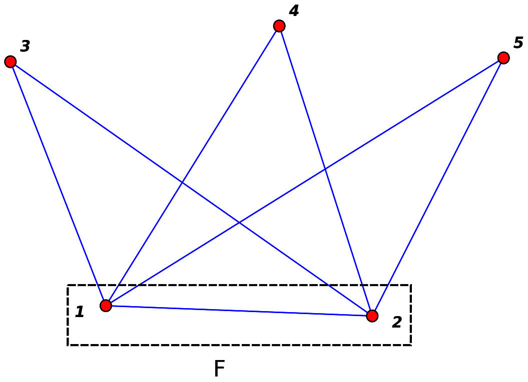

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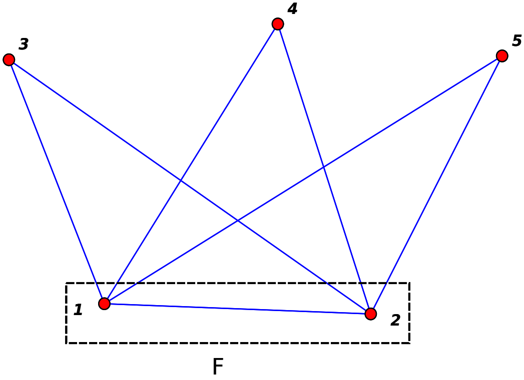
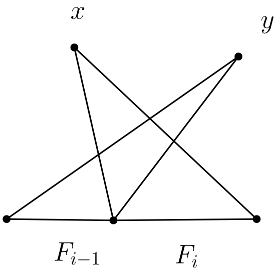


Path Constraints

- Let $\tau(Y) = (-1)^k$ where k transpositions are required to sort tuple Y.
- For $i > 1$, let $e_i = F_i \setminus F_{i-1}$, $l_i = F_{i-1} \setminus F_i$.
- Let $\sigma_1 = 1$, and for $i > 1$ let $\sigma_i = \tau(F_{i-1}, e_i)\tau(F_i, l_i)\sigma_{i-1}$. If F_i and F_{i-1} are both facets, then

$$\sigma_i \chi(F_i, x) = \sigma_{i-1}\chi(F_{i-1}, y) \qquad x \notin F_i, y \notin F_{i-1}$$



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(No-Path)

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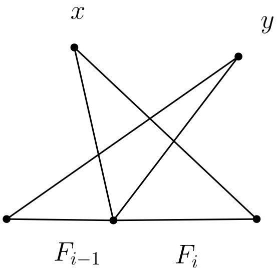

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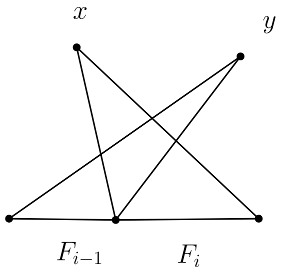

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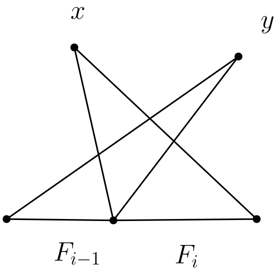

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(No-Path)

How many cases?

d	n	length	rev.	drops	
d	n	k	m	l	#
4	11	7	0	0	50
4	11	7	1	1	200
4	11	7	2	2	354
4	11	7	3	3	96
4	12	8	0	0	160
4	12	8	1	1	1258
4	12	8	2	2	5172
4	12	8	3	3	7398
4	12	8	4	4	1512
5	11	7	1	0	98
5	11	7	2	1	98
5	12	8	1	0	1079
5	12	8	2	1	3184
5	12	8	3	2	2904

d	n	k	m	l	#
6	12	7	1	0	11
6	13	8	1	0	293
6	13	8	2	1	452

hard cases

d	n	k	m	l	#
4	12	8	0	0	2
5	12	8	1	0	15
5	12	8	2	1	6
6	13	8	1	0	138
6	13	8	2	1	63

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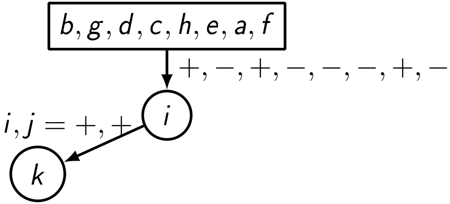
Parallel Backtracking

$$\boxed{b, g, d, c, h, e, a, f}$$

Parallel Backtracking

$$\begin{tikzcd}
\boxed{b, g, d, c, h, e, a, f} \arrow[d, "{+, -, +, -, -, -, +, -}"] \\
i
\end{tikzcd}$$

Parallel Backtracking



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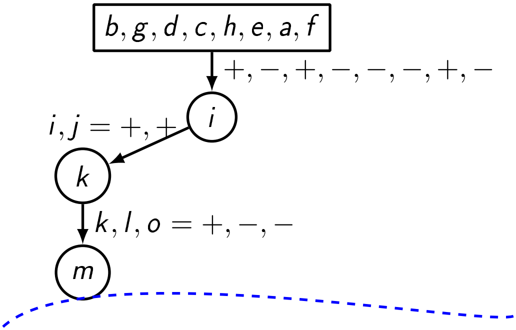

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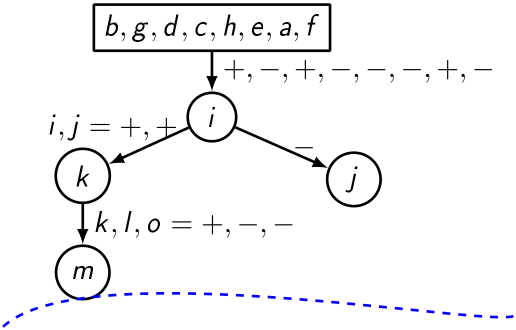

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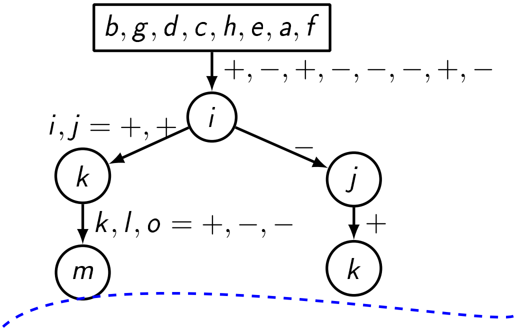

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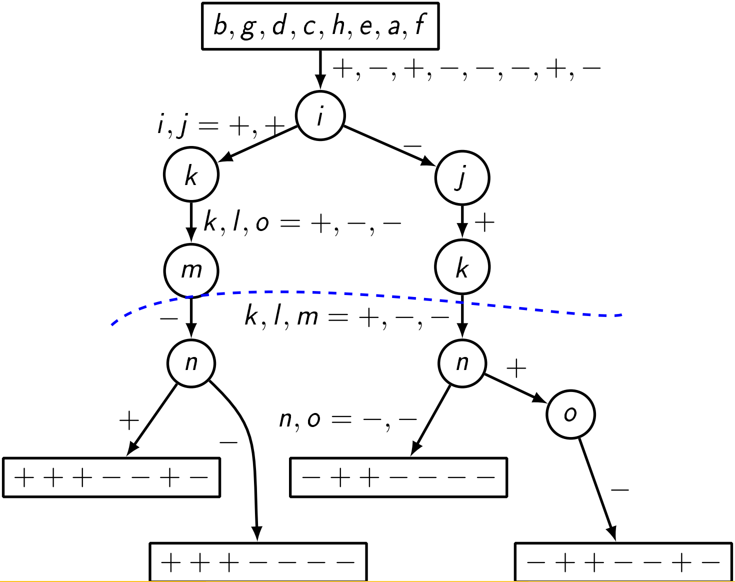

Updated Bounds

$n - d$

d	4	5	6	7	8
4	4	5	5	**6**	**7**
5	4	5	6	7	
6	4	5	**6**	7	
7	4	5	6	$[7, 10]$	