The Stability of Walrasian General Equilibrium under a Replicator Dynamic

Herbert Gintis and Antoine Mandel*

September 24, 2013

Abstract

We prove the stability of equilibrium in a completely decentralized Walrasian general equilibrium economy in which prices are fully controlled by economic agents, with production and trade occurring out of equilibrium.

Journal of Economic Literature Classifications:
C62—Existence and Stability Conditions of Equilibrium
D51—Exchange and Production Economies
D58—Computable and Other Applied General Equilibrium Economies

1 Introduction

Walras (1874) developed a general model of competitive market exchange, but provided only an informal argument for the existence of a market-clearing equilibrium for this model. Wald (1951) provided a proof of existence for a simplified version of Walras' model, and this proof was substantially generalized by Debreu (1952), Arrow and Debreu (1954), Gale (1955), Nikaido (1956), McKenzie (1959), Negishi (1960), and others.

Walras was well aware that his arguments had to backed by a theory of price adjustment that would ensure stability of equilibrium. He considered that the key force leading to equilibrium to be face-to-face competition, which he would result in the continual updating of prices by economic agents until equilibrium was attained. However, Walras believed that a model where economic agents individually update their prices would be analytically intractable, whereas a simple centralized

*The authors contributed equally to this research. Gintis: Santa Fe Institute and Central European University; Mandel: Paris School of Economics, University of Paris 1 Panthéon-Sorbonne. The authors acknowledge the support of Institute for New Economic Thinking grant INO1200022.

model of price adjustment, the auctioneer playing the role of a representative agent, would more easily lend itself to a proof of the stability of equilibrium.

The stability of the Walrasian economy became has been a central research focus in the years following the existence proofs (Arrow and Hurwicz 1958, 1959, 1960; Arrow, Block and Hurwicz 1959; Nikaido 1959; McKenzie 1960; Nikaido and Uzawa 1960). Following Walras' tâtonnement process, these models assumed that there is no production or trade until equilibrium prices are attained, and out of equilibrium, there is a price profile shared by all agents, the time rate of change of which is a function of excess demand. proving stability were successful only by assuming narrow and implausible conditions without microeconomic foundations (Fisher 1983). Indeed, Scarf (1960) and Gale (1963) provided simple examples of unstable Walrasian equilibria under a tâtonnment dynamic.

Several researchers then explored the possibility that allowing trading out of equilibrium could sharpen stability theorems (Uzawa 1959, 1961, 1962; Negishi 1961; Hahn 1962; Hahn and Negishi 1962, Fisher 1970, 1972, 1973), but these efforts did not provide micro-level conditions that ensure stability. Moreover, Sonnenschein (1973), Mantel (1974, 1976), and Debreu (1974) showed that any continuous function, homogeneous of degree zero in prices, and satisfying Walras' Law, is the excess demand function for some Walrasian economy. These results showed that no general stability theorem could be obtained based on the tâtonnement process. Indeed, subsequent analysis showed that chaos in price movements is the generic case for the tâtonnement adjustment processes (Saari 1985, Bala and Majumdar 1992).

A novel approach to the dynamics of large-scale social systems, evolutionary game theory, was initiated by Maynard Smith and Price (1973), and adapted to dynamical systems theory in subsequent years (Taylor and Jonker 1978, Friedman 1991, Weibull 1995). The application of these models to economics involved the shift from biological reproduction to behavioral imitation as the mechanism for the replication of successful agents.

We apply this framework by treating the Walrasian economy as the stage game of an evolutionary process. We assume each agent is endowed in each period with goods he must trade to obtain the various goods he consumes. There are no inter-period exchanges. An agent's trade strategy consists of a set of *private prices* for the good he produces and the goods he consumes, such that, according to the individual's private prices, a trade is acceptable if the value of goods received is at least as great as the value of the goods offered in exchange. The exchange process of the economy is hence defined as a multipopulation game where agents use private prices as strategies.

Competition materializes when a seller who slightly undercuts his competitors increases his sales and hence his income or when a buyer who overbids his competitors increases his purchases and hence his utility. In other words, when markets are in disequilibrium, there are always agents who have incentives to change their prices. More precisely, under rather mild conditions on the trading process, Walrasian equilibria are the only strict Nash equilibria of private prices games. Thus, if we assume that the strategies of traders are updated according to the replicator dynamic, the stability of equilibrium is guaranteed (Weibull 1995)).

Hence, the adjunction of private prices to the general equilibrium model allow on the one hand to propose a model closer to Walras original insights where prices are controlled by economic agents as in actual markets and on the other hand to revisit the issue of stability using game-theoretic and evolutionary principles to analyze the dynamic adjustment of prices to their equilibrium values.

The paper is organized as follows: Section 2 introduces our model economy, Section 3 defines a class of bargaining games based on private prices in this economy. Section 4 shows how learning dynamics in this games induce price dynamics in the economy and reviews the stability properties of evolutionary dynamics. Section 5 proves the stability of equilibrium in a stylized setting where the kinds of goods consumed and sold by each agent are independent of relative prices and section 6 extends this result to an arbitrary exchange economy. Section 7 discusses in greater detail the necessary conditions for competition to entail a stable price adjustment process in a setting with private prices. Section 8 analyzes a concrete agent-based instantiation of Walrasian competitive dynamics, showing that economic agents with extremely limited computational capacity can achieve a market-clearing equilibrium quite rapidly, showing that our analytical model is of practical relevance. Section 9 reviews the results of our analysis.

2 The Walrasian Economy

We consider an economy with a finite number of goods, indexed by $h = 1, \ldots, l$, and a finite number of agents, indexed by $i = 1, \ldots, m$. Agent i has $\mathbf{R}^l_+$ as consumption set, a utility function $u_i \colon \mathbf{R}^l_+ \to \mathbf{R}_+$ and an initial endowment $e_i \in \mathbf{R}^l_+$. We shall denote this economy by $\mathcal{E}(u, e)$ and consider the following standard concepts for its analysis:

- An allocation $\overline{x} \in (\mathbf{R}^l_+)^m$ of goods is *feasible* if it belongs to the set $\mathcal{A}(e) = \left\{ x \in (\mathbf{R}^l_+)^m \middle| \sum_{i=1}^m x_i \leq \sum_{i=1}^m e_i \right\}$.
- The demand of an agent i is the mapping $d_i \colon \mathbf{R}^l_+ \times BR_+ \to \mathbf{R}^l_+$ that associates to a price $p \in \mathbf{R}^l_+$ and an income $w \in \mathbf{R}_+$, the utility-maximizing individual allocations satisfying the budget constraint

$$d_i(p, w) = \text{argmax}\{u(x_i) \big| x_i \in \mathbf{R}^l_+, p \cdot x_i \leq w\}.$$

- A feasible allocation $\overline{x} \in \mathcal{A}(e)$ is an *equilibrium allocation* if there exists a price $\overline{p} \in \mathbf{R}^l_+$ such that for all i, $\overline{x}_i \in d_i(\overline{p}, \overline{p} \cdot e_i)$. The price $\overline{p}$ is then called an *equilibrium price*. We shall denote the set of such equilibrium prices by $\mathbf{E}(u, e)$.

- Additionally, a feasible allocation $x^* \in \mathcal{A}(e)$ and a price $p^* \in \mathbf{R}^l_+$ form a quasi-equilibrium if $u_i(x_i) > u_i(x_i^*)$ implies $p^* \cdot x_i > p^* \cdot x_i^*$.

Our focus is on the stability of equilibrium. Of course, this presupposes there actually are equilibria in the economy. Therefore we place ourselves throughout the paper in a setting where sufficient conditions for the existence of an equilibrium hold.

First, in order to ensure the existence of a quasi-equilibrium (Florenzano 2005), we assume that utility functions satisfy the following standard assumptions (the strict concavity is not necessary for the existence of a quasi-equilibrium but implies demand mappings are single-valued, which will prove useful below).

Assumption 1 (Utility) *For all $i = 1, \ldots, m$, u_i is continuous, strictly concave, and locally non-satiated.*

Second, to ensure that every quasi-equilibrium is an equilibrium allocation, it suffices to assume that at a quasi-equilibrium the agents do not receive the minimal possible income (Hammond 1993, Florenzano 2005). This condition is satisfied under the survival assumption (i.e when all initial endowments are in the interior of the consumption set) as well as in settings with corner endowments such as those investigated in Scarf (1960) and Gintis (2007). Formally, the assumption can be stated as follows.

Assumption 2 (Income) *For every quasi-equilibrium (p^*, x^*), and for every $i = 1, \ldots, m$, there exists $x_i \in \mathbf{R}^l_+$ such that $p^* \cdot x_i^* > p^* \cdot x_i$.*

It is standard to show that under Assumptions (1) and (2) the economy $\mathcal{E}(u, e)$ has at least an equilibrium (Florenzano 2005). Hence, we shall assume in the following they do hold. Moreover, we shall restrict attention to the generic case where the economy has a finite set of equilibria Balasko 2009) .

3 Exchange Processes with Private Prices

In his Elements of Pure Economics (1874), Walras envisages equilibrium emerging as the outcome of free competition among economic agents. He characterizes free competition as the combination of (i) the free entry and exit in the market, (ii) the capacity of producers to choose their level of production and, (iii) the freedom of traders to set and modify *their* prices. Dockes and Potier (2005) provide an extensive discussion Walras' views about competition. The striking point for our latter analysis is that Walras emphasizes that prices are actually private (note his usage of a possessive article below) and set in a decentralized manner by the traders. In his own words:

> "As buyers, traders make their demand by outbidding each other. As sellers, traders make their offers by underbidding each other. . . The markets that are best organized from the competitive standpoint are those in which. . . the terms of every exchange are openly announced and an opportunity is given to sellers to lower their prices and to buyers to raise their bids (Walras 1984, paragraph 41).

Though he saw decentralized competition as the driving force towards equilibrium, it seems Walras considered that an aggregate representation of price adjustment based on the centralized tâtonnement process would provide a simpler model in which one could prove the convergence towards equilibrium and its stability. Modern work on the question unfortunately showed that the tâtonnement process lacked those desirable features.

We propose an alternative model of price adjustment based on individual learning by traders who both set prices and trade at these prices out-of-equilibrium. Treating traders as price-setters out of equilibrium is consonant both with Walras' description of free competition and the way actual trading takes place. Nor is this assumption at odds with the law of one price or the notion that economic agents are price-takes under perfect competition, for both of these conditions hold in market equilibrium, and hence are long-run properties of our model.

The main building block of our approach is a game-theoretic representation of exchange processes based on private prices . Namely, we consider that each agent i in the economy is characterized by a private price $p_i \in \mathbf{R}^l_+$ whose coordinates represent the prices at which he is willing to sell the goods he supplies to the market and the maximum prices he is willing to pay for the goods he demands. The allocation of goods in the economy is then determined by the distribution of private prices as well as by the initial endowments. In other words, we represent the exchange process as a game where agents use private prices as strategies. Formally, we shall associate to the economy $\mathcal{E}(u, e)$ the class of games $\mathcal{G}(u, e, \xi)$ where:

- Each agent has a finite set of prices $P \subset \mathbf{R}^l_+$ as strategy set.
- The game form is defined by an exchange mechanism $\xi\colon P^m \to \mathcal{A}(e)$ that associates to a profile of private prices $\pi = (p_1, \dots, p_m)$ an attainable allocation $\xi(\pi) = (\xi_1(\pi), \dots, \xi_m(\pi)) \in \mathcal{A}(e)$.

- The payoff ϕ_i: $P^m \to \mathbf{R}_+$ of player i is evaluated according to the utility of the allocation it receives, that is $\phi_i(\pi) = u_i(\xi_i(\pi))$.

A wide range of exchange processes can be embedded in this framework: processes based on sequences of bilateral trades as in Gintis (2007,2012), processes based on a central clearing system such as double auctions, processes based on simultaneous and multilateral exchanges as usually considered in general equilibrium models with out-of-equilibrium features (Grandmont 1977, Benassy 2005) , or processes akin to the tâtonnement where no trade takes place unless excess demand vanishes (that is $\xi_i(\pi) = d_i(\overline{p}, \overline{p} \cdot e_i)$ if for all i $\pi_i = \overline{p} \in \mathbf{E}(u, e)$ and $\xi_i(\pi) = e_i$ otherwise).

4 Learning and Price Dynamics

Once individual prices are actually represented and recognized as strategic variables, it becomes clear that their evolution over time shall depend on the way agents react to the utility these prices yield. In a setting that is supposed to represent a competitive economy and hence to encompass a very large number of agents, it seems illusory to consider that any single agent has the information processing and the computational abilities to choose the best response to the complete strategy profile. Assuming boundedly rational and learning agents seem a much better model, provided the resources and the technology of the economy (the initial endowments in our setting) evolve slowly enough with respect to the frequency of transactions. Indeed, learning shall then have the time to bite.

There is a number of competing models of individual learning in games, from fictitious play to reinforcement learning through bayesian updating or stochastic imitation (Fudenberg and Levine 1997). Yet, the replicator equation forms the backbone of most of the theory. Indeed, hopkins02 shows that the expected motion of stochastic fictitious play and reinforcement learning with experimentation can both be written as a perturbed form of the evolutionary replicator dynamics, Helbing (1996) proves a similar result for stochastic imitation models and Shalizi (2009) proves that Bayesian updating also leads to the replicator equation. Detailed studies of convergence speed and error bounds are provided by Benaim and Weibull (2003) for stochastic imitation models, Hofbauer and Sandholm (2002) for stochastic fictitious play and by Ianni (2013) for reinforcement learning. The latter deals in particular with games admitting strict equilibria that are of concern for the present paper. Approximation by the replicator dynamic is in particular valid for the stochastic imitation models used by Gintis (2007) to simulate the dynamics of private prices in games with bilateral exchange processes based on private prices. Formally, in the game $\mathcal{G}(u, e, \xi)$, the replicator dynamics defines the evolution of populations of players by identifying a mixed strategy in

$$\Delta_i = \left\{ \sigma_i \in \mathbf{R}_+^P \mid \sum_p \ \sigma_{i,p} = 1 \right\}$$

with a population of players of type i whose share $\sigma_{i,p}$ uses the private price $p \in P$. The replicator dynamic then prescribes that the share of agents using price p in the population i should grow proportionally to the utility it is expected to yield in the exchange process under the assumption that trade partners are drawn uniformly in each of the populations. This yields the system of differential equations defined for all $i = 1, \ldots, n$ and $p \in P$ by:

$$\frac{\partial \sigma_{i,p}}{\partial t} = \sigma_{i,p} \left(E_{\sigma_{-i}}(u_i(\xi(p, \cdot))) - E_{\sigma}(u_i(\xi(\cdot))) \right) \tag{1}$$

where $E_{\sigma_{-i}}(u_i(\xi(p, \cdot)))$ represents the expected utility of the strategy p given the mixed strategy profile σ_{-i}, that is

$$E_{\sigma_{-i}}(u_i(\xi(p, \cdot))) = \sum_{\rho \in P^{m-1}} (\prod_{j \neq i} \sigma_{j,\rho_j}) u_i(\xi(p, \rho),$$

and $E_{\sigma}(u_i(\xi(\cdot))$ represents the expected utility of the mixed strategy σ_i given the mixed strategy profile σ_{-i}; that is

$$E_{\sigma}(u_i(\xi(\cdot)) = \sum_{\pi \in P^m} (\prod_{j=1}^{m} \sigma_{j,\pi_j}) u_i(\xi(\pi)).$$

In multi-population games such as $\mathcal{G}(u, e, \xi)$,, the replicator dynamic has very salient equilibrium selection properties: a strategy profile is (locally) asymptotically stable for the replicator dynamics if and only if it is a strict Nash equilibrium of the underlying game[1] (see Weibull 1995 and Appendix A below). In other words, one has:

Proposition 1 *A price profile $\pi \in P^m$ is asymptotically stable for the replicator dynamic if and only if for all $i = 1 \cdots m$, and all $p \neq \pi_i$, one has $u_i(\pi) > u_i(p, \pi_{-i})$.*

Therefore, to prove the asymptotic stability of the economic equilibria of the economy $\mathcal{E}(u, e)$, it suffices to show that these equilibria can be identified with strict Nash equilibria of the game $\mathcal{G}(u, e, \xi)$. This is the purpose of the remaining of this paper. Note that on top of asymptotic stability, this would guarantee that general economic equilibria are both risk-dominant (as strict equilibria) and Pareto-dominant (given the first welfare theorem).

[1]Similar results for a broader class of dynamics follow from the application of the results recalled in the appendix.

5 An Axiomatic Characterization of Stable Exchange Processes

From here on, we consider as price set $P = K^{l-1} \times \{1\}$, where $K \subset \mathbf{R}_+$ is a finite set of commodity prices with minimum $p_{\min} > 0$ and maximum $p_{\max} > p_{\min}$ while good l is used as a numeraire and its price is fixed equal to 1. We also consider that the price set P contains each of the finite number of equilibrium prices of the economy.

As underlined in section 3, a wide range of exchange processes can be represented as games of the form $\mathcal{G}(u, e, \xi)$. On the one hand, it is clear that equilibrium stability can't hold for every exchange process ξ, on the other hand a stability result that would hold for a single process won't do justice to the variety of market structures that can be deemed competitive. In order to avoid both pitfalls, we adopt an axiomatic approach and characterize a large class of exchange processes ξ, such that the only strict Nash equilibria of the game $\mathcal{G}(u, e, \xi)$ are strategy profiles $\overline{\pi}$ such that each agent uses the same general equilibrium price, that is there exists $\overline{p} \in \mathbf{E}(u, e)$ such that for all $i \in \{1, \ldots, m\}$, $\pi_i = \overline{p}$. In other words, we give sufficient conditions for the general equilibria of the economy $\mathcal{E}(u, e)$ to be the only asymptotically stable states of the replicator dynamics in the game $\mathcal{G}(u, e, \xi)$.

It turns out that the analysis is simpler in a setting where for each agent the set of goods is partitioned between consumption goods (those the agent consumes) and production goods (those the agent is endowed with). We first focus on this particular case and treat the general case in the next session. Hence, we assume within this section that the following assumption holds.

Assumption 3 (Goods partition) *For $i = 1, \ldots, m$, there exists a partition $\{P_i, C_i\}$ of $\{1, \ldots, l\}$ such that:*

1. *for all $h \in P_i$, $e_{i,h} > 0$;*
2. *there exists $v_i : V_{C_i} \to \mathbf{R}_+$ such that*[2]*:*
   (a) *for all $u_i(x) = v_i(pr_{V_{C_i}}(x))$*
   (b) $v_i(x) > 0 \Rightarrow \forall h \in C_i,\ x_h > 0.$

That is P_i are the goods produced/sold by agent i and C_i those he consumes/buys. Accordingly, we define the set of buyers of good h as $B_h := \{i \mid h \in C_i\}$, and the set of sellers of good h as $S_h := \{i \mid h \in P_i\}$.

[2]For a subset $C \subset \{1, \cdots, l\}$, we define the vector subspace V_C as $V_C := \{x \in \mathbf{R}^l \mid \forall h \notin C,\ x_h = 0\}$ and p_{V_C} as the projection on V_C.

Now the core of our approach to prove stability is as follows : general equilibria will be stable if agents have incentives (i) to agree on a common price in order to increase the size of the market and the number of opportunities for Pareto improving trades and (ii) to deviate from the common price by underbidding or overbidding their competitors when there is excess supply or excess demand.

Incentives to agree on a common price are related to the constraints private prices put on trading possibilities. In this respect, the less stringent condition on market access we can assume is to consider that access is restricted to traders that have a possibility to find a trade partner: buyers whose private price for a good is above the lowest private price of sellers, and conversely sellers whose private price for a good is below the highest private price of buyers (see respectively conditions 1 and 2 in definition 1 below). This subsumes more stringent conditions, such as buying only from sellers using a price below one's own, that can be enforced by a given exchange process ξ but can not be expressed at the level of generality at which we place ourselves.

We must also account for the budgetary constraint an agent's private price put on his allocation by assuming an agent's allocation is less than its utility maximizing allocation for his private prices (see condition 3 in definition 1 below). This latter condition is also in line with the fact that, out of equilibrium, there is no public price with regards to which agents can act as price takers: they must evaluate goods according to their private prices and determine their behavior accordingly.

Formally, given a private price profile π, we define the set of acceptable buyers as those whose prices is above the lowest buying price, that is $\mathcal{B}_h(\pi) := \{i \in B_h \mid \pi_{i,h} \geq \min_{j \in S_h} \pi_{j,h}\}$. the set of acceptable sellers as those whose prices is below the highest buying price, that is $\mathcal{S}_h(\pi) := \{i \in S_h \mid \pi_{j,h} \leq \max_{i \in B_h} \pi_{i,h}\}$ and the feasible income as $w_i(\pi) = \sum_{\{h \mid i \in \mathcal{S}_h(\pi)\}} \pi_{i,h} e_h$. The set of price feasible allocations $\mathcal{A}'(e, \pi)$ is then defined as follows.

Definition 1 *The set of price feasible allocations $\mathcal{A}'(e, \pi)$ is the subset of feasible allocations $\mathcal{A}(e, \pi)$ such that for every $x \in \mathcal{A}'(e, \pi)$ one has:*

1. $\forall h \in C_i$, $x_{i,h} > 0 \Rightarrow i \in \mathcal{B}_h(\pi)$;
2. $\sum_{i \in \mathcal{B}_h(\pi)} x_{i,h} \leq \sum_{j \in \mathcal{S}_h(\pi)} e_{j,h}$;
3. $x_i \leq d_i(\pi_i, w_i(\pi))$.

Following the above discussion, condition 1 states that only buyers with acceptable prices have access to the market. Conversely, condition 2 states that the relevant supply is this from agents who have access to the market. Condition 3 expresses the fact that demand are computed at private prices[3].

From a broader point of view, definition 1 expresses the fact that as agents agree on the valuation of goods, i.e as private prices become more similar, the size of the market grows. In particular when a uniform price profile is reached, i.e when $\pi \in P^m$ is such that there exists $p \in P$ such that for all $i \in \{1, \ldots, m\}$ $\pi_i = p$, than the market includes each agent, and the constraint his private price put on a given agent is simply his private budgetary constraint.

Now, agents will have incentives to agree on a common price if they actually gain from an increased size of the market. To express mathematically this idea, it is useful to see the set of price feasible allocations $\mathcal{A}'(e, \pi)$ as a bargaining set defined by the agents' private prices. Then, the fact that an agent gains form an increase in the market size (provided his budget constraint is not being strengthened) simply appears as the transcription in our context of the monotonicity condition that is standard in the bargaining literature since the seminal paper by Kalai and Smorodinsky (1975)

Assumption 4 (Monotonicity) *If π and π' are such that $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$ then for all $i = 1 \cdots n$, one has $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$.*

Assumption 4 will guarantee that agents have incentives to reach a uniform price profile. If this price is an equilibrium price, that is if $\overline{\pi} \in P^m$ is such that for all $i \in \{1, \ldots, m\}, \overline{\pi}_i = \overline{p}$ where $\overline{p} \in \mathbf{E}(u, e)$, we shall call $\overline{\pi}$ an equilibrium price profile. Our approach to the issue of stability is then based on the identification of equilibrium price profiles of $\mathcal{G}(u, e, \xi)$ with equilibria of the economy $\mathcal{E}(u, e)$. That is we shall assume that at an equilibrium price profile, the corresponding equilibrium allocation prevails.

Assumption 5 (Equilibrium) *If $\overline{\pi} \in P^m$ is such that for all $i \in \{1, \ldots, m\}$, $\overline{\pi}_i = \overline{p}$ where $\overline{p} \in \mathbf{E}(u, e)$, then one has for all $i \in \{1, \ldots, m\}$:*

$$\xi_i(\overline{\pi}) = d_i(\overline{p}, \overline{p} \cdot e_i).$$

Note that this is in fact a very weak efficiency requirement on the exchange process ξ. Indeed, condition (3) in the definition of $\mathcal{A}'(\pi, e)$ implies that for all $i \in \{1, \ldots, m\}, \xi_i(\overline{\pi}) \leq d_i(\overline{p}, \overline{p} \cdot e_i)$. As moreover $(d_i(\overline{p}, \overline{p} \cdot e_i))_{i=1,\cdots,m}$ indeed is a price feasible allocation, assuming that for all $i \in \{1, \ldots, m\}$, one has $\xi_i(\overline{\pi}) = d_i(\overline{p}, \overline{p} \cdot e_i)$, in fact amounts to consider that the exchange process is efficient (in the sense that it picks up a Pareto optimal allocation) at equilibrium price profiles.

[3]it would be equivalent for our purposes to assume that each agent satisfies its private budget constraint, that is $\pi_i \cdot x_i \leq w_i(\pi)$.

This is much weaker than what is usually assumed of bargaining solutions, which are thought to be efficient for any given of the bargaining problem.

Eventually, emergence and stability of equilibrium will depend on the effectiveness of competition in the economy. At a non-equilibrium uniform price profile, that is if $\pi \in P^m$ is such that for all $i \in \{1, \ldots, m\}$, $\pi_i = p$ where $p \notin \mathbf{E}(u, e)$, Walras' law implies that there necessarily is excess demand for some good. We shall restrict attention to exchange processes that are competitive in the sense that in case of excess demand, at least one agent has incentives to deviate from the common price. This is the counterpart of Walras' description of buyers raising their bids to price out competitors. Namely, we shall assume that:

Assumption 6 (Competition) *If $\pi = (p, \cdots, p) \in P^m$ is a uniform price profile and there exists an $h \in \{1, \cdots, l\}$ such that*

$$\sum_{i=1}^{m} d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^{m} e_{i,h},$$

then there exists $i \in \{1, \cdots, m\}$ and $p' \in P$ such that $u_i(\xi_i(p', \pi_{-i})) \geq u_i(\xi_i(\pi))$.

An extensive discussion of the conditions under which exchange processes actually are competitive in the latter sense is developed in section 7. Basically, an exchange process will be competitive if it implements some form of priority for lowest bidding sellers and highest bidding buyers. It is then clear that when there is excess supply a seller who slightly undercuts his competitors increases his sales and hence his income while when there is excess demand, a buyer who slightly overbids his competitors overcomes rationing. Note that in our framework, it does not make sense for a seller to increase his price before buyers do as he would price himself out of the market.

Thus, out of equilibrium, there always are agents who have incentives to deviate. Conversely, at equilibrium, an agent who deviates either prices himself out of the market or decreases his consumption and hence his utility. Hence, only Walrasian equilibria can be strict/stable Nash equilibria of the private price game. A detailed proof is given below.

Proposition 2 *Under Assumptions (3) – (6), $\pi \in \Pi$ is a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$ if and only if $\overline{\pi}$ is an equilibrium price profile.*

Proof: *According to assumption 5, if $\overline{\pi} = (\overline{p}, \cdots, \overline{p})$ is an equilibrium price profile then for all i, $\xi_i(\overline{\pi}) = d_i(\overline{p}, w_i(\overline{\pi}))$, where $w_i(\overline{\pi}) = \overline{p} \cdot e_i$. Assume then agent i deviates to a price $p \neq \overline{p}$.*

As for production goods, one has:

- *If $p_h > \overline{p}_h$, one has $i \notin \mathcal{S}_h(p', \overline{\pi}_{-i})$ so that agent i can no longer sale good h and does not rise any income on the good h market.*

- *If $p_h < \overline{p}_h$, one has $p_h e_{i,h} < \overline{p}_h e_{i,h}$.*

- *If $p_h = \overline{p}_h$, one has $p_h e_{i,h} = \overline{p}_h e_{i,h}$*

Hence, either one has for all $h \in P_i$, $p_h = \overline{p}_h$, and $w_i(p', \overline{\pi}_{-i}) = w_i(\overline{\pi})$, either one has $w_i(p', \overline{\pi}_{-i}) < w_i(\overline{\pi})$.

Looking then at consumption goods, one has:

- *Either there exists $h \in C_i$ such that $p_h < \overline{p}_h$, so that $i \notin \mathcal{B}_h(p', \overline{\pi}_{-i})$ and $\xi_{i,h}(p', \overline{\pi}_{-i}) = 0$, which implies according to assumption $3.2b$ that $u_i(\xi_{i,h}(p', \overline{\pi}_{-i})) = 0 < u_i(d_i(\overline{p}, \overline{p} \cdot e_i))$.*

- *Either one has for all $h \in C_i$, $p_h \geq \overline{p}_h$, with the inequality being strict for some h. As according to the preceding, one always has $w_i(p', \overline{\pi}_{-i}) \leq w_i(\overline{\pi})$, it must be that $u_i(d_i(p', w_i(p', \overline{\pi}_{-i}))) < u_i(d_i(\overline{p}, w_i(\overline{\pi})))$ so that $u_i(\xi_i(p', \overline{\pi}_{-i})) \leq u_i(d_i(p', w_i(p', \overline{\pi}_{-i}))) < u_i(d_i(\overline{p}, w_i(\overline{\pi}))) = u_i(\xi_i(\overline{\pi}))$.*

- *Either for all $h \in C_i$, $p_h = \overline{p}_h$, so that one necessarily has $w_i(p', \overline{\pi}_{-i}) < w_i(\overline{\pi})$ according to the preceding. Again, one has $u_i(d_i(p', w_i(p', \overline{\pi}_{-i}))) < u_i(d_i(\overline{p}, w_i(\overline{\pi})))$ and therefore $u_i(\xi_i(p', \overline{\pi}_{-i})) < u_i(\xi_i(\overline{\pi}))$.*

To sum up, if agent i deviates to price p' either his income or his choice set or both are strictly reduced: he cannot by definition obtain an allocation as good as $d_i(\overline{p}, w_i(\overline{\pi}))$ and hence is strictly worse off.

Suppose then π is not an equilibrium price profile.

- *If π is a uniform price profile and there exists an $h \in \{1, \cdots, l\}$ such that $\sum_{i=1}^{m} d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^{m} e_{i,h}$, then assumption 6 implies π is not a (strict) Nash equilibrium.*

Hence, from here on we can assume that π is not a uniform price profile.

- *If there exists $h \in \{1, \cdots, \ell\}$ and $i \in S_h / \mathcal{S}_h(\pi)$, by setting $p'_h := \max_{i \in B_h} \pi_{i,h}$ and $p'_j := \pi_{i,j}$ for all $j \neq h$ one obtains a population $\pi' := (p', \pi_{-i})$ such that $w_i(\pi') \geq w_i(\pi)$ while the other constraints in the definition of $\mathcal{A}'$ can only be relaxed. Therefore, $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$. According to assumption 4, it can not be that π is a strict Nash equilibrium. Similar arguments apply whenever there exists $h \in \{1, \cdots, \ell\}$ and $i \in B_h / \mathcal{B}_h(\pi)$.*

- *Otherwise, one must have* $B_h = \mathcal{B}_h(\pi)$, $S_h = \mathcal{S}_h(\pi)$, *and there must exist* $i, i' \in \{1, \cdots, m\}$ *and* $h \in \{1, \cdots l\}$ *such that* $\pi_{i,h} \neq \pi_{i',h}$, *e.g* $\pi_{i,h} < \pi_{i',h}$. *One then has:*

  - *If* $i, i' \in S_h$ *by setting* $\pi'_{i,h} = \pi_{i',h}$ *and* $\pi'_{j,k} = \pi_{j,k}$ *for* $j \neq i'$ *or* $k \neq h$, *one has* $w_i(\pi') \geq w_i(\pi)$ *and hence* $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$ *so that* π *can not be a strict Nash equilibrium according to assumption* 4. *The same holds true if* $i \in S_h$ *and* $i' \in B_h$.

  Hence, from here on we can assume that for all $i, i' \in S_h$, $\pi'_{i,h} = \pi_{i',h}$. *Then:*

  - *If* $i, i' \in B_h$ *by setting* $\pi'_{i',h} = \pi_{i,h}$ *and* $\pi'_{j,k} = \pi_{j,k}$ *for* $j \neq i$ *or* $k \neq h$, *the private budget constraint of agent* i' *(condition 3 in definition of* $\mathcal{A}'$*) is relaxed and one then has* $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$ *so that* π *can not be a strict Nash equilibrium according to assumption* 4.
  - *If* $i \in B_h$ *and* $i' \in S_h$, *it must be, given that* $S_h = \mathcal{S}_h(\pi)$, *that there exist* $i'' \in B_h$ *such that* $\pi_{i',h} \leq \pi_{i'',h}$ *and the preceding argument applies to* $i, i'' \in B_h$ *such that* $\pi_{i,h} < \pi_{i'',h}$.

Summing up, π *cannot be a strict Nash equilibrium of* $\mathcal{G}(u, e, \xi)$ *if it is not an equilibrium price profile in the Walrasian sense. This ends the proof.*

Through proposition 1, a direct corollary of proposition 2 is the asymptotic stability of equilibrium for the replicator dynamics.

Proposition 3 *Under assumptions (3) through (6), the only asymptotically stable strategy profiles for the replicator dynamic in* $\mathcal{G}(u, e, \xi)$ *are those for which each agent uses an equilibrium price* $\overline{p} \in \mathcal{E}(u, e)$ *and agent* i *is allocated his equilibrium allocation* $d_i(\overline{p}, \overline{p} \cdot e_i)$.

This result strongly contrasts with the lack of generic stability of the tâtonnement. Both processes are driven by competition, the adaptation of prices to market condition. Yet, a fundamental difference is that in the tâtonnement process, there is a single public price whose variations induce massive side effects whereas in our setting price changes are individual and, taken individually, only have a marginal impact on the economy.

6 Extension to an Arbitrary Economy

Propositions 2 and 3 require that the goods an agent buys and sells be fixed independently of relative prices. This assumption can be relaxed by localizing the notions of acceptable buyers and sellers defined in the preceding section. In the following, we do not assume that assumption 3 holds and consequently adapt the definitions and proofs of the preceding section (incidentally, we slightly overload some of the notations). Given a population $\pi \in P^m$, we define:

- the set of buyers of good h as $B_h(\pi) := \{i \mid d_{i,h}(\pi_i) > e_{i,h}\}$;
- the set of sellers of good h as $S_h(\pi) := \{i \mid d_{i,h}(\pi_i) \leq e_{i,h}\}$;
- the set of acceptable buyers as those agents whose prices is above the lowest selling price that is $\mathcal{B}_h(\pi) := \{i \in \{1, \cdots m\} \mid \pi_{i,h} \geq \min_{j \in S_h(\pi)/\{i\}} \pi_{j,h}\}$;
- the set of acceptable sellers as those agents whose prices is below the highest buying price, that is $\mathcal{S}_h(\pi) := \{i \in \{1, \cdots m\}/\{i\} \mid \pi_{j,h} \leq \max_{i \in B_h(\pi)} \pi_{i,h}\}$;
- the feasible income as $w_i(\pi) = \sum_{\{h \mid i \in \mathcal{S}_h(\pi)\}} \pi_{i,h} e_h$.

The key differences with the preceding section is that here an acceptable buyer (resp. seller) is not necessarily a buyer (resp. seller). In particular, at an uniform price profile, each agent is both an acceptable buyer and an acceptable seller. In fact, whether an agent will end up being a net buyer or a net seller for a given good doesn't depend solely on his excess demand but on the complete demand profile in the economy. In other words, we assume that an agent can't restrict his trades to fulfill in priority his own excess demand. He must bring all his endowment to the market and let the exchange process determine the allocation. The set of price feasible allocations $\mathcal{A}''(e, \pi)$ is then defined in a similar way as in definition (1), but for the fact that conditions bear on excess demands rather than on demands.

Definition 2 *The set of price feasible allocations $\mathcal{A}''(e, \pi)$is the subset of feasible allocations $\mathcal{A}(e, \pi)$ such that for every $x \in \mathcal{A}''(e, \pi)$ one has:*

1. $\forall h,\ \forall i \in B_h(\pi),\ x_{i,h} > e_{i,h} \Rightarrow i \in \mathcal{B}_h(\pi)$;
2. $\sum_{i \in \mathcal{B}_h(\pi) \cap B_h(\pi)} (x_{i,h} - e_{i,h}) \leq \sum_{j \in \mathcal{S}_h(\pi)} (e_{j,h} - x_{j,h})$
3. $x_i \leq d_i(\pi_i, w_i(\pi))$;

Assumption (4) also has an exact counterpart:

Assumption 7 (Monotonicity Bis) *If π and π' are such that $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$ then for all $i = 1 \cdots n$, one has $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$.*

Assumptions 5, 6 and 7 then suffice to establish the counterpart of proposition 2 but for the two following caveats. First, the definition of acceptable sellers and buyers prevent an agent from being the sole buyer and seller of a given good. Therefore, we have to assume that (at least at equilibrium) there are at least a buyer and a distinct seller for every good. Second, in our framework the change of the private price of a commodity he neither consumes nor is endowed with has no effect whatsoever on an agent's utility, so that two strategies that differ only for such a good yield exactly the same utility. This might prevent the identification of Walrasian equilibria with strict equilibria. In order to avoid this failure, one shall assume that either each agent consumes or is endowed with each good or that an agent's strategy space is reduced to meaningful prices: these of commodities he consumes or sells. Assuming both conditions fulfilled, we can proceed with the proof of the following proposition , which is very similar to that of proposition 2 and hence given in the appendix.

Proposition 4 *Under Assumptions* 5, 6 *and 7,* $\overline{\pi} \in \Pi$ *is a strict Nash equilibrium of* $\mathcal{G}(u, e, \xi)$ *if and only if there exists* $\overline{p} \in \mathbf{E}(u, e)$ *such that for all* $i \in \{1, \ldots, m\}$, $\overline{\pi}_i = \overline{p}$.

7 Characterization of Free Competition

It remains to analyze how restrictive is assumption 6 about buyers' incentive to increase prices when there is excess demand. Walras' description of buyers raising their bids is the report of an empirical observation, the "natural" expression of competition. We shall investigate here how this idea of competition can be grounded in the exchange process ξ.

Let us first consider a basic example with two agents and two goods. The first agent derives utility from consumption of good 2 only, e.g his utility function is $u_1(x_1, x_2) = x_2$, and is endowed with a quarter unit of good 1, i.e his initial endowment is $e_1 = (\frac{1}{4}, 0)$. The second agent has Cobb-Douglas preferences $u_2(x_1, x_2) = x_1 x_2$ and initial endowment $e_2 = (\frac{1}{4}, \frac{3}{4})$. Good 2 is the numéraire and is price is fixed equal to one. It is straightforward to check that the only equilibrium is such that the price equals $(1, 1)$, agent 1 is allocated $(0, \frac{1}{4})$ and agent 2 is allocated $(\frac{1}{2}, \frac{1}{2})$. It is also clear that if both agents adopt $(1, 1)$ as private price, the only efficient price feasible allocation is the equilibrium one. Yet, if both agents adopt $(p, 1)$ as private price, then the demands of agent 1 and 2 respectively are $d_1(p) = (0, \frac{p}{4})$ and $d_2(p) = (\frac{1}{8} + \frac{3}{8p}, \frac{p}{8} + \frac{3}{8})$. Whenever $p < 1$, there is excess demand for good 1 and the only efficient allocation is $(0, \frac{p}{4})$ to agent 1 and $(\frac{1}{2}, \frac{3-p}{4})$ to agent 2. Let us examine assumption 6 in this setting. Agent 1 who is not rationed would be worse off if he decreased his private price for good 1 and hence its income, he cannot increase it unilaterally as he would price himself out of the market. Agent 2 cannot decrease his private price for good 1 as he would price himself out of the market. One would expect that "competition" induces him to increase his price for good 1 but he has no incentive to do so as this would only decrease his purchasing power and hence his utility. The key issue is that agent 2 actually faces no competition on the good 1 market as there is no other buyer he could outbid by increasing his price.

Let us then consider a more competitive situation by "splitting in two" agent 2. That is we consider an economy with three agents and two goods. The first agent still has utility function $u_1(x_1, x_2) = x_2$, and initial endowment $e_1 = (\frac{1}{4}, 0)$. The second and third agent have Cobb-Douglas preferences $u_2(x_1, x_2) = u_3(x_1, x_2) = x_1 x_2$ and initial endowment $e_2 = e_3 = (\frac{1}{8}, \frac{3}{8})$. As above, It is straightforward to check that the only equilibrium is such that the price equals $(1, 1)$, agent 1 is allocated $(0, \frac{1}{4})$ while agents 2 and 3 are allocated $(\frac{1}{4}, \frac{1}{4})$. It is also clear that if both agents adopt $(1, 1)$ as private price, the only efficient price feasible allocation is the equilibrium one. Now, if each agent adopts $(p, 1)$ as private price, then the demands respectively are $d_1(p) = (0, \frac{p}{4})$ and $d_2(p) = d_3(p) = (\frac{1}{16} + \frac{3}{16p}, \frac{p}{16} + \frac{3}{16})$. Whenever $p < 1$, there is excess demand for good 1. It seems natural to assume that the exchange process would then allocate its demand $d_1(p) = (0, \frac{p}{4})$ to agent 1 who is not rationed. As far as agents 2 and 3 are concerned, the allocation is efficient provided they are allocated no more than their demand $\frac{1}{16} + \frac{3}{16p}$ in good 1. It seems sensible to consider that the allocation is symmetric and hence that both agents are allocated $(\frac{1}{4}, \frac{3-p}{8})$ and so are rationed in good 1. As before agent 1 has no incentive to change his private price for good 1 and neither agents 2 nor 3 can further decrease their private price for good 1. However, if the exchange process implements a form of competition between buyers by fulfilling in priority the demand of the agent offering the highest price for the good, then agents 2 and 3 have an incentive to increase their private prices for good 1. Indeed assume that agent 2 increases his private price for good 1, so $q > p$. He will then be allocated his demand $(\frac{1}{16} + \frac{3}{16q}, \frac{q}{16} + \frac{3}{16})$ and be better off than at price p provided that

$(\frac{1}{16}+\frac{3}{16q})(\frac{q}{16}+\frac{3}{16}) > \frac{1}{4}.\frac{3-p}{8}$. It is straightforward to check that this equation holds for $q = p$ (whenever $p < 1$) and hence by continuity in a neighborhood of p. In particular, there exists $q > p$ such that agent 2 is better of adopting q as private price for good 1.

The above examples show that necessary conditions for assumption 6 to hold are potential and actual competition among buyers, which are respectively ensured *via* the presence of more than one buyer for every good and the priority given to highest bidding buyers in the trading process. It turns out these two conditions are in fact sufficient.

Since Edgeworth (1881) , notably in (Debreu 1963), the seminal way to ensure competition is effective in a general equilibrium economy is to consider that the economy consists of sufficiently many replicates of a given set of primitive types. For our purposes, it suffices to assume that the economy $\mathcal{E}(u, e)$ is a 2-fold replicate of some underlying simple economy.

Assumption 8 (Replicates) *For every $i \in \{1 \cdots m\}$ there exists $i' \in \{1 \cdots m\}/\{i\}$ such that $u_i = u_{i'}$ and $e_i = e_{i'}$. Types i and i' are called replicates.*

In our framework, a companion assumption is to consider that the exchange process is symmetric with respect to replicates using the same private price. That is:

Assumption 9 (Symmetry) *For every $\pi \in P^m$, if $i, i' \in \{1 \cdots m\}$ are replicates such that $\pi_i = \pi_{i'}$, then $\xi_i(\pi) = \xi_{i'}(\pi)$.*

Then, to ground in the exchange process ξ actual effects of competition, we shall assume that if there is excess demand for one good the highest bidding seller has priority access to the market. That is a buyer who deviates upwards from a uniform price profile has its demand fulfilled in priority.

Assumption 10 (High Bidders Priority) *Let $\pi = (p, \cdots, p) \in P^m$ be a uniform price profile, $i, i' \in \{1, \cdots, m\}$ be replicates and π' a price profile such that $\pi'_{i,h} > p_h$ for every $h \in C_i$ and $\pi_{j,k} = p_k$ otherwise. Then defining*

$$X_i(\pi') = \{x_i \in \mathbf{R}^l_+ \mid x_i \leq \xi_i(\pi) + \xi_{i'}(\pi) \text{ and } \pi'_i x_i \leq \pi'_i e_i\},$$

we have

$$u_i(\xi_i(\pi')) \geq \max_{x_i \in X_i(\pi')} u_i(x_i).$$

That is, given that agent i gains priority over his replicate by increasing his price, everything goes as if he could pick any allocation satisfying his private budget constraint in the pool formed by adding the allocations he and his replicates were formerly allocated.

As announced, the latter conditions suffice to ensure that competition holds in the sense of assumption (6). Namely, one has

Proposition 5 *If assumptions* 3, 4, 8, 9, *and* 10 *hold, then assumption* 6 *holds.*

Proof: *Let $\pi = (p, \cdots, p) \in P^m$ be a uniform price profile such that for some $h \in \{1, \cdots, l\}$, one has $\sum_{i=1}^m d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^m e_{i,h}, .$ We shall prove that there exists $p' \in P$ such that $u_i(\xi_i(p', \pi_{-i})) \geq u_i(\xi_i(\pi))$. If there exists i such that for some $h \in C_i$, $\xi_{i,h}(\pi) = 0$, the proof is straightforward according to assumption (3). Otherwise, let us then consider an agent i such that $\xi_{i,h}(\pi) < d_{i,h}(\pi_i, w_i(\pi))$. It is then clear, given condition 3 in the definition of $\mathcal{A}'$, that $u_i(\xi_i(\pi)) < u_i(d_{i,h}(\pi_i, w_i(\pi))$. Hence it must either be that:*

- *the private budget constraint of agent i is not binding, that is one has $p \cdot \xi_i(\pi) < p \cdot e_i$,*
- *or there are budget neutral utility improving shifts in consumption, that is there exists $v \in \mathbf{R}^l_+$ such that $v_h = 0$ for all $h \notin C_i$ and $p \cdot v = 0$ such that for all sufficiently small $t > 0$, $u_i(\xi_i(\pi) + tv) > u_i(\xi_i(\pi))$.*

In the case where agent i private budget constraint is not binding it is clear, given assumption 10, that agent i can still afford and obtain $\xi_i(\pi)$ if he shifts to a price $p' \in P$ such that $p'_h > p_h$ for every $h \in C_i$, $p'_h >= p_h$ otherwise and p' is sufficiently close to p. In the latter case where for all sufficiently small $t > 0$, one has $u_i(\xi_i(\pi) + tv) > u_i(\xi_i(\pi))$, denoting by i', the replicate of i (who receives the same allocation according to assumption (9)) one has for $t > 0$ sufficiently small: $(\xi_i(\pi) + tv) \leq \xi_i(\pi) + \xi_{i'}(\pi)$. Then for any p' such that $p'_h > p_h$ for every $h \in C_i$, and $p'_h >= p_h$, let us set $x_i(p') = \dfrac{p' \cdot e_i}{p \cdot e_i}(\xi_i(\pi) + tv)$. One clearly has $x_i(p') \leq \xi_i(\pi) + \xi_{i'}(\pi)$ and $p' \cdot x_i(p') \leq p' \cdot e_i$. This implies according to assumption (10) that $\xi_i(\pi')) \geq u_i(x_i(p'))$ (where p' is defined as in assumption (10)). Moreover, for p' sufficiently close to p, one has using the continuity of the utility that $u_i(x_i(p')) > u_i(\xi_i(\pi))$. This ends the proof.

8 A Markov Implementation of Walrasian Dynamics

Consider a market economy with many goods, many agents and one institution—the marketplace.[4] We assume each agent produces one good, in fixed amount, using only personal labor, but consumes a variety of goods. Agents are endowed with private prices, and they have no information about the economy other than that gathered from private experience in trade. These assumptions lack realism, but each can be replaced by more realistic assumptions when necessary. For instance, we could add a labor market, a capital market, firms, and a central bank without difficulty (Gintis 2007).

We assume there are n goods. Each agent consumes a subset of goods, but not his production good. We write the set of goods as $G = \{g_k | k = 1, \ldots, n\}$. A producer of good g_k uses personal labor and no other inputs to produce an amount q_k of good g_k which depreciates to zero if it remains in inventory at the close of a trading period. A trade inventory for an agent can include any good acquired through production or trade.

The Markov process is initialized by creating N agents, each of whom is randomly assigned a production good g_k. Each agent A is assigned a private price vector $(p_1^A, \ldots, p_n^A)$ by choosing each price from a uniform distribution on the open unit interval, then normalizing so that the price of the n^{th} good is unity. Each g_k producer is then randomly assigned a set $H \subseteq G$, $g_k \notin H$ of consumption goods.

The utility function of each agent is rendered unique by randomly setting several parameters of a hybrid CES utility function, explained in Appendix C. These utility functions, which are generalizations of a functional form widely used in economic models, do not satisfy the gross substitutability assumption (Arrow, Block and Hurwitz 1959), so stability in the tâtonnement dynamic generally does not hold.

For each good $g_k \in G$ there is a market m_k of g_k producers. In each period, the agents in the economy are randomly ordered and permitted one-by-one to initiate trades. When the g_k-producer A is the currently active agent, for each good g_h for which A has positive demand, A encounters a random member B in market m_h who consumes g_k. A then offers B the maximum quantity y_k of g_k, subject to the constraints $y_k \leq \mathbf{i}_k^A$, where $\mathbf{i}_k^A$ represents A's current inventory of good g_k, and $y_k \leq p_h^A x_h^A / p_k^A$, where x_h^A is A's current demand for g_h. This means that if A's offer is accepted, A will receive in value at least as much as he gives up, according to A's private prices. A then offers to exchange y_k for an amount $y_h = p_k^A y_k / p_h^A$ of good g_h; that is, he offers B an equivalent value of good g_k, the valuation being

[4] This model and its dynamics is more fully described in Gintis (2013)

at A's prices. B accepts this offer provided the exchange is weakly profitable at B's private prices; that is, provided $p_k^B y_k \geq p_h^B y_h$. However, B adjusts the amount of each good traded downward if necessary, while preserving their ratio, to ensure that what he receives does not exceed his demand, and what he gives is compatible with his inventory of g_h.[5] If A fails to trade with this agent, he still might secure a trade giving him g_k, because A $\in m_k$ may also be on the receiving-end of trade offers from g_h-agents at some point during the period. If a g_k-agent exhausts his supply of g_k, he leaves the market for the remainder of the period.

After each trading period, traders consume their inventories, and agents replenish the amount of their production good in inventory. Moreover, each agent updates his private price vector on the basis of his trading experience over the period, raising the price of a consumption or production good by 0.05% if his inventory is empty (that is, if he failed to purchase any of the consumption good or sell all of his production good), and lowering price by 0.05% otherwise (that is, if he succeeded in obtaining his consumption good or sold all his production inventory). We allow this adjustment strategy to evolve endogenously according to an imitation processes.

After a number of trading periods, the population of agents is updated using a standard replicator dynamic, in which agents who have high scores for trading and consuming have a high probability of reproducing, while unsuccessful trades are eliminated from the economy. In all cases, the new agents inherit the price vector of its parent, perhaps mutated a bit. The resulting updating process is a discrete approximation of a monotonic dynamic in evolutionary game theory. In differential equation systems, all monotonic dynamics have the same properties as the simplest, which is the replicator dynamic (Taylor and Jonker 1978, Samuelson and Zhang 1992). Other monotonic approximations, including choosing a pair of agents in m_k and let the lower-scoring agent copy the price vector of the higher-scoring agent, produce similar dynamical results.

The result of the dynamic specified by the above conditions is the change over time in the distribution of private prices. The general result is that the system of private prices, which at the outset are randomly generated, in rather short time evolves to a set of *quasi-public* prices with very low inter-agent variance. Over the long term, these quasi-public prices move toward their equilibrium, market-clearing levels.

We illustrate this dynamic assuming $n = 54$, and $N = 300$, so there are 16200 agents in the economy. A g_k-agent produces one unit of good k per period. We assume that there are equal numbers of producers of each good from the outset,

[5]For related analyses of trading algorithms in a multiple market setting, see Rubinstein and Wolinsky (1985), Gale (1987), and Binmore and Herrero (1988).

although we allow migration from less profitable to more profitable sectors, so in the long run profit rates are close to equal in all sectors. The complexity of the utility functions do not allow us to calculate equilibrium properties of the system perfectly, but we will assume that market-clearing prices are approximately equal to unit costs, given that unit costs are fixed, agents can migrate from less to more profitable sectors, and utility functions do not favor one good or style over another, on average. Population updating occurs every ten periods, and the number of encounters per sector is 10% of the number of agents in the sector. The mutation rate is $\mu = 0.01$ and the error correction is $\epsilon = 0.01$.

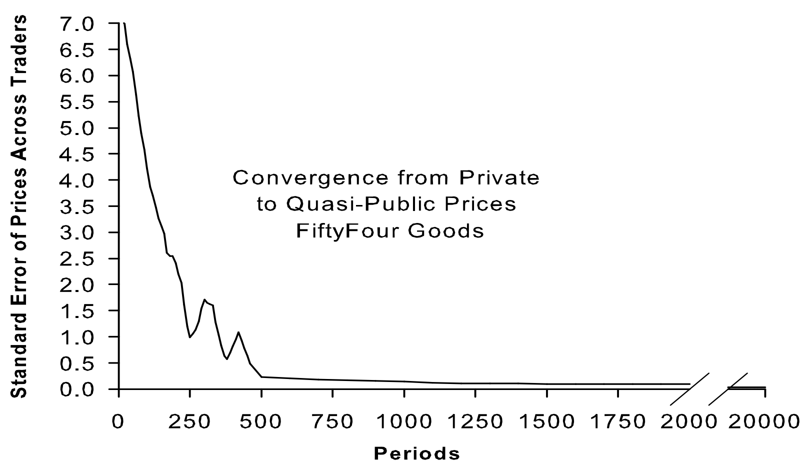

Figure 1: Convergence of private prices to quasi-public prices in a typical run with fifty-four goods.

The results of a typical run of this model is illustrated in Figures 1 and 2. Figure 1 shows the passage from private to quasi-public prices over the first 20,000 trading periods of a typical run. The mean standard error of prices is computed as follows. For each good g we measure the standard deviation of the price of g across all g-agents, where for each agent, the price of the numeraire good g_9 unity. Figure 1 shows the average of the standard errors for all goods. The passage from private to quasi-public prices is quite dramatic, the standard error of prices across individuals falling by an order of magnitude within 300 periods, and falling another order of magnitude over the next 8500 periods. The final value of this standard error is 0.029, as compared with it initial value of 6.7.

Figure 2 shows the movement of the absolute value of excess demand over 50,000 periods for fifty-four goods. Using this measure, after 1500 periods excess

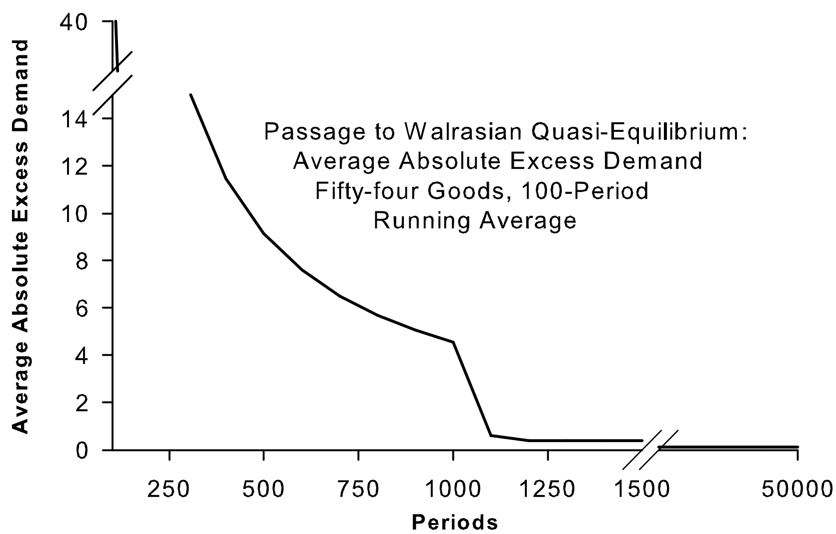

Figure 2: The path of aggregate excess demand over 50,000 periods.

demand has decrease by two orders of magnitude, and it decreases another order of magnitude by the end of the run.

The distinction between low-variance private prices and true public prices is significant, even when the standard error of prices across agents is extremely small, because stochastic events such as technical changes propagate very slowly when prices are highly correlated private prices, but very rapidly when all agents react in parallel to price movement. In effect, with private prices, a large part of the reaction to a shock is a temporary reduction in the correlation among prices, a reaction that is impossible with public prices, as the latter are always perfectly correlated.

There is nothing special about the parameters used in the above example. Of course adding more goods or styles increases the length of time until quasi-public prices become established, as well as the length of time until market quasi-equilibrium is attained. Increasing the number of agents increases the length of both of these time intervals.

9 Conclusion

We have shown that the equilibrium of a Walrasian market system is a strict Nash equilibrium of an exchange game in which the requirements of the exchange process are quite mild and easily satisfied. Assuming producers update their private price profiles periodically by adopting the strategies of more successful peers, we have a multipopulation game in which strict Nash equilibria are asymptotically stable in the replicator dynamic. Conversely, all stable equilibria of the replicator dynamic are strict Nash equilibria of the exchange process and hence Walrasian equilibria of the underlying economy.

The major innovation of our model is the use of private prices, one set for each agent, in place of the standard assumption of a uniform public price faced by all agents, and the replacement of the tátonnement process with a replicator dynamic. The traditional public price assumption would not have been useful even had a plausible stability theorem been available using such prices. This is because there is no mechanism for prices to change in a system of public prices—no agent can alter the price schedules faced by the large number of agents with whom any one agent has virtually no contact.

The private price assumption is the only plausible assumption for a fully decentralized market system not in equilibrium, because there is in fact no natural way to define a common price system except in equilibrium. With private prices, each individual is free to alter his price profile at will, market conditions alone ensuring that something approximating a uniform system of prices will prevail in the long run.

There are many general equilibrium models with private prices in the literature, based for the most part on strategic market games (Shapley and Shubik 1977, Sahi and Yao 1989, Giraud 2003) in which equilibrium prices are set on a market-by-market basis to equate supply and demand, and it is shown that under appropriate conditions the Nash equilibria of the model are Walrasian equilibria. These are equilibrium models, however, without known dynamical properties, and unlike our approach they depend on an extra-market mechanism to balance demand and supply.

The equations of our dynamical system are too many and too complex to solve analytically or to estimate numerically. However, it is possible to construct a discrete version of the system as a finite Markov process. The link between stochastic Markov process models and deterministic replicator dynamics is well documented in the literature. Helbing (1996) shows, in a fairly general setting, that mean-field approximations of stochastic population processes based on imitation and mutation lead to the replicator dynamic. Moreover, Benaim and Weibull (2003) show that large population Markov process implementations of the stage game have approximately the same behavior as the deterministic dynamical system implementations based on the replicator dynamic. This allows us to study the behavior of the dynamical market economy for particular parameter values. For sufficiently large population size, the discrete Markov process captures the dynamics of the Walrasian economy extremely well with near certainty (Benaim and Weibull 2003). While analytical solutions for the discrete system exist (Kemeny and Snell 1960, Gintis 2009), they also cannot be practically implemented. However, the dynamics of the Markov process model can be studied for various parameter values by computer simulation (Gintis 2007, 2012).

Macroeconomic models have been especially handicapped by the lack of a general stability model for competitive exchange. The proof of stability of course does not shed light on the fragility of equilibrium in the sense of its susceptibility to exogenous shocks and its reactions to endogenous stochasticity. These issues can be studied directly through Markov process simulations, and may allow future macroeconomists to develop analytical microfoundations for the control of excessive market volatility.

Appendix A: Asymptotic stability and replicator dynamics

Let G be an n-player game with finite strategy sets $\{S_i|i = 1, \ldots, n\}$, the cardinal of which is denoted by $k_i = |S_i|$, with strategies indexed by $h = 1, \ldots, k_i$ and payoff functions $\{\pi_i|i = 1, \ldots, n\}$. Let $\Delta_i = \{\sigma_i \in \mathbf{R}^{k_i}|\forall h, \sigma_{i,h} \geq 0$ and $\sum_{h=1}^{k_i} \sigma_{i,h} = 1\}$ the which is the mixed strategy space of agent i, and let $\Delta = \prod_{i=1}^{m} \Delta_i$. In an evolutionary game setting, an element $\sigma_i \in \Delta_i$ represents a population of players i with a share $\sigma_{i,h}$ of the population playing strategy $h \in S_i$.

Dynamics for such population of players $(\sigma_1, \ldots, \sigma_N) \in \Delta$ are defined by specifying, a growth rate function $g : \Delta \to \mathbf{R}^{\sum_{i=1}^{m} k_i}$, for all $i = 1, \ldots, n$ and $h = 1, \ldots, k_i$:

$$\frac{\partial \sigma_{i,h}}{\partial t} = \sigma_{i,h} g_{i,h}(\sigma) \tag{2}$$

We shall restrict attention to growth-rate functions that satisfy a regularity condition and maps Δ into itself (Weibull 1995).

Definition 3 *A regular growth-rate function is a Lipschitz continuous function g defined in a neighborhood of Δ such that for all $\sigma \in \Delta$ and all $i = 1, \ldots, n$ we have has $g_i(\sigma) \cdot \sigma_i \neq 0$.*

The dynamics of interest in a game-theoretic setting are those that satisfy minimal properties of monotonicity with respect to payoffs. Strategies of player i in $B_i(\sigma) := \{s \in S_i|u_i(s, \sigma_{-i}) > u_i(\sigma)\}$ that have above average payoffs against σ_{-i}, have a positive growth-rate in the following sense:

Definition 4 *A regular growth-rate function g is weakly payoff-positive if for all $\sigma \in \Delta$ and $i = 1, \ldots, n$,*

$$B_i(\sigma) \neq 0 \Rightarrow g_{i,h} > 0 \text{ for some } s_{i,h} \in B_i(\sigma), \tag{3}$$

where $s_{i,h}$ denotes the h^{th} pure strategy of player i.

Among the class of weakly-payoff positive dynamics, the replicator dynamic is by far the most commonly used to represent the interplay between population dynamics and strategic interactions. It corresponds to the system of differential equations defined for all $i = 1, \ldots, n$ and $h = 1, \ldots, |S_i|$ by:

$$\frac{\partial \sigma_{i,h}}{\partial t} = \sigma_{i,h}(\pi_i(s_{i,h}, \sigma_{-i}) - \pi_i(\sigma)). \tag{4}$$

That is thus the system of differential equation corresponding to the growth rate function $g_{i,h}(\sigma) = \pi_i(s_{i,h}, \sigma_{-i}) - \pi_i(\sigma)$.

It is standard to show that the system of differential equations (2) associated with a regular and weakly-payoff monotonic growth function has a unique solution defined at all times for every initial condition in Δ. We will generically denote the solution mapping by $\psi : \mathbf{R}_+ \times \Delta \to \Delta$, so $\psi(t, \sigma_0)$ gives the value at time t of the solution to (2) with initial condition $\sigma(0) = \sigma_0$. Stability properties of (2), are then defined in terms of this solution mapping:

Definition 5 *A strategy profile $\sigma^* \in \Delta$ is called Lyapunov stable if every neighborhood V of σ^* contains a neighborhood W of σ^* such that $\psi(t, \sigma) \in V$ for all $\sigma \in W \cap \Delta$.*

Definition 6 *A strategy profile $\sigma^* \in \Delta$ is called asymptotically stable if it is Lyapunov stable and there exists a neighborhood V of σ^* such that for all $\sigma \in V \cap \Delta$:*

$$\lim_{t \to +\infty} \psi(t, \sigma) = \sigma^*.$$

Appendix B: Proof of proposition 4.

The proof of proposition 4 proceeds as follows.

Proof: *According to assumption 5, if $\overline{\pi} = (\overline{p}, \cdots, \overline{p})$ is an equilibrium price profile then for all i, $\xi_i(\overline{\pi}) = d_i(\overline{p}, w_i(\overline{\pi}))$, where $w_i(\overline{\pi}) = \overline{p} \cdot e_i$. Assume then agent i deviates to a price $p \neq \overline{p}$ and let $\pi' = (p, \pi_{-i})$. Then, one has:*

- *If there is h such that $p_h > \overline{p}_h$, one has $i \notin \mathcal{S}_h(\pi')$ and agent i can no longer sale good h and does not rise any income on the good h market.*

- *If there is h such that $p_h < \overline{p}_h$, one clearly has $p_h e_{i,h} < \overline{p}_h e_{i,h}$.*

Given that at the equilibrium price profile $\overline{\pi}$, every agent is an acceptable seller for every good so that $w_i(\overline{\pi}) = \sum_{h=1}^{\ell} \pi_{i,h} e_{i,h}$, it follows that $w_i(\pi') \leq w_i(\overline{\pi})$, the inequality being strict unless $e_{i,h} = 0$ for every h such that $p_h \neq \overline{p}_h$.

If the latter condition holds, there must be h such that $p_h \neq \overline{p}_h$ and $e_{i,h} = 0$ (so that for any price profile ρ, $i \in B_h(\rho)$). For any h such that $p_h < \overline{p}_h$, then $i \notin \mathcal{B}_h(\pi')$ and one necessarily has $\xi_{i,h}(p', \pi_{-i}) = 0$ according to the definition of $\mathcal{A}$ (and given that $e_{i,h} = 0$). Hence agent i can only consume goods h such that $p_h \geq \overline{p}_h$.

To sum up, if agent i deviates to price p' his income and his choice set are both reduced and one of them is strictly reduced. He cannot by definition obtain an allocation as good as $d_i(\overline{p}, \overline{p} \cdot e_i)$ and hence is strictly worse off.

Suppose then π is not an equilibrium price profile.

If π is a uniform price profile and there exists an $h \in \{1, \cdots, l\}$ such that $\sum_{i=1}^{m} d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^{m} e_{i,h}$, then assumption 6 implies π is not a (strict) Nash equilibrium.

Hence, from here on we can assume that π is not a uniform price profile. Then:

- *If there exists $h \in \{1, \cdots, \ell\}$ and $i \in S_h(\pi)/\mathcal{S}_h(\pi)$, by setting $p'_h := \max_{i \in B_h(\pi)/\{i\}} \pi_{i,h}$ and $p'_j := \pi_{i,j}$ for all $j \neq h$ one obtains a population $\pi' := (p', \pi_{-i})$ such that $w_i(\pi') \geq w_i(\pi)$ while the other constraints in the definition of $\mathcal{A}''$ can only be relaxed. Therefore $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$. According to assumption 7, it can not be that π is a strict Nash equilibrium. Similar arguments apply whenever there exists $h \in \{1, \cdots, \ell\}$ and $i \in B_h(\pi)/\mathcal{B}_h(\pi)$.*
- *Otherwise, one must have $B_h(\pi) \subset \mathcal{B}_h(\pi)$, $S_h(\pi) \subset \mathcal{S}_h(\pi)$, and there must exist $i, i' \in \{1, \cdots, m\}$ and $h \in \{1, \cdots l\}$ such that $\pi_{i,h} \neq \pi_{i',h}$, e.g $\pi_{i,h} < \pi_{i',h}$. One then has:*
    - *If $i, i' \in S_h(\pi)$ by setting $\pi'_{i,h} = \pi_{i',h}$ and $\pi'_{j,k} = \pi_{j,k}$ for $j \neq i'$ or $k \neq h$, one has $w_i(\pi') \geq w_i(\pi)$ while the other constraints in the definition of $\mathcal{A}''$ can only be relaxed. Therefore $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$ so that π can not be a strict Nash equilibrium according to assumption 7. The same holds true if $i \in S_h(\pi)$ and $i' \in B_h(\pi)$.*

  Hence, from here on we can assume that for all $i, i' \in S_h$, $\pi'_{i,h} = \pi_{i',h}$. Then:

– *If $i, i' \in B_h(\pi)$ by setting $\pi'_{i',h} = \pi_{i,h}$ and $\pi'_{j,k} = \pi_{j,k}$ for $j \neq i$ or $k \neq h$, the private budget constraint of agent i' (condition 3 in the definition of $\mathcal{A}''$) is relaxed and one then has $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$ so that π can not be a strict Nash equilibrium according to assumption 7.*
– *If $i \in B_h(\pi)$ and $i' \in S_h(\pi)$, it must be, given that $S_h(\pi) = \mathcal{S}_h(\pi)$, that there exist $i'' \in B_h$ such that $\pi_{i',h} \leq \pi_{i'',h}$ and the preceding argument applies to $i, i'' \in B_h(\pi)$ such that $\pi_{i,h} < \pi_{i'',h}$.*

Summing up, π cannot be a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$ if it is not an equilibrium price profile in the Walrasian sense. This ends the proof.

Appendix C: A Hybrid CES Utility Function

The utility function of each agent is the product of powers of CES utility functions of the following form. For each consumer, we partition the n consumer goods into k segments (k is chosen randomly from $1 \ldots n/2$) of randomly chosen sizes $m_1, \ldots, m_k$, with $m_j > 1$ for all j, and $\sum_j m_j = n$. We randomly assign goods to the various segments, and for each segment, we generate a CES consumption with random weights and elasticity. Total utility is the product of the k CES utility functions to random powers f_j such that $\sum_j f_j = 1$. In effect, no two consumers have the same utility function.

For example, consider a segment using goods $x_1, \ldots, x_m$ with private prices $p_1, \ldots, p_m$ and elasticity of substitution s, and suppose the power of this segment in the overall utility function is f. It is straightforward to show that the agent spends a fraction f of his income M on goods in this segment, whatever prices he faces. The utility function associated with this segment is then

$$u(x_1, \ldots, x_n) = \left(\sum_{l=1}^{m} \alpha_l x_l^{\gamma} \right)^{1/\gamma}, \tag{5}$$

where $\gamma = (s-1)/s$, and $\alpha_1, \ldots, \alpha_m > 0$ satisfy $\sum_l \alpha_l = 1$. The income constraint is $\sum_{l=1}^{m} p_l x_l = f_i M$. Solving the resulting first order conditions for utility maximization, and assuming $\gamma \neq 0$ (that is, the utility function segment is not Cobb-Douglas), this gives

$$x_i = \frac{M f_i}{\sum_{l=1}^{m} p_l \phi_{il}^{1/(1-\gamma)}}, \tag{6}$$

where

$$\phi_{il} = \frac{p_i \alpha_l}{p_l \alpha_i} \quad \text{for } i, l = 1, \ldots, m.$$

When $\gamma = 0$ (which occurs with almost zero probability), we have a Cobb-Douglas utility function with exponents α_l, so the solution becomes

$$x_i = \frac{M f_i \alpha_i}{p_i}. \tag{7}$$

The utility function of each agent is the product of powers of CES utility functions of the following form. For each consumer, we partition the n consumer goods into k segments (k is chosen randomly from $1 \ldots n/2$) of randomly chosen sizes $m_1, \ldots, m_k$, with $m_j > 1$ for all j, and $\sum_j m_j = n$. We randomly assign goods to the various segments, and for each segment, we generate a CES consumption with random weights and an elasticity randomly drawn from a uniform distribution. Total utility is the product of the k CES utility functions to random powers f_j such that $\sum_j f_j = 1$. In effect, no two consumers have the same utility function.

For example, consider a segment using goods $x_1, \ldots, x_m$ with prices $p_1, \ldots, p_m$ and (constant) elasticity of substitution s, and suppose the power of this segment in the overall utility function is f. It is straightforward to show that the agent spends a fraction f of his income M on goods in this segment, whatever prices he faces. The utility function associated with this segment is then

$$u(x_1, \ldots, x_n) = \left(\sum_{l=1}^{m} \alpha_l x_l^{\gamma} \right)^{1/\gamma}, \tag{8}$$

where $\gamma = (s-1)/s$, and $\alpha_1, \ldots, \alpha_m > 0$ satisfy $\sum_l \alpha_l = 1$. The income constraint is $\sum_{l=1}^{m} p_l x_l = f_i M$. Solving the resulting first order conditions for utility maximization, and assuming $\gamma \neq 0$ (that is, the utility function segment is not Cobb-Douglas), this gives

$$x_i = \frac{M f_i}{\sum_{l=1}^{m} p_l \phi_{il}^{1/(1-\gamma)}}, \tag{9}$$

where

$$\phi_{il} = \frac{p_i \alpha_l}{p_l \alpha_i} \quad \text{for } i, l = 1, \ldots, m.$$

When $\gamma = 0$ (which occurs with almost zero probability), we have a Cobb-Douglas utility function with exponents α_l, so the solution becomes

$$x_i = \frac{M f_i \alpha_i}{p_i}. \tag{10}$$

References

Arrow, Kenneth and Leonid Hurwicz, "On the Stability of the Competitive Equilibrium, I," *Econometrica* 26 (1958):522–552.

____ and ____ , "Competitive Stability under Weak Gross Substitutability: Nonlinear Price Adjustment and Adaptive Expectations," Technical Report, Office of Naval Research Contract Nonr-255, Department of Economics, Stanford University (1959). Technical Report No. 78.

____ and ____ , "Some Remarks on the Equilibria of Economics Systems," *Econometrica* 28 (1960):640–646.

____ , H. D. Block, and Leonid Hurwicz, "On the Stability of the Competitive Equilibrium, II," *Econometrica* 27 (1959):82–109.

Arrow, Kenneth J. and Gérard Debreu, "Existence of an Equilibrium for a Competitive Economy," *Econometrica* 22,3 (1954):265–290.

Bala, V. and M. Majumdar, "Chaotic Tatonnement," *Economic Theory* 2 (1992):437–445.

Balasco, Yves, *The Equilibrium Manifold. Postmodern Developments in the Theory of General Economic Equilibrium* (Cambridge: MIT Press, 2009).

Benaim, Michel and Jürgen W. Weibull, "Deterministic Approximation of Stochastic Evolution in Games," *Econometrica* 71,3 (2003):873–903.

Bénassy, Jean-Pascal, *The Macroeconomics of Imperfect Competition and Nonclearing Markets: A Dynamic General Equilibrium Approach* (The MIT Press, 2005).

Binmore, Kenneth and M. J. Herrero, "Matching and Bargaining in Dynamic Markets," *Review of Economic Studies* LV (1988):17–31.

Debreu, Gérard, "A Social Equilibrium Existence Theorem," *Proceedings of the National Academy of Sciences* 38 (1952):886–893.

Debreu, Gërard, "A Limit Theorem on the Core of an Economy," *International Economic Review* 4,3 (1963):235–246.

Debreu, Gérard, "Excess Demand Function," *Journal of Mathematical Economics* 1 (1974):15–23.

Edgeworth, Francis Ysidro, *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences* (London: Kegan Paul, 1881).

Fisher, Franklin M., "Quasi-competitive Price Adjustment by Individual Firms: A Preliminary Paper," *Journal of Economic Theory* 2 (1970):195–206.

——, "On Price Adustments without an Auctioneer," *Review of Economic Studies* 39 (1972):1–15.

——, "Stability and Competitive Equilibrium in Two Models of Search and Individual Price Adjustment," *Journal of Economic Theory* 6 (1973):446–470.

——, *Disequilibrium Foundations of Equilibrium Economics* (Cambridge: Cambridge University Press, 1983).

Florenzano, M., *General Equilibrium Analysis: Existence and Optimality Properties of Equilibria* (Berlin: Springer, 2005).

Friedman, Daniel, "Evolutionary Games in Economics," *Econometrica* 59,3 (May 1991):637–666.

Fudenberg, Drew and David K. Levine, *The Theory of Learning in Games* (Cambridge, MA: MIT Press, 1997).

Gale, David, "The Law of Supply and Demand," *Math. Scand.* 30 (1955):155–169.

——, "A Note on Global Instability of Competitive Equilibrium," *Naval Research Logistics Quarterly* 10 (March 1963):81–87.

Gale, Douglas, "Limit Theorems for Markets with Sequential Bargaining," *Journal of Economic Theory* 43 (1987):20–54.

Gintis, Herbert, "The Dynamics of General Equilibrium," *Economic Journal* 117 (October 2007):1289–1309.

——, *Game Theory Evolving* Second Edition (Princeton: Princeton University Press, 2009).

——, "The Dynamics of Pure Market Exchange," in Masahiko Aoki, Kenneth Binmore, Simon Deakin, and Herbert Gintis (eds.) *Complexity and Institutions: Norms and Corporations* (London: Palgrave, 2012).

——, "Markov Models of Social Exchange: Theory and Applications," *ACM Transactions in Intelligent Systems and Technology* 4,3,53 (June 2013).

Giraud, Gaël, "Strategic Market Games: An Introduction," *Journal of Mathematical Economics* 39 (2003):355–375.

Grandmont, Jean Michel, "Temporary General Equilibrium Theory," *Econometrica* (1977):535–572.

Hahn, Frank, "A Stable Adjustment Process for a Competitive Economy," *Review of Economic Studies* 29 (1962):62–65.

—— and Takashi Negishi, "A Theorem on Non-Tâtonnement Stability," *Econometrica* 30 (1962):463–469.

Hammond, Peter J., "Irreducibility, Resource Relatedness and Survival in Equilibrium with Individual Nonconvexities in General Equilibrium," in R. Becker, M. Boldrin, R. Jones, and W. Thomson (eds.) *Growth and Trade* (San Diego: Academic Press, 1993) pp. 73–115.

Helbing, Dirk, "A Stochastic Behavioral Model and a 'Microscopic' Foundation of Evolutionary Game Theory," *Theory and Decision* 40 (1996):149–179.

Hofbauer, Josef and William H. Sandholm, "On the Global Convergence of Stochastic Fictitious Play," *Econometrica* 70,6 (2002):2265–2294.

Ianni, Antonella, "Learning Strict Nash Equilibria Through Reinforcement," *Journal of Mathematical Economics* (2013).

Kalai, Ehud and Meir Smorodinsky, "Other Solutions to Nash's Bargaining Problem," *Econometrica* 43,3 (May 1975):513–518.

Kemeny, John G. and J. Laurie Snell, *Finite Markov Chains* (Princeton: Van Nostrand, 1960).

Mantel, Rolf, "On the Characterization of Aggregate Excess Demand," *Journal of Economic Theory* 7 (1974):348–53.

—— , "Homothetic Preferences and Community Excess Demand Functions," *Journal of Economic Theory* 12 (1976):197–201.

Maynard Smith, John and George R. Price, "The Logic of Animal Conflict," *Nature* 246 (2 November 1973):15–18.

McKenzie, L. W., "On the Existence of a General Equilibrium for a Competitive Market," *Econometrica* 28 (1959):54–71.

——— , “Stability of Equilibrium and Value of Positive Excess Demand,” *Econometrica* 28 (1960):606–617.

Negishi, Takashi, “Welfare Economics and the Existence of an Equilibriumfor a Competitive Economy,” *Metroeconomica* 12 (1960):92–97.

——— , “On the Formation of Prices,” *International Economic Review* 2 (1961):122–126.

Nikaido, Hukukaine, “On the Classical Multilateral Exchange Problem,” *MetroEconomica* 8 (1956):135–145.

——— , “Stability of Equilibrium by the Brown-von Neumann Differential Equation,” *MetroEconomica* 27 (1959):645–671.

——— and Hirofumi Uzawa, “Stability and Nonnegativity in a Walrasian Tâtonnement Process,” *International Economic Review* 1 (1960):50–59.

Rubinstein, Ariel and Asher Wolinsky, “Equilibrium in a Market with Sequential Bargaining,” *Econometrica* 53 (1985):1133–1150.

Saari, Donald G., “Iterative Price Mechanisms,” *Econometrica* 53,5 (September 1985):1117–1131.

Sahi, S. and S. Yao, “The Non-cooperative Equilibria of a Trading Economy with Complete Markets and Consistent Prices,” *Journal of Mathematical Economics* 18 (1989):315–346.

Samuelson, Larry and Jianbo Zhang, “Evolutionary Stability in Asymmetric Games,” *Journal of Economic Theory* 57,2 (1992):363–391.

Scarf, Herbert, “Some Examples of Global Instability of Competitive Equilibrium,” *International Economic Review* 1 (1960):157–172.

Shalizi, Cosmo, “Dynamics of Bayesian Updating with Dependent Data and Misspecified Model,” *Electronic Journal of Statistics* 3 (2009):1039–10–74.

Shapley, Lloyd and Martin Shubik, “Trade Using one Commodity as a Means of Payment,” *Journal of Political Economy* 85 (1977):937–968.

Sonnenschein, Hugo, “Do Walras’ Identity and Continuity Characterize the Class of Community Excess Demand Functions?,” *Journal of Ecomonic Theory* 6 (1973):345–354.

Taylor, Peter and Leo Jonker, "Evolutionarily Stable Strategies and Game Dynamics," *Mathematical Biosciences* 40 (1978):145–156.

Uzawa, Hirofumi, "Walras' Tâtonnement in the Theory of Exchange," *Review of Economic Studies* 27 (1959):182–194.

——— , "The Stability of Dynamic Processes," *Econometrica* 29,4 (October 1961):617–631.

——— , "On the Stability of Edgeworth's Barter Process," *International Economic Review* 3,2 (May 1962):218–231.

Wald, Abraham, "On Some Systems of Equations of Mathematical Economics," *Econometrica* 19,4 (October 1951[1936]):368–403.

Walras, Leon, *Elements of Pure Economics* (London: George Allen and Unwin, 1874).

Weibull, Jörgen W., *Evolutionary Game Theory* (Cambridge, MA: MIT Press, 1995).

c\papers\Stability of GE\EvStabilityOfGE.tex\\ September 24, 2013