History based pivot rules for acyclic USOs on hypercubes

David Avis (Kyoto, McGill)
joint work with
Yoshikazu Aoshima, Theresa Deering, Yoshitake Matsumoto
and Sonoko Moriyama

January 19, 2011
Warning!

This problem may be addictive ...
Works for low dimensions ...
Let's try $d = 10$
Dear Victor,

Please post this offer of $1000 to the first person who can find a counterexample to the least entailed rule or prove it to be polynomial. The least entailed rule enter the improving variable which has been entailed least often.

Sincerely,

Norman Zadeh
Thursday, 10:30am
Subexponential Lower Bounds for the Simplex Algorithm
Oliver Friedmann
”... We also give a subexponential lower bound for Zadehs pivoting rule which among all improving pivoting steps enters the variable that has been entered least often. ”
Thursday, 10:30am
Subexponential Lower Bounds for the Simplex Algorithm
Oliver Friedmann
"... We also give a subexponential lower bound for Zadehs pivoting rule which among all improving pivoting steps enters the variable that has been entered least often. ”

Norman Zadeh will come tomorrow ....
Long unpublished gem

Long unpublished gem

Victor Klee (1925-2007)

Vic Klee at Oberwolfach in 1981
(photo: L. Danzer)
How Good Is the Simplex Algorithm?

Victor Klee*

Department of Mathematics, University of Washington, Seattle, Washington

AND

George J. Minty†

Department of Mathematics, Indiana University, Bloomington, Indiana

1. Introduction

By constructing long “increasing” paths on appropriate convex polytopes, we show that the simplex algorithm for linear programs (at least with its most commonly used pivot rule, Dantzig [1]) is not a “good algorithm” in the sense of Jack Edmonds. That is, the number of pivots or iterations that may be required is not majorized by any polynomial function of the two parameters that specify the size of the program. In particular, $2^d - 1$ iterations may be required in solving a linear program whose feasible region, defined by $d$ linear inequality constraints in $d$ nonnegative variables or by $d$ linear equality constraints in $2d$ nonnegative variables, is projectively equivalent to a $d$-dimensional cube. Further, for each $d$ there are positive constants $\alpha_d$ and

The start of Polyhedral Computation?
Norm Zadeh creator of Perfect Ten Magazine at his Beverly Hills Mansion November 2001 with his perfect 10 models (photo: Jonas Mohr)
Hot property: Norm Zada, publisher of Perfect 10 magazine has listed his house in Beverly Park at $24.5 million.
Sold on 11/16/2010
$16,500,000

72 BEVERLY PARK Dr
Beverly Hills, CA 90210

BEDS: 11
BATHS: 18
SQ. FT.: 20,000
$/SQ. FT.: $825
LOT SIZE: 6.79 Acres
PROPERTY TYPE: Residential, Single Family
STYLE: Architectural
VIEW: Canyon, City Lights, Mountain, Yes
YEAR BUILT: 2000
COMMUNITY: Beverly Hills Post Office
COUNTY: Los Angeles
MLS#: 09-352603
SOURCE: TheMLS
STATUS: Closed

The absolute best opportunity to purchase a pristine almost new Beverly Park compound in years. Trophy contemporary estate by the Landry Design Group sited on the highest elevation in Beverly Park. The free-flowing approx 20,000 sq. ft. estate includes a new 6,100 sq. ft. guest house +
The Simplex Method and LP digraphs

Linear Programming (LP) A:
\[
\min f := x + 2y + 4z \\
\begin{cases}
-1 \leq x \leq 1 \\
-1 \leq y \leq 1 \\
-1 \leq z \leq 1
\end{cases}
\]

The Simplex Method:
Algorithm of searching a sink of LP digraphs by some pivotting rules.

Strongly polynomial-time algorithms for LP?
Good characterizations for LP digraphs?
Basic problem

Can we efficiently find the sink of an LP-digraph by following a directed path from any given vertex, using a given edge selection rule (pivoting)?
Necessary conditions for LP digraphs

- Unique Sink Orientation (USO) ['01 Szabo, Welzl]
- Acyclicity
- Holt Klee Property ['99 Holt, Klee]
- Shelling Property ['09 Avis, Moriyama]
Necessary conditions for LP digraphs

Unique Sink Orientation (USO)  
[‘01 Szabo, Welzl]

Each subgraph $G(P, H)$ of $G(P)$ induced by a face $H$ of $P$ has a unique sink (and then a unique source).
Necessary conditions for LP digraphs

Acyclicity

G(P) has no directed cycle.

[Diagram showing an acyclic and a non-acyclic graph with arrows indicating direction. The acyclic graph is on the left with black arrows, and the non-acyclic graph is on the right with black and red arrows, indicating cycles.]
Holt Klee property [‘99 Holt, Klee]

G(P) has a USO, and for every k-dimensional face H of P there are k disjoint paths from the unique source to the unique sink in G(P,H).
Klee-Minty Examples

• 3-cube (Chvátal, P.47)

\[
\text{maximize} \quad 100x_1 + 10x_2 + x_3 \\
\text{s.t.} \quad x_1 \leq 1 \\
20x_1 + x_2 \leq 100 \\
200x_1 + 20x_2 + x_3 \leq 10000 \\
x_1, x_2, x_3 \geq 0
\]
Klee-Minty Examples

- 3-cube (Chvátal, P.47)

\[
\begin{align*}
\text{maximize} & \quad 100x_1 + 10x_2 + x_3 \\
\text{s.t.} & \quad x_1 \leq 1 \\
& \quad 20x_1 + x_2 \leq 100 \\
& \quad 200x_1 + 20x_2 + x_3 \leq 10000 \\
& \quad x_1, x_2, x_3 \geq 0 \\
\end{align*}
\]

- Vertices:

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 100 & 8000 \\
1 & 0 & 0 & 1 & 80 & 8200 \\
1 & 80 & 0 & 1 & 0 & 9800 \\
0 & 100 & 0 & 0 & 0 & 10000 \\
\end{array}
\]
### Pivot Sequence (Dantzig’s rule)

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>80</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>9800</td>
</tr>
<tr>
<td>1</td>
<td>80</td>
<td>8200</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>10000</td>
</tr>
</tbody>
</table>
Pivot Sequence (Dantzig’s rule)

\[ x_1 \ x_2 \ x_3 \\
0 \ 0 \ 0 \\
1 \ 0 \ 0 \\
1 \ 80 \ 0 \\
\]

- \( x_1 \) pivots 2 \( n \) - 1 times.

\[ x_n \] stays out of basis for \( 2^{n-1} \) iterations.
Pivot Sequence (Dantzig’s rule)

\[
\begin{array}{cccc}
  x_1 & x_2 & x_3 \\
  0 & 0 & 0 \\
  1 & 0 & 0 \\
  1 & 80 & 0 \\
  0 & 100 & 0 \\
  0 & 100 & 8000 \\
  1 & 0 & 9800 \\
  1 & 80 & 8200 \\
  0 & 0 & 10000 \\
\end{array}
\]

- \( x_1 \) stays out of basis for \( 2^{n-1} \) iterations.
- \( x_n \) stays out of basis for \( 2^{n-1} \) iterations.
- \( x_1 \) pivots \( 2^{n-1} \) times.
Klee-Minty construction
Klee-Minty path
n-cube USOs

- Vertices $V = \{0, 1, \ldots, 2^n - 1\} = \{00..00, 00..01, \ldots, 11..11\}$
n-cube USOs

- Vertices $V = \{0, 1, \ldots, 2^n - 1\} = \{00..00, 00..01, \ldots, 11..11\}$
- Facets $F_1, F_2, \ldots, F_{2n}$. For $i = 1, \ldots, n$,
  
  $F_i = \{(x_1, x_2, \ldots, x_n)| x_i = 0\}$, $F_{n+i} = \{(x_1, x_2, \ldots, x_n)| x_i = 1\}$.
n-cube USOs

- Vertices $V = \{0, 1, \ldots, 2^n - 1\} = \{00..00, 00..01, \ldots, 11..11\}$
- Facets $F_1, F_2, \ldots, F_{2n}$. For $i = 1, \ldots, n$,
  
  \[ F_i = \{(x_1, x_2, \ldots, x_n) | x_i = 0\}, \quad F_{n+i} = \{(x_1, x_2, \ldots, x_n) | x_i = 1\}. \]

- Cobasis $C(v) = \{i : v \in F_i, i = 1, \ldots, 2n\}, \quad v \in V$
n-cube USOs

- Vertices $V = \{0, 1, \ldots, 2^n - 1\} = \{00..00, 00..01, \ldots, 11..11\}$
- Facets $F_1, F_2, \ldots, F_{2n}$. For $i = 1, \ldots, n$,
  \[ F_i = \{(x_1, x_2, \ldots, x_n) | x_i = 0\}, \quad F_{n+i} = \{(x_1, x_2, \ldots, x_n) | x_i = 1\}. \]
- Cobasis $C(v) = \{i : v \in F_i, i = 1, \ldots, 2n\}, \quad v \in V$
- Basis $B(v) = \{i : v \notin F_i, i = 1, \ldots, 2n\}, \quad v \in V$
n-cube USOs

- Vertices $V = \{0, 1, \ldots, 2^n - 1\} = \{00..00, 00..01, \ldots, 11..11\}$
- Facets $F_1, F_2, \ldots, F_{2n}$. For $i = 1, \ldots, n$,
  
  $$F_i = \{(x_1, x_2, \ldots, x_n) | x_i = 0\}, \quad F_{n+i} = \{(x_1, x_2, \ldots, x_n) | x_i = 1\}.$$
- Cobasis $C(v) = \{i : v \in F_i, i = 1, \ldots, 2n\}, \quad v \in V$
- Basis $B(v) = \{i : v \notin F_i, i = 1, \ldots, 2n\}, \quad v \in V$
- Note $i \in B(v)$ iff $n + i \in C(v)$. 
n-cube USOs

- Vertices $V = \{0, 1, \ldots, 2^n - 1\} = \{00..00, 00..01, \ldots, 11..11\}$
- Facets $F_1, F_2, \ldots, F_{2n}$. For $i = 1, \ldots, n$,
  $$F_i = \{(x_1, x_2, \ldots, x_n)|x_i = 0\}, \quad F_{n+i} = \{(x_1, x_2, \ldots, x_n)|x_i = 1\}.$$
- Cobasis $C(v) = \{i : v \in F_i, i = 1, \ldots, 2n\}, \quad v \in V$
- Basis $B(v) = \{i : v \notin F_i, i = 1, \ldots, 2n\}, \quad v \in V$
- Note $i \in B(v)$ iff $n + i \in C(v)$.
- A pivot interchanges a pair of indices $i$ and $n + i$ between $B(v)$ and $C(v)$. (flips bit $i$ of $v$)
3-cube acyclic USO

- Vertices $V = \{0, 1, \ldots, 7\} = \{000, 001, \ldots, 111\}$
3-cube acyclic USO

- Vertices $V = \{0, 1, ..., 7\} = \{000, 001, ..., 111\}$
- $F_i = \{(x_1, x_2, x_3) | x_i = 0\}$, $F_{3+i} = \{(x_1, x_2, x_3) | x_i = 1\}$
3-cube acyclic USO

- Vertices $V = \{0, 1, ..., 7\} = \{000, 001, ..., 111\}$
- $F_i = \{(x_1, x_2, x_3)|x_i = 0\}$, $F_{3+i} = \{(x_1, x_2, x_3)|x_i = 1\}$
- $C(6) = C(110) = \{4, 5, 3\}$, $B(6) = B(110) = \{1, 2, 6\}$

v = 6 pivots to vertices 2, 4, 7 by flipping bits 1, 2, 3
Pivots correspond to moves in the 4, 2, 1 directions
3-cube acyclic USO

- Vertices $V = \{0, 1, ..., 7\} = \{000, 001, ..., 111\}$
- $F_i = \{(x_1, x_2, x_3)|x_i = 0\}$, $F_{3+i} = \{(x_1, x_2, x_3)|x_i = 1\}$
- $C(6) = C(110) = \{4, 5, 3\}$, $B(6) = B(110) = \{1, 2, 6\}$
- $\nu = 6$ pivots to vertices 2, 4, 7 by flipping bits 1, 2, 3
3-cube acyclic USO

- Vertices $V = \{0, 1, \ldots, 7\} = \{000, 001, \ldots, 111\}$
- $F_i = \{(x_1, x_2, x_3)|x_i = 0\}$, $F_{3+i} = \{(x_1, x_2, \ldots, x_3)|x_i = 1\}$
- $C(6) = C(110) = \{4, 5, 3\}$, $B(6) = B(110) = \{1, 2, 6\}$
- $v = 6$ pivots to vertices 2, 4, 7 by flipping bits 1, 2, 3
- Pivots correspond to moves in the 4, 2, 1 directions
History based rules

Choose the improving variable that satisfies:

- Least number of times to enter basis (Zadeh)
History based rules

Choose the improving variable that satisfies:

- Least number of times to enter basis (Zadeh)
- Least recently considered (Cunningham)
History based rules

Choose the improving variable that satisfies:

- Least number of times to enter basis (Zadeh)
- Least recently considered (Cunningham)
- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
- Least recently basic (Johnson)
- All of the above break Klee-Minty type constructions
- We try to find an acyclic USO for which a given rule follows a Hamiltonian path
History based rules

Choose the improving variable that satisfies:

- Least number of times to enter basis (Zadeh)
- Least recently considered (Cunningham)
- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
- Least recently basic (Johnson)

We try to find an acyclic USO for which a given rule follows a Hamiltonian path
History based rules

Choose the improving variable that satisfies:

• Least number of times to enter basis (Zadeh)
• Least recently considered (Cunningham)
• Least recently entered (Fathi-Tovey)
• Least number of iterations in basis (A-M-M)
• Least used direction (A-M-M)
History based rules

Choose the improving variable that satisfies:

- Least number of times to enter basis (Zadeh)
- Least recently considered (Cunningham)
- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
- Least recently basic (Johnson)
History based rules

Choose the improving variable that satisfies:

- Least number of times to enter basis (Zadeh)
- Least recently considered (Cunningham)
- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
- Least recently basic (Johnson)
- All of the above break Klee-Minty type constructions
History based rules

Choose the improving variable that satisfies:

- Least number of times to enter basis (Zadeh)
- Least recently considered (Cunningham)
- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
- Least recently basic (Johnson)
- All of the above break Klee-Minty type constructions
- We try to find an acyclic USO for which a given rule follows a Hamiltonian path
Least entered rule (Zadeh)

<table>
<thead>
<tr>
<th>Vertex</th>
<th>(orientation, direction)-pair</th>
<th>Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 0</td>
<td>+4 -4 +3 -3 +2 -2 +1 -1</td>
<td>+1, +2, +3, +4</td>
</tr>
<tr>
<td>1 0 0 1</td>
<td>0 0 0 0 0 0 0 0</td>
<td>+2, +3, +4</td>
</tr>
<tr>
<td>3 0 0 1 1</td>
<td>0 0 0 0 1 0 0 1</td>
<td>-1, +4</td>
</tr>
<tr>
<td>2 0 0 1 0</td>
<td>0 0 0 0 1 0 1 1</td>
<td>+3, +4</td>
</tr>
<tr>
<td>1 0 1 0</td>
<td>1 0 0 0 1 0 1 1</td>
<td>-2</td>
</tr>
<tr>
<td>8 1 0 0 0</td>
<td>1 0 0 0 0 1 1 1</td>
<td></td>
</tr>
</tbody>
</table>
Least recently basic (Johnson)
Least recently considered (Cunningham)

Vertex Sequence Options

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Sequence</th>
<th>Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 0 0 0 + 2, - 2, + 1, - 1, + 3, - 3, + 4, - 4</td>
<td>+ 2</td>
</tr>
<tr>
<td>2</td>
<td>0 0 1 0 - 2, + 1, - 1, + 3, - 3, + 4, - 4</td>
<td>+ 3</td>
</tr>
<tr>
<td>6</td>
<td>0 1 1 0 - 3, + 4, - 4, + 2, - 2, + 1, - 1</td>
<td>+ 4</td>
</tr>
<tr>
<td>14</td>
<td>1 1 1 0 - 4, + 2, - 2, + 1, - 1, + 3, - 3</td>
<td>- 3</td>
</tr>
<tr>
<td>10</td>
<td>1 0 1 0 + 4, - 4, + 2, - 2, + 1, - 1, + 3</td>
<td>- 2</td>
</tr>
<tr>
<td>8</td>
<td>1 0 0 0 + 1, - 1, + 3, - 3, + 4, - 4, + 2</td>
<td></td>
</tr>
</tbody>
</table>
Least recently entered (Fathi-Tovey)

<table>
<thead>
<tr>
<th>Vertex</th>
<th>(orientation, direction)-pair</th>
<th>Options</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+ 4</td>
<td>- 4</td>
</tr>
<tr>
<td>0 0 0 0</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>1 0 0 1</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>5 0 1 0</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>13 1 1 0</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>15 1 1 1</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>11 1 0 1</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>9 1 0 0 1</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>8 1 0 0 0</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

The graph shows the least recently entered vertices and their orientation options.
Least number of iterations in basis (A-M-M)
Least used direction (A-M-M-M)

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Direction</th>
<th>Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0000</td>
<td>0 0 0 0 0</td>
<td>1, 2, 3, 4</td>
</tr>
<tr>
<td>1 0001</td>
<td>0 0 0 1 1</td>
<td>2, 3, 4</td>
</tr>
<tr>
<td>3 0011</td>
<td>0 0 1 1 4</td>
<td></td>
</tr>
<tr>
<td>11 1011</td>
<td>1 0 1 1 2</td>
<td></td>
</tr>
<tr>
<td>9 1001</td>
<td>1 0 2 1 1</td>
<td></td>
</tr>
<tr>
<td>8 1000</td>
<td>1 0 2 2</td>
<td></td>
</tr>
</tbody>
</table>
Least used direction

For n-cube $H_n$, directions $i = 1, ..., n$

- $nv(i)$ = the number of times that direction $i$ has been taken.
Least used direction

For n-cube $H_n$, directions $i = 1, ..., n$

- $nv(i)$ = the number of times that direction $i$ has been taken.
- Initialize: $nv(i) = 0$ for $i = 1, ..., n$
Least used direction

For n-cube $H_n$, directions $i = 1, \ldots, n$

- $nv(i)$ = the number of times that direction $i$ has been taken.
- Initialize: $nv(i) = 0$ for $i = 1, \ldots, n$
- Update: From current vertex $y$ choose an outgoing edge to a facet $F_j$ minimizing $nv(j)$
Least used direction

For n-cube $H_n$, directions $i = 1, \ldots, n$

- $nv(i) =$ the number of times that direction $i$ has been taken.
- Initialize: $nv(i) = 0$ for $i = 1, \ldots, n$
- Update: From current vertex $y$ choose an outgoing edge to a facet $F_j$ minimizing $nv(j)$
- Set $nv(j) = nv(j) + 1$.
- Special case of Zadeh’s rule.
Unique $H_3$

Hamilton path using least used direction rule
It satisfies the Holt-Klee condition

Least visited rule

```
nv(1) = nv(2) = nv(4) = 0
4: nv(1) = nv(2) = 0, nv(4) = 1
2: nv(1) = 0, nv(2) = 1, nv(4) = 1
 1: nv(1) = 1, nv(2) = 1, nv(4) = 1
2: nv(1) = 1, nv(2) = 2, nv(4) = 1
4: nv(1) = 1, nv(2) = 2, nv(4) = 2
2: nv(1) = 1, nv(2) = 3, nv(4) = 2
 1
```
Unique $H_4$

Hamilton path using least used direction rule
It satisfies the Holt-Klee condition

$[3, 4, 3, 2, 1, 2, 1, 3, 1, 0, 2, 2, 1, 3, 2, 2]$
$[9, 8, 12, 4, 6, 14, 10, 11, 15, 13, 5, 7, 3, 2, 0, 1]$

nv(1) = nv(2) = nv(4) = nv(8) = 0
1: nv(1) = 1, nv(2) = 0, nv(4) = 1, nv(8) = 0
4: nv(1) = 1, nv(2) = 0, nv(4) = 1, nv(8) = 1
8: nv(1) = 1, nv(2) = 0, nv(4) = 1, nv(8) = 1
2: nv(1) = 1, nv(2) = 1, nv(4) = 1, nv(8) = 1
8: nv(1) = 1, nv(2) = 1, nv(4) = 1, nv(8) = 2
4: nv(1) = 1, nv(2) = 1, nv(4) = 2, nv(8) = 2
1: nv(1) = 2, nv(2) = 1, nv(4) = 2, nv(8) = 2
4: nv(1) = 2, nv(2) = 1, nv(4) = 3, nv(8) = 2
2: nv(1) = 2, nv(2) = 2, nv(4) = 3, nv(8) = 2
8: nv(1) = 2, nv(2) = 2, nv(4) = 3, nv(8) = 3
2: nv(1) = 2, nv(2) = 3, nv(4) = 3, nv(8) = 3
8: nv(1) = 2, nv(2) = 3, nv(4) = 4, nv(8) = 3
4: nv(1) = 2, nv(2) = 3, nv(4) = 4, nv(8) = 2
1: nv(1) = 3, nv(2) = 2, nv(4) = 4, nv(8) = 2
2: nv(1) = 3, nv(2) = 3, nv(4) = 4, nv(8) = 2
1

Least visited rule
David's example
Acyclic, USO, non-HK
One $H_4$ cube
Another candidate for $H_5$

Yoshitake’s example
Acyclic, USO, non-HK
No H_4 cubes

Least visited rule
### Computational results: least used direction

<table>
<thead>
<tr>
<th>dimension</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of Hamilton paths</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Holt-Klee</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
### Computational results: least used direction

<table>
<thead>
<tr>
<th>dimension</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of Hamilton paths</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Holt-Klee</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- For $n \leq 4$, each example extends to the next dimension
Computational results: least used direction

<table>
<thead>
<tr>
<th>dimension</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of Hamilton paths</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Holt-Klee</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- For $n \leq 4$, each example extends to the next dimension
- Since HK fails for $n = 5$, these examples are not LP-digraphs
Computational results: least used direction

But things do not go well for $n \geq 6$ ...

<table>
<thead>
<tr>
<th>dimension</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of Hamilton paths</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Holt-Klee</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We did a computer search of all acyclic USOs that contain Hamiltonian paths.
Least times to enter basis (Zadeh’s rule)

Facets $F_i, i = 1, \ldots, 2n$

- $nv(i)$ = the number of times that $F_i$ has been visited.
Least times to enter basis (Zadeh’s rule)

Facets $F_i, i = 1, ..., 2n$

- $nv(i)$ = the number of times that $F_i$ has been visited.
- Initialize: $nv(i) = 0$ for all $i$
Least times to enter basis (Zadeh’s rule)

Facets $F_i, i = 1, \ldots, 2n$

- $nv(i)$ is the number of times that $F_i$ has been visited.
- Initialize: $nv(i) = 0$ for all $i$
- Update: From current vertex $y$ choose an outgoing edge to a facet $F_j$ minimizing $nv(j)$
Least times to enter basis (Zadeh’s rule)

Facets $F_i$, $i = 1, \ldots, 2n$

- $nv(i) =$ the number of times that $F_i$ has been visited.
- Initialize: $nv(i) = 0$ for all $i$
- Update: From current vertex $y$ choose an outgoing edge to a facet $F_j$ minimizing $nv(j)$
- Set $nv(j) = nv(j) + 1$. 
Computational results: least times to enter basis

The deluge!

<table>
<thead>
<tr>
<th>dimension</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>$\geq 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ham. paths</td>
<td>1</td>
<td>2</td>
<td>17</td>
<td>1,072</td>
<td>3,262,342</td>
<td>$\geq 42,500,000,000$</td>
</tr>
<tr>
<td>Holt-Klee</td>
<td>1</td>
<td>2</td>
<td>12</td>
<td>79</td>
<td>360</td>
<td>none yet</td>
</tr>
</tbody>
</table>
### Computational results: all rules

<table>
<thead>
<tr>
<th>Dimension</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least-entered (Zadeh)</td>
<td>1</td>
<td>2</td>
<td>17</td>
<td>1,072</td>
<td>3,262,342</td>
<td>&gt; 10^{10}</td>
</tr>
<tr>
<td>Least-used-direction</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Least-recently-entered</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Least-recently-considered</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Least-recently-basic</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Least-iterations-in-basis</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table:** Hamiltonian paths produced by history based pivot rules
How do we get the results?

Williamson Hoke’s theorem (1988)

- Given an oriented $n$-cube $H$, let $d_k =$ number of vertices with in-degree $k$
How do we get the results?

Williamson Hoke’s theorem (1988)

- Given an oriented $n$-cube $H$, let $d_k =$ number of vertices with in-degree $k$
- Theorem: $H$ is an AUSO if and only if

$$d_k = \binom{n}{k}, \quad k = 0, 1, ..., n$$

Eg. Exactly $n$ vertices have in-degree one.
Williamson Hoke’s theorem (1988)

• Given an oriented $n$-cube $H$, let $d_k = \text{number of vertices with in-degree } k$

• Theorem: $H$ is an AUSO if and only if

$$d_k = \binom{n}{k}, \quad k = 0, 1, \ldots, n$$

• Eg. Exactly $n$ vertices have in-degree one.
How do we get the results (ctd)?

- Hopeless trying to generate all AUSOs directly
How do we get the results (ctd)?

- Hopeless trying to generate all AUSOs directly
- We generate a Hamiltonian path (HP) on an unoriented cube (Klee’s maxim)
How do we get the results(continued)?

• Hopeless trying to generate all AUSO(s) directly
• We generate a Hamiltonian path (HP) on an unoriented cube (Klee’s maxim)
• A HP defines the orientation of all edges of an acyclic orientation
How do we get the results (ctd)?

- Hopeless trying to generate all AUSOs directly
- We generate a Hamiltonian path (HP) on an unoriented cube (Klee’s maxim)
- A HP defines the orientation of all edges of an acyclic orientation
- We generate only HPs consistent with Zadeh’s rule using $nv$ sequence
How do we get the results (ctd)?

- Hopeless trying to generate all AUSOs directly
- We generate a Hamiltonian path (HP) on an unoriented cube (Klee’s maxim)
- A HP defines the orientation of all edges of an acyclic orientation
- We generate only HPs consistent with Zadeh’s rule using $nv$ sequence
- Reject partial HP if it violates W-H theorem
Canonical paths lemma

For any history based pivot rule generating a HP there is a labelling of the cube s.t.

• HP starts at vertex 0
• The order of first used-directions is $+1, +2, \ldots, +d$.
• Except for Zadeh's rule we can assume that the first $d$ pivots are $+1, +2, \ldots, +d$.
• For Zadeh's rule we cannot: Eg. $+1, +2, -1, +3, \ldots$ is valid.
Canonical paths lemma

For any history based pivot rule generating a HP there is a labelling of the cube s.t.

- HP starts at vertex 0
For any history based pivot rule generating a HP there is a labelling of the cube s.t.

- HP starts at vertex 0
- The order of first used-directions is $+1, +2, \ldots, +d$. 
Canonical paths lemma

For any history based pivot rule generating a HP there is a labelling of the cube s.t.

- HP starts at vertex 0
- The order of first used-directions is $+1, +2, \ldots, +d$.
- Except for Zadeh’s rule we can assume that the first $d$ pivots are $+1, +2, \ldots, +d$. 
Canonical paths lemma

For any history based pivot rule generating a HP there is a labelling of the cube s.t.

- HP starts at vertex 0
- The order of first used-directions is $+1, +2, ..., +d$.
- Except for Zadeh’s rule we can assume that the first $d$ pivots are $+1, +2, ..., +d$.
- For Zadeh’s rule we cannot:
  Eg. $+1, +2, -1, +3, ...$ is valid.
In a canonical HP the indegree of a vertex is 1 if and only if it is reached by a signed direction that is being used for the first time.
In a canonical HP the indegree of a vertex is 1 if and only it is reached by a signed direction that is being used for the first time.

- \((v, v')\) is an edge of HP that uses the direction \(+t\) for the first time.
In a canonical HP the indegree of a vertex is 1 if and only it is reached by a signed direction that is being used for the first time.

- \((v, v')\) is an edge of HP that uses the direction \(+ t\) for the first time.
- Previous vertices in the path must have zero on the \(t\)-th bit.
- The indegree of \(v'\) is one.
- Each of \(d\) directions yields one such vertex.
- By WH there are no others.
- Eg: \(+1,+2,+3,-1,...\) does give a HP by Zadeh.
In a canonical HP the indegree of a vertex is 1 if and only it is reached by a signed direction that is being used for the first time.

- \((v, v')\) is an edge of HP that uses the direction \(+t\) for the first time.
- Previous vertices in the path must have zero on the \(t\)-th bit.
- Neighbours of \(v'\) on the hypercube except \(v\) have one on this bit, so they cannot have been visited.
Indegree 1 vertices

In a canonical HP the indegree of a vertex is 1 if and only it is reached by a signed direction that is being used for the first time.

- \((v, v')\) is an edge of HP that uses the direction \(+t\) for the first time.
- Previous vertices in the path must have zero on the \(t\)-th bit.
- Neighbours of \(v'\) on the hypercube except \(v\) have one on this bit, so they cannot have been visited.
- The indegree of \(v'\) is one.
Indegree 1 vertices

In a canonical HP the indegree of a vertex is 1 if and only it is reached by a signed direction that is being used for the first time.

- \((v, v')\) is an edge of HP that uses the direction \(+t\) for the first time.
- Previous vertices in the path must have zero on the \(t\)-th bit.
- Neighbours of \(v'\) on the hypercube except \(v\) have one on this bit, so they cannot have been visited.
- The indegree of \(v'\) is one.
- Each of \(d\) directions yields one such vertex.
In a canonical HP the indegree of a vertex is 1 if and only it is reached by a signed direction that is being used for the first time.

- \((v, v')\) is an edge of HP that uses the direction \(+t\) for the first time.
- Previous vertices in the path must have zero on the \(t\)-th bit.
- Neighbours of \(v'\) on the hypercube except \(v\) have one on this bit, so they cannot have been visited.
- The indegree of \(v'\) is one.
- Each of \(d\) directions yields one such vertex.
- By WH there are no others.
Indegree 1 vertices

In a canonical HP the indegree of a vertex is 1 if and only it is reached by a signed direction that is being used for the first time.

• \((v, v')\) is an edge of HP that uses the direction \(+t\) for the first time.
• Previous vertices in the path must have zero on the \(t\)-th bit.
• Neighbours of \(v'\) on the hypercube except \(v\) have one on this bit, so they cannot have been visited.
• The indegree of \(v'\) is one.
• Each of \(d\) directions yields one such vertex.
• By WH there are no others.
• Eg: \(+1, +2, +3, -1, \ldots\) does give a HP by Zadeh
Non-existence of Hamiltonian paths

Theorem

• For $d \geq 3$ the following rules do not generate any Hamiltonian paths on AUSO cubes: least-iterations-in-basis, least-recently-basic, least-recently considered.
Non-existence of Hamiltonian paths

Theorem

• For $d \geq 3$ the following rules do not generate any Hamiltonian paths on AUSO cubes: least-iterations-in-basis, least-recently-basic, least-recently considered.

• For $d \geq 5$ the least-recently-entered rule does not generate any Hamiltonian paths on AUSO cubes.
Proof of non-existence of Hamiltonian paths

- The least-iterations-in-basis, least-recently-basic, least-recently considered, rules start with $+1, +2, \ldots, +d, -1, \ldots$
Proof of non-existence of Hamiltonian paths

- The least-iterations-in-basis, least-recently-basic, least-recently considered, rules start with $+1,+2,...,+d,-1,...$
- These $d + 1$ vertices have indegree one violating Williamson-Hoke
Non-existence: least-recently-entered, \( d \geq 5 \)

**Theorem 1.1.** The least-recently entered rule does not have any Hamiltonian paths on a \( d \)-cube for \( d \geq 5 \).

**Proof.** Suppose \( P \) is a Hamiltonian path produced by Algorithm 1 for the least-recently entered rule when \( d \geq 5 \). We will show that \( P \) must begin with the sequence of vertices \( Q = Q_1, Q_2, Q_3, Q_4 \) where \( Q_1 = \{0, 1, 3, \ldots, 2^d - 1\} \), \( Q_2 = \{2^d - 1, 2^d - 2, 2^d - 3, \ldots, 2^d - 2d + 2\} \), and \( Q_3 = \{\sum_{i=0}^{d-1} 2^i, \sum_{i=0}^{d-2} 2^i, \ldots, \sum_{i=0}^{d-k} 2^i \} \). 

Since \( Q \) should be represented as \( \sum_{i=0}^{d-1} 2^i \) requires \( 2^d = 2d + 1 \). It means the next vertex can be \( 2^d - 1 \). In other words, there exists \( \exists (d - 1, d - 2) \) not to be equal to 0, \( \sum_{i=0}^{d} 2^i - 2^k \) should be represented as \( \sum_{i=0}^{d} 2^i = \sum_{i=0}^{d-k} 2^i - \sum_{i=0}^{d-k+1} 2^i \) for certain \( l \).

Therefore, the \( (k, j) \) equal \( (d - 1, d - 2) \) or \( (d - 2, d - 1) \), and \( d - 1 > k \) requires \( k = d - 2 \).

We can prove the inductive step similarly. If the path is continued by \( 2^d - 1, 2^d - 2, \ldots, 2^d - k \), the next vertex should be equal to \( \sum_{i=0}^{d} 2^i - \sum_{i=0}^{d-k-2} 2^i + 2^j (d - 2 - k \leq j \leq d - 2) \) or \( \sum_{i=0}^{d-1} 2^i - \sum_{i=0}^{d-k} 2^i - 2^j (j = d - 1) \). By Corollary 7, two neighbours of it are in \( \{2^d - 1, 2^d - 2, \ldots, 2^d - 2d + 2\} \).

Using binary numbers, \( 2^d - \sum_{k=0}^{d-2k} 2^{d-2} \) can be denoted \( 100 \ldots 011 \ldots 11 \), where we have \( k \) 1s.

\[ \sum_{i=0}^{d-1} 2^i = \sum_{i=0}^{d} 2^i - \sum_{i=0}^{d-k} 2^i - 2^j \]

At the vertex \( 2^d - 1 \), the history information function becomes

\[
\begin{align*}
f(x) = \begin{cases} 
  d + x & (i f x > 0) \\
  1 & (i f x = -d) \\
  2d + 1 - x & (i f x < 0)
\end{cases}
\end{align*}
\]

Although its minimum value is 1, when \( x = -d \), and the second smallest value is 2, when \( x = -1 \), we can not use either the direction \(-d \) or \(+1 \), since they lead to visited vertices. That leads us to use the direction \(+2 \), whose value is third smallest. Likewise, to avoid visiting an already visited vertex, we have to follow the sequence \( \{2^d - 1, 2^d - 2, 2^d - 3, \ldots, 2^d - 2d + 2\} \) as a subsequence and does not contain the vertex \( 2^d - 1 \) or \( 2^d - 2 \). These four vertices lie on a 2-face which has two sources, \( 2^d - 2 \) and \( 2^d - 2 - 2^d - 2d + 2 \), a contradiction. It remains to show that \( P \) begins as specified.

- \( Q_1 = 0, 1, 3 \cdots, 2^d - 1 \). This follows from Lemma 7.
- \( Q_2 = 2^d - 1, 2^d - 2, \ldots, 2^d - 2d + 2 \). 
We prove this by mathematical induction. For the basic step, we will show only \( 2^d - 2d + 2 \) can come right after \( 2^d - 1 \). When we visited the vertex \( 2^d - 1 \), all of the bits are 1. It means the next vertex can be represented as \( 2^d - 2d + 2 \) or \( 2^d - 2d + 3 \) (\( d - 1 > k \geq 0 \)). By Corollary 7, vertex \( \sum_{i=0}^{d-1} 2^i \) should have two visited neighbours, one of which is obviously the vertex \( 2^d - 1 \). In other words, there exists \( j \neq k \) such that \( \sum_{i=0}^{d-1} 2^i - 2^j \in \{0, 1, 3 \cdots, 2^d - 1\} = \{v \mid \exists x, x = \sum_{i=0}^2 2^i \} \cup \{0\} \).

Since \( d \geq 3 \) forces \( \sum_{i=0}^{d-1} 2^i - 2^j \) not to be equal to 0, \( \sum_{i=0}^{d-1} 2^i - 2^j \) should be represented as \( \sum_{i=0}^{d-1} 2^i = \sum_{i=0}^{d-k} 2^i \) for certain \( l \).

Therefore, the \( (k, j) \) equal \( (d - 1, d - 2) \) or \( (d - 2, d - 1) \), and \( d - 1 > k \) requires \( k = d - 2 \).
How few times can a variable enter the basis?

- In Klee-Minty examples, one variable never enters the basis...
How few times can a variable enter the basis?

- In Klee-Minty examples, one variable never enters the basis...
- ...and one variable enter $2^{n-1}$ times.
How few times can a variable enter the basis?

- In Klee-Minty examples, one variable never enters the basis...
- ...and one variable enter $2^{n-1}$ times.
- Zadeh’s rule tries to balance this.
How few times can a variable enter the basis?

• In Klee-Minty examples, one variable never enters the basis...
• ...and one variable enter $2^{n-1}$ times.
• Zadeh’s rule tries to balance this.
• **Theorem**
  Let $H$ be a AUSO $n$-cube with a H.P. followed by Zadeh’s rule. Each variable enters the basis at least $\frac{2^{n-2}}{n} - 1$ times.
Proof of the lower bound

- Variable $-d$ enters the basis min number $k$ times
Proof of the lower bound

- Variable $-d$ enters the basis min number $k$ times
- At the sink $\sum_{i=1}^{2^n} n\nu(i) = 2^n - 1$
Proof of the lower bound

- Variable $-d$ enters the basis min number $k$ times
- At the sink $\sum_{i=1}^{2^n} n\nu(i) = 2^n - 1$
- Pivot $\pm d$ is blocked at $v$ if $v$’s twin already visited
Proof of the lower bound

- Variable $-d$ enters the basis min number $k$ times
- At the sink $\sum_{i=1}^{2^n} n\nu(i) = 2^n - 1$
- Pivot $\pm d$ is blocked at $v$ if $v$’s twin already visited
- Any pivot with $n\nu(i) \geq k + 2$ must be blocked
Proof of the lower bound

- Variable $-d$ enters the basis min number $k$ times
- At the sink $\sum_{i=1}^{2^n} n\nu(i) = 2^n - 1$
- Pivot $\pm d$ is blocked at $v$ if $v$’s twin already visited
- Any pivot with $n\nu(i) \geq k + 2$ must be blocked
- Number of blocked pivots is at most $2^{n-1}$
Proof of the lower bound

- Variable $-d$ enters the basis $min$ number $k$ times
- At the sink $\sum_{i=1}^{2^n} nv(i) = 2^n - 1$
- Pivot $\pm d$ is blocked at $v$ if $v$’s twin already visited
- Any pivot with $nv(i) \geq k + 2$ must be blocked
- Number of blocked pivots is at most $2^{n-1}$
- $\sum_{i=1}^{2^n} nv(i) \leq 2n(k + 2) - 1 + 2^{n-1}$
Proof of the lower bound

- Variable $-d$ enters the basis $\min$ number $k$ times
- At the sink $\sum_{i=1}^{2^n}nv(i) = 2^n - 1$
- Pivot $\pm d$ is blocked at $v$ if $v$'s twin already visited
- Any pivot with $nv(i) \geq k + 2$ must be blocked
- Number of blocked pivots is at most $2^{n-1}$
- $\sum_{i=1}^{2^n} nv(i) \leq 2n(k + 2) - 1 + 2^{n-1}$
- Combining, $k \geq \frac{2^{n-2}}{n} - 2$
Hamiltonian paths are special

- The theorem does not generalize to arbitrary exponential length Zadeh paths
Hamiltonian paths are special

- The theorem does not generalize to arbitrary exponential length Zadeh paths
- Let $C_1$ and $C_2$ be copies of an AUSO with an exponential length Zadeh path.
And the open problems are ...
And the open problems are ...

- Show Hamiltonian paths exist in all dimensions for Zadeh’s rule
And the open problems are ...

- Show Hamiltonian paths exist in all dimensions for Zadeh's rule
- Are there exponential lower bounds for all rules?
And the open problems are ...

- Show Hamiltonian paths exist in all dimensions for Zadeh’s rule
- Are there exponential lower bounds for all rules?
- Are any of these rules subexponential for LPs?