History based pivot rules for acyclic USOs on hypercubes

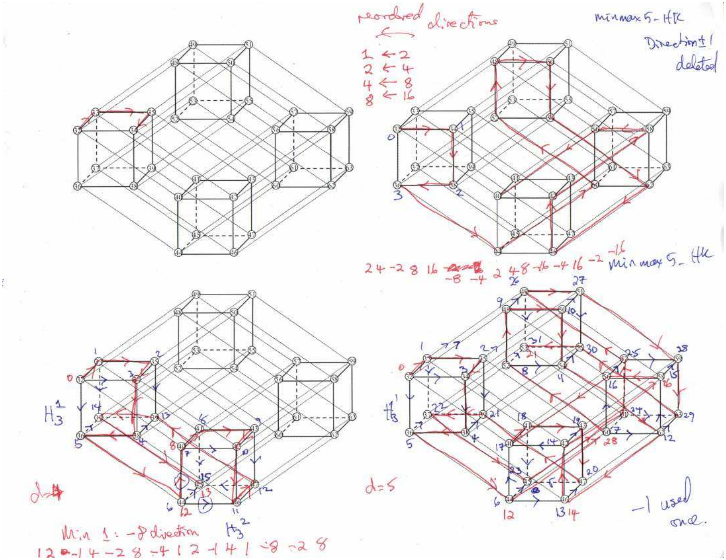
David Avis (Kyoto, McGill)
joint work with
Yoshikazu Aoshima, Theresa Deering, Yoshitake Matsumoto and Sonoko Moriyama

January 19, 2011

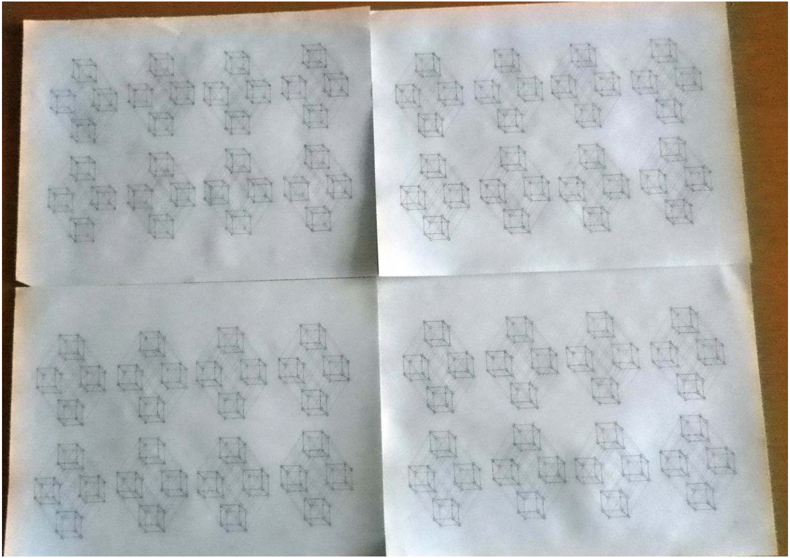
Warning !

This problem may be addictive ...

Works for low dimensions ...



Let's try $d = 10$



Starting point

(Courtesy: G. Ziegler)

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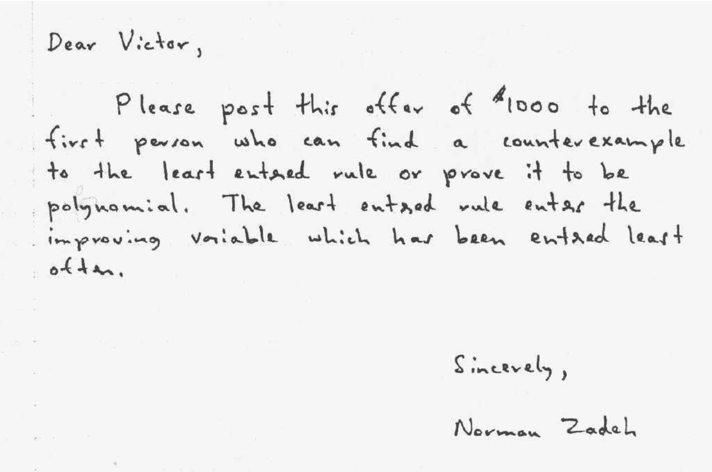

Dear Victor,

Please post this offer of $1000 to the first person who can find a counterexample to the least entered rule or prove it to be polynomial. The least entered rule enters the improving variable which has been entered least often.

Sincerely,

Norman Zadeh

Reward claimed!

- Thursday, 10:30am
  Subexponential Lower Bounds for the Simplex Algorithm
  Oliver Friedmann
  "... We also give a subexponential lower bound for Zadehs pivoting rule which among all improving pivoting steps enters the variable that has been entered least often. "

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- Norman Zadeh will come tomorrow ....

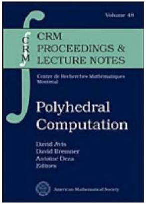
Long unpublished gem

- N. Zadeh, "What is the worst case behavior of the simplex algorithm," *Technical Report 27*, Dept. Operations Research, Stanford University, 1980.

Long unpublished gem

- N. Zadeh, "What is the worst case behavior of the simplex algorithm," *Technical Report 27*, Dept. Operations Research, Stanford University, 1980.
- Now published with postscript in:
  Polyhedral Computation, CRM-AMS Proceedings vol 48, eds. D.A., D. Bremner and A. Deza, 2009.

Polyhedral Computation



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David Avis (Editor), David Bremner (Editor), Antoine Deza (Editor)
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Victor Klee (1925-2007)

Vic Klee at Oberwolfach in 1981
(photo: L. Danzer)

Klee-Minty paper (1970)

How Good Is the Simplex Algorithm?

VICTOR KLEE*

Department of Mathematics, University of Washington, Seattle, Washington

AND

GEORGE J. MINTY†

Department of Mathematics, Indiana University, Bloomington, Indiana

1. INTRODUCTION

By constructing long "increasing" paths on appropriate convex polytopes, we show that the simplex algorithm for linear programs (at least with its most commonly used pivot rule, Dantzig [*1*]) is not a "good algorithm" in the sense of Jack Edmonds. That is, the number of pivots or iterations that may be required is not majorized by any polynomial function of the two parameters that specify the size of the program. In particular, $2^d - 1$ iterations may be required in solving a linear program whose feasible region, defined by d linear inequality constraints in d nonnegative variables or by d linear equality constraints in $2d$ nonnegative variables, is projectively equivalent to a d-dimensional cube. Further, for each d there are positive constants α_d and

The start of Polyhedral Computation?

Norm Zadeh

Norm Zadeh creator of Perfect Ten Magazine at his Beverly Hills Mansion November 2001 with his perfect 10 models (photo:Jonas Mohr)

For Sale!

1 of 6 Next Photo

(Erhard Pfeiffer)

Email

Hot property: Norm Zada, publisher of Perfect 10 magazine has listed his house in Beverly Park at $24.5 million

Sold!

72 BEVERLY PARK Dr, Beverly Hills, CA 90210 | MLS# 09-352603   http://www.redfin.com/CA/Beverly-Hills/72-Beverly-Park-90210...

- **Sold on 11/16/2010**

  $16,500,000

72 BEVERLY PARK Dr
Beverly Hills, CA 90210

BEDS:	**11**
BATHS:	**18**
SQ. FT.:	**20,000**
$/SQ. FT.:	**$825**
LOT SIZE:	**6.79 Acres**
PROPERTY TYPE:	**Residential, Single Family**
STYLE:	**Architectural**
VIEW:	**Canyon, City Lights, Mountain, Yes**
YEAR BUILT:	**2000**
COMMUNITY:	**Beverly Hills Post Office**
COUNTY:	**Los Angeles**
MLS#:	**09-352603**
SOURCE:	**TheMLS**
STATUS:	**Closed**

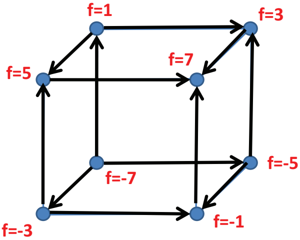
The absolute best opportunity to purchase a pristine almost new Beverly Park compound in years. Trophy contemporary estate by the Landry Design Group sited on the highest elevation in Beverly Park. The free-flowing approx 20,000 sq. ft. estate includes a new 6,100 sq. ft. guest house +

LP-digraphs

The Simplex Method and LP digraphs

Linear Programming (LP) **A**:

$$\min f := x + 2y + 4z$$
$$\begin{cases} -1 \le x \le 1 \\ -1 \le y \le 1 \\ -1 \le z \le 1 \end{cases}$$



LP digraph for A

The Simplex Method:

Algorithm of searching a sink of LP digraphs by some pivotting rules.

Strongly polynomial-time algorithms for LP?

Good characterizations for LP digraphs?

Basic problem

Can we efficiently find the sink of an LP-digraph by following a directed path from any given vertex, using a given edge selection rule (pivoting)?

Necessary conditions for LP digraphs

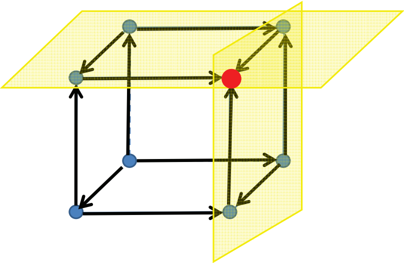
- Unique Sink Orientation (USO) [‘01 Szabo,Welzl]
- Acyclicity
- Holt Klee Property [‘99 Holt, Klee]
- Shelling Property [‘09 Avis, Moriyama]

Necessary conditions for LP digraphs

Unique Sink Orientation (USO)

['01 Szabo,Welzl]

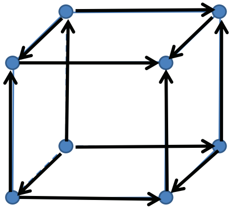
Each subgraph G(P,H) of G(P) induced by a face H of P has a unique sink (and then a unique source).



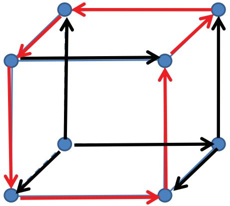
Necessary conditions for LP digraphs

Acyclicity

G(P) has no directed cycle.



Acyclic

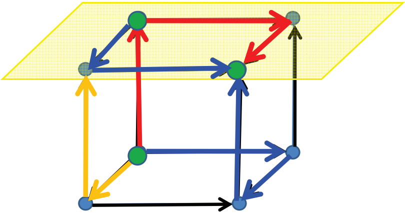


Not acyclic

Necessary conditions for LP digraphs

Holt Klee property ['99 Holt, Klee]

G(P) has a USO, and for every k-dimensional face H of P there are k disjoint paths from the unique source to the unique sink in G(P,H).



Klee-Minty Examples

- 3-cube (Chvátal, P.47)

$$
\begin{array}{lrl}
\textit{maximize} & 100x_1 + 10x_2 + x_3 & \\
s.t. & x_1 & \leq 1 \\
 & 20x_1 + x_2 & \leq 100 \\
 & 200x_1 + 20x_2 + x_3 & \leq 10000 \\
 & x_1, x_2, x_3 & \geq 0
\end{array}
$$

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$$

- Vertices:

0 0 0	0 100 8000
1 0 0	1 80 8200
1 80 0	1 0 9800
0 100 0	0 0 10000

Pivot Sequence (Dantzig's rule)

- x_1 x_2 x_3
  0 0 0
  1 0 0
  1 80 0
  0 100 0
  0 100 8000
  1 0 9800
  1 80 8200
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- x_n stays out of basis for 2^{n-1} iterations.

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- x_1 x_2 x_3
  0 0 0
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  1 80 0
  0 100 0
  0 100 8000
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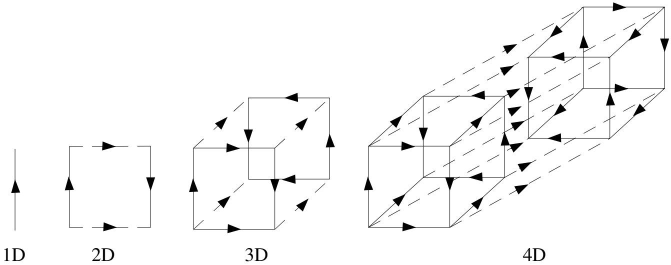
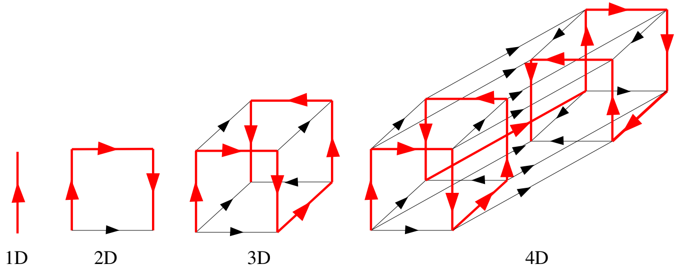
  1 80 8200
  0 0 10000
- x_n stays out of basis for 2^{n-1} iterations.
- x_1 pivots 2^{n-1} times.

Klee-Minty construction



Klee-Minty path



n-cube USOs

- Vertices $V = \{0, 1, ..., 2^n - 1\} = \{00..00, 00..01, ..., 11..11\}$

n-cube USOs

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- Facets $F_1, F_2, ..., F_{2n}$. For $i = 1, ..., n$,

$$F_i = \{(x_1, x_2, ..., x_n) | x_i = 0\},\ \ F_{n+i} = \{(x_1, x_2, ..., x_n) | x_i = 1\}.$$

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- Cobasis $C(v) = \{i : v \in F_i, i = 1, ..., 2n\}, v \in V$

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- Note $i \in B(v)$ iff $n + i \in C(v)$.

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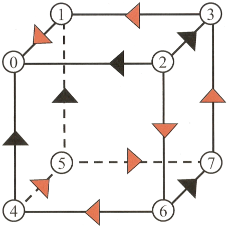
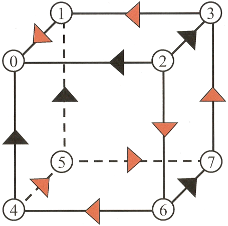
- Basis $B(v) = \{i : v \notin F_i, i = 1, ..., 2n\}, v \in V$
- Note $i \in B(v)$ iff $n + i \in C(v)$.
- A *pivot* interchanges a pair of indices i and $n + i$ between $B(v)$ and $C(v)$. (flips bit i of v)

3-cube acyclic USO



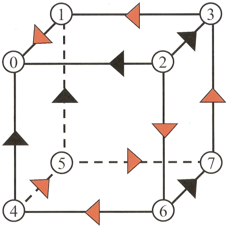
- Vertices $V = \{0, 1, ..., 7\} = \{000, 001, ..., 111\}$

3-cube acyclic USO



- Vertices $V = \{0, 1, ..., 7\} = \{000, 001, ..., 111\}$
- $F_i = \{(x_1, x_2, x_3)|x_i = 0\}, \;\; F_{3+i} = \{(x_1, x_2, , x_3)|x_i = 1\}$

3-cube acyclic USO



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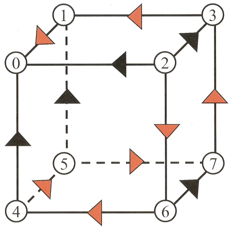
- $C(6) = C(110) = \{4, 5, 3\}$, $B(6) = B(110) = \{1, 2, 6\}$

3-cube acyclic USO



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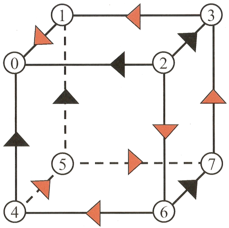

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- $v = 6$ pivots to vertices 2,4,7 by flipping bits 1,2,3
- Pivots correspond to moves in the 4,2,1 *directions*

History based rules

Choose the improving variable that satisfies:

- Least number of times to enter basis (Zadeh)

History based rules

Choose the improving variable that satisfies:

- Least number of times to enter basis (Zadeh)
- Least recently considered (Cunningham)

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- Least recently entered (Fathi-Tovey)

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- Least number of iterations in basis (A-M-M)

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- Least used direction (A-M-M)

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- Least recently basic (Johnson)

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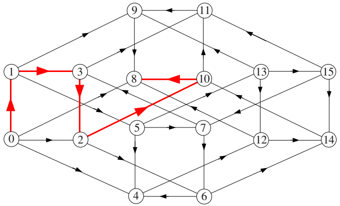
- All of the above break Klee-Minty type constructions

History based rules

Choose the improving variable that satisfies:

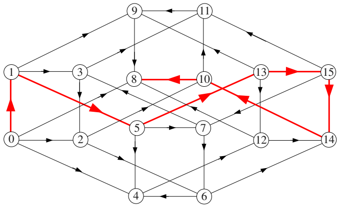
- Least number of times to enter basis (Zadeh)
- Least recently considered (Cunningham)
- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
- Least recently basic (Johnson)
- All of the above break Klee-Minty type constructions
- We try to find an acyclic USO for which a given rule follows a Hamiltonian path

Least entered rule (Zadeh)



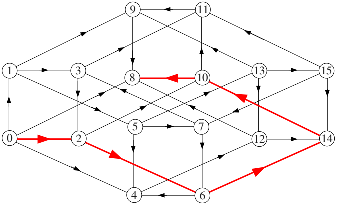
Vertex		(orientation, direction)-pair								Options
		+ 4	-4	+ 3	-3	+ 2	-2	+ 1	-1	
0	0 0 0 0	0	0	0	0	0	0	0	0	**+1**, +2, +3, +4
1	0 0 0 1	0	0	0	0	0	0	1	0	**+2**, +3, +4
3	0 0 1 1	0	0	0	0	1	0	1	0	**-1**, +4
2	0 0 1 0	0	0	0	0	1	0	1	1	+3, **+4**
10	1 0 1 0	1	0	0	0	1	0	1	1	**-2**
8	1 0 0 0	1	0	0	0	1	1	1	1	

Least recently basic (Johnson)



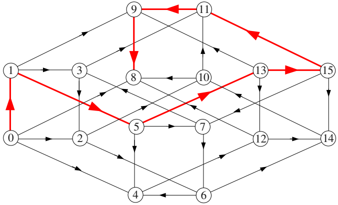
Vertex		+ 4	- 4	+ 3	- 3	+ 2	- 2	+ 1	- 1	Options
		(orientation, direction)-pair								
0	0 0 0 0		✓		✓		✓		✓	**+1**, +2, +3, +4
1	0 0 0 1		✓		✓		✓	✓		+2, **+3**, +4
5	0 1 0 1		✓	✓			✓	✓		+2, **+4**
13	1 1 0 1	✓		✓			✓	✓		**+2**
15	1 1 1 1	✓		✓		✓		✓		**-1**
14	1 1 1 0	✓		✓		✓			✓	**-3**
10	1 0 1 0	✓			✓	✓			✓	**-2**
8	1 0 0 0	✓			✓		✓		✓	

Least recently considered (Cunningham)



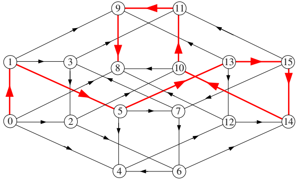
Vertex		Sequence	Options
0	0 0 0 0	+ 2, - 2, + 1, - 1, + 3, - 3, + 4, - 4	**+ 2**
2	0 0 1 0	- 2, + 1, - 1, + 3, - 3, + 4, - 4	**+ 3**
6	0 1 1 0	- 3, + 4, - 4, + 2, - 2, + 1, - 1	**+ 4**
14	1 1 1 0	- 4, + 2, - 2, + 1, - 1, + 3, - 3	**- 3**
10	1 0 1 0	+ 4, - 4, + 2, - 2, + 1, - 1, + 3	**- 2**
8	1 0 0 0	+ 1, - 1, + 3, - 3, + 4, - 4, + 2	

Least recently entered (Fathi-Tovey)



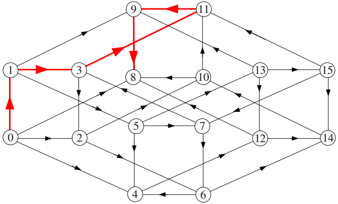
Vertex		(orientation, direction)-pair								Options
		+ 4	- 4	+ 3	- 3	+ 2	- 2	+ 1	- 1	
0	0 0 0 0		✓		✓		✓		✓	**+1**, +2, +3, +4
1	0 0 0 1		✓		✓		✓	✓		+2, **+3**, +4
5	0 1 0 1		✓	✓			✓	✓		+2, **+4**
13	1 1 0 1	✓		✓			✓	✓		**+2**
15	1 1 1 1	✓		✓		✓		✓		-1, **-3**, -4
11	1 0 1 1	✓			✓	✓		✓		**-2**
9	1 0 0 1	✓			✓		✓	✓		**-1**
8	1 0 0 0	✓			✓		✓		✓	

Least number of iterations in basis (A-M-M)



Vertex		(orientation, direction)-pair								Options
		+ 4	- 4	+ 3	- 3	+ 2	- 2	+ 1	- 1	
0	0 0 0 0	0	1	0	1	0	1	0	1	**+1**, +2, +3, +4
1	0 0 0 1	0	2	0	2	0	2	1	1	+2, **+3**, +4
5	0 1 0 1	0	3	1	2	0	3	2	1	+2, **+4**
13	1 1 0 1	1	3	2	2	0	4	3	1	**+2**
15	1 1 1 1	2	3	3	2	1	4	4	1	**-1**
14	1 1 1 0	3	3	4	2	2	4	4	2	**-3**
10	1 0 1 0	4	3	4	3	3	4	4	3	**+1**, -2
11	1 0 1 1	5	3	4	4	4	4	5	3	**-2**
9	1 0 0 1	6	3	4	5	4	5	6	3	**-1**
8	1 0 0 0	7	3	4	6	4	6	6	4	

Least used direction (A-M-M)



Vertex		Direction 4	3	2	1	Options
0	0 0 0 0	0	0	0	0	**1**, 2, 3, 4
1	0 0 0 1	0	0	0	1	**2**, 3, 4
3	0 0 1 1	0	0	1	1	**4**
11	1 0 1 1	1	0	1	1	**2**
9	1 0 0 1	1	0	2	1	**1**
8	1 0 0 0	1	0	2	2	

Least used direction

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- nv(i)= the number of times that direction i has been taken.

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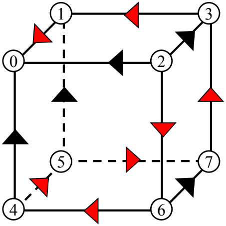
- Set nv(j)=nv(j)+1.
- Special case of Zadeh's rule.

Unique H_3

Hamilton path using least used direction rule
It satisfies the Holt-Klee condition



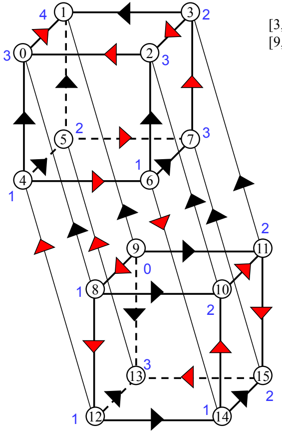
Least visited rule

nv(1) = nv(2) = nv(4) = 0
4: nv(1) = nv(2) = 0, nv(4) = 1
2: nv(1) = 0, nv(2) = 1, nv(4) = 1
1: nv(1) = 1, nv(2) = 1, nv(4) = 1
2: nv(1) = 1, nv(2) = 2, nv(4) = 1
4: nv(1) = 1, nv(2) = 2, nv(4) = 2
2: nv(1) = 1, nv(2) = 3, nv(4) = 2
1

Unique H_4

Hamilton path using least used direction rule

It satisfies the Holt-Klee condition



[3, 4, 3, 2, 1, 2, 1, 3, 1, 0, 2, 2, 1, 3, 2, 2]
[9, 8, 12, 4, 6, 14,10, 11, 15, 13, 5, 7, 3, 2, 0, 1]

nv(1) = nv(2) = nv(4) = nv(8) = 0
1: nv(1) = 1, nv(2) = nv(4) = nv(8) = 0
4: nv(1) = 1, nv(2) = 0, nv(4) = 1, nv(8) = 0
8: nv(1) = 1, nv(2) = 0, nv(4) = 1, nv(8) = 1
2: nv(1) = 1, nv(2) = 1, nv(4) = 1, nv(8) = 1
8: nv(1) = 1, nv(2) = 1, nv(4) = 1, nv(8) = 2
4: nv(1) = 1, nv(2) = 1, nv(4) = 2, nv(8) = 2
1: nv(1) = 2, nv(2) = 1, nv(4) = 2, nv(8) = 2
4: nv(1) = 2, nv(2) = 1, nv(4) = 3, nv(8) = 2
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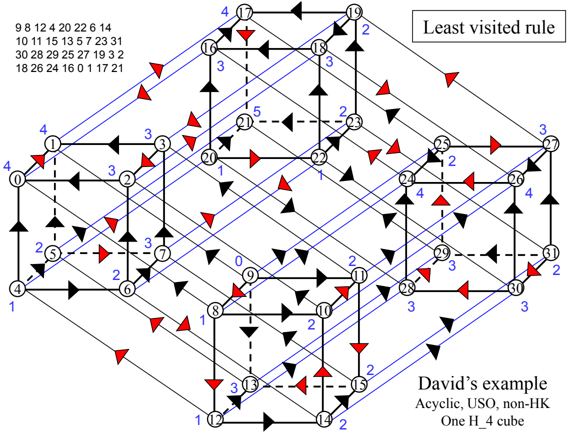
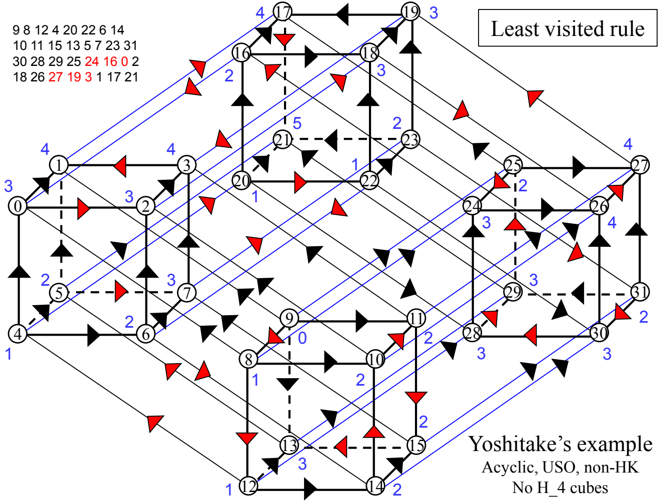
Least visited rule

H_5



9 8 12 4 20 22 6 14
10 11 15 13 5 7 23 31
30 28 29 25 27 19 3 2
18 26 24 16 0 1 17 21
Least visited rule
David's example
Acyclic, USO, non-HK
One H_4 cube

Another candidate for H_5



Computational results: least used direction

dimension	2	3	4	5
number of Hamilton paths	1	1	1	2
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- Since HK fails for $n = 5$, these examples are not LP-digraphs

Computational results: least used direction

But things do not go well for $n \geq 6$ ...

dimension	2	3	4	5	6	7	8
number of Hamilton paths	1	1	1	2	0	0	0
Holt-Klee	1	1	1	0	0	0	0

We did a computer search of all acyclic USOs that contain Hamiltonial paths.

Least times to enter basis (Zadeh's rule)

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Computational results: least times to enter basis

The deluge!

dimension	2	3	4	5	6	7
Ham. paths	1	2	17	1,072	3,262,342	$\geq$ 42,500,000,000
Holt-Klee	1	2	12	79	360	none yet

Computational results: all rules

Dimension	2	3	4	5	6	7
Least-entered(Zadeh)	1	2	17	$1,072$	$3,262,342$	$> 10^{10}$
Least-used-direction	1	1	1	2	0	0
Least-recently-entered	1	1	1	0	0	0
Least-recently-considered	1	0	0	0	0	0
Least-recently-basic	1	0	0	0	0	0
Least-iterations-in-basis	1	0	0	0	0	0

Table: Hamiltonian paths produced by history based pivot rules

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Williamson Hoke's theorem (1988)

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- Eg. Exactly n vertices have in-degree one.

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- Reject partial HP if it violates W-H theorem

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- Except for Zadeh's rule we can assume that the first d pivots are $+1, +2, ..., +d$.
- For Zadeh's rule we cannot:
  Eg. +1, +2, -1, +3, ... is valid.

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In a canonical HP the indegree of a vertex is 1 if and only it is reached by a signed direction that is being used for the first time.

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- By WH there are no others.
- Eg: +1,+2,+3,-1,... does give a HP by Zadeh

Non-existence of Hamiltonian paths

Theorem

- For $d \geq 3$ the following rules do not generate any Hamiltonian paths on AUSO cubes: least-iterations-in-basis, least-recently-basic, least-recently considered.

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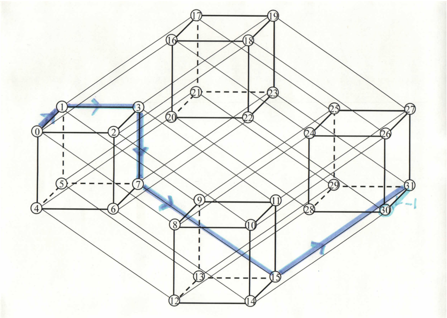
- For $d \geq 3$ the following rules do not generate any Hamiltonian paths on AUSO cubes: least-iterations-in-basis, least-recently-basic, least-recently considered.
- For $d \geq 5$ the least-recently-entered rule does not generate any Hamiltonian paths on AUSO cubes.

Proof of non-existence of Hamiltonian paths



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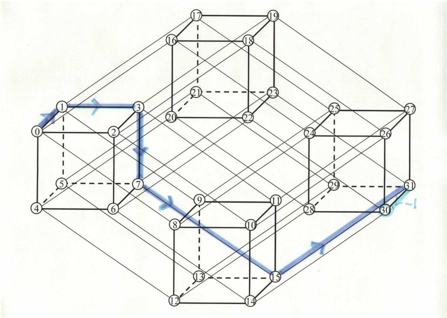

- The least-iterations-in-basis, least-recently-basic, least-recently considered, rules start with +1,+2,...,+d,-1,...
- These $d+1$ vertices have indegree one violating Williamson-Hoke

Non-existence: least-recently-entered, $d \geq 5$

Theorem 1.1. The least-recently entered rule does not have any Hamiltonian paths on a d-cube for $d \geq 5$.

Proof. Suppose P is a Hamiltonian path produced by Algorithm 1 for the least-recently entered rule when $d \geq 5$. We will show that P must begin with the sequence of vertices $Q = Q_1, Q_2, Q_3, Q_4$ where $Q_1 = \{0, 1, 3, \cdots, 2^d - 1\}$, $Q_2 = \{2^d - 1 - 2^{d-2}, 2^d - 1 - 2^{d-2} - 2^{d-3}, \cdots, 2^{d-1}\}$, $Q_3 = \{2^{d-1}+2, 2, 6, 14, \cdots, 2^d - 2\}$ and $Q_4 = \{2^d - 2 - 2^{d-2}, 2^d - 2 - 2^{d-2} - 2^{d-3}, \cdots, 2^{d-1}+2+4+8, 2^{d-1}+2+4\}$.

Q includes the vertices $\{2^{d-1}+2, 2^{d-1}+2+4+8, 2^{d-1}+2+4\}$ as a subsequence and does not contain the vertex $2^{d-1}+2+8$. These four vertices lie on a 2-face which has two sources, $2^{d-1}+2$ and $2^{d-1}+2+4+8$, a contradiction. It remains to show that P begins as specified.

- $Q_1 = 0, 1, 3 \cdots, 2^d - 1$. This follows from Lemma **??**.
- $Q_2 = 2^d - 1, 2^d - 2^{d-2} - 1, 2^d - 2^{d-2} - 2^{d-3} - 1, \cdots, 2^{d-1}$.

  We prove this by mathematical induction. For the basic step, we will show only $2^d - 2^{d-2} - 1$ can come right after $2^d - 1$. When we visited the vertex $2^d - 1$, all of the bits are 1. It means the next vertex can be represented as $2^d - 2^k - 1 = \sum_{i=0}^{d-1} 2^i - 2^k$ $(d-1 > k \geq 0)$. By Corollary **??**, vertex $\sum_{i=0}^{d-1} 2^i - 2^k$ should have two visited neighbours, one of which is obviously the vertex $2^d - 1$. In other words, there exists $j \neq k$ such that $\sum_{i=0}^{d-1} 2^i - 2^k - 2^j \in \{0, 1, 3 \cdots, 2^d - 1\} = \left\{v | \exists l \ s.t. \ v = \sum_{i=0}^{l} 2^i\right\} \cup \{0\}$. Since $d \geq 3$ forces $\sum_{i=0}^{d-1} 2^i - 2^k - 2^j$ not to be equal to 0, $\sum_{i=0}^{d-1} 2^i - 2^k - 2^j$ should be represented as $\sum_{i=0}^{l} 2^i = \sum_{i=0}^{d-1} 2^i - \sum_{i=l+1}^{d-1} 2^i$ for certain l. Therefore, the (k, j) equal $(d-1, d-2)$ or $(d-2, d-1)$, and $d - 1 > k$ requires $k = d - 2$.

  We can prove the inductive step similarly. If the path is continued by $2^d - 1, 2^d - 1 - 2^{d-2}, \cdots, 2^d - 1 - \left\{\sum_{i=d-2-k}^{d-2} 2^i\right\}$, the next vertex should be equal to $\sum_{i=0}^{d-1} 2^i - \sum_{i=d-2-k}^{d-2} 2^i + 2^j$ $(d-2-k \leq j \leq d-2)$ or $\sum_{i=0}^{d-1} 2^i - \sum_{i=d-2-k}^{d-2} 2^i - 2^j$ $(j = d-1$ or $j < d-2-k)$. By Corollary **??**, two neighbours of it are in $\left\{0, 1, 3 \cdots, 2^d - 1, 2^d - 1 - 2^{d-2}, \cdots, 2^d - 1 - \sum_{i=d-2-k}^{d-2} 2^i\right\}$ Using binary numbers, $2^d - \left\{\sum_{k=0}^{i} 2^{d-(2+k)}\right\} - 1$ can be denoted $100 \ldots 0011 \ldots 11$, where we have $k+1$ 0s.
- $Q_3 = \{2^{d-1} + 2, 2, 6, 14, \cdots, 2^d - 2\}$

  At the vertex 2^{d-1}, the history information function becomes

  $$f(x) = \begin{cases} d + x \text{ (if } x > 0 \text{)} \\ 1 \text{ (if } x = -d \text{)} \\ 2d + 1 - x \end{cases}$$

  Although its minimum value is 1, when $x = -d$, and the second smallest value is 2, when $x = +1$, we can not use either the direction $-d$ or $+1$, since they lead to visited vertices. That leads us to use the direction $+2$, whose value is third smallest. Likewise, to avoid visiting an already visited vertex, we have to follow the sequence $\{2^{d-1} + 2, 2, 6, 14, \cdots, 2^d - 2\}$

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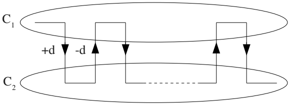
- Theorem
  Let H be a AUSO n-cube with a H.P. followed by Zadeh's rule.
  Each variable enters the basis at least $\frac{2^{n-2}}{n} - 1$ times.

Proof of the lower bound

- Variable $-d$ enters the basis min number k times



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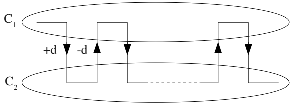

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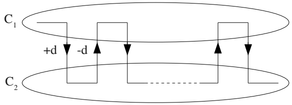

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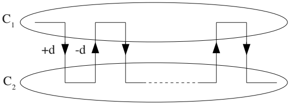

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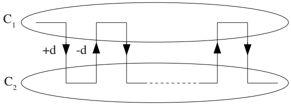

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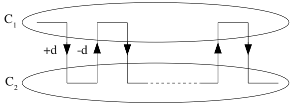

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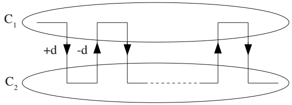
- Combining, $k \geq \frac{2^{n-2}}{n} - 2$



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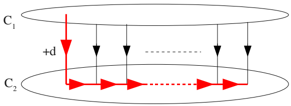
- The theorem does not generalize to arbitrary exponential length Zadeh paths
- Let C_1 and C_2 be copies of an AUSO with an exponential length Zadeh path.



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- Are any of these rules subexponential for LPs?