THE TOPOLOGICAL VERTEX

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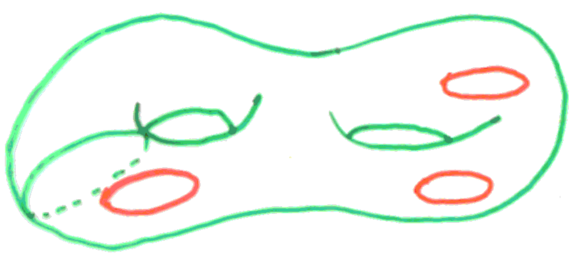
hep-th/0305132

TOPOLOGICAL A-MODEL:

COUNTING OF HOLOMORPHIC MAPS FROM RIEMANN SURFACES INTO CALABI YAU X, POSSIBLY WITH BOUNDARIES



$$\Sigma_{g,h} \longrightarrow (X, L)$$

ON LAGRANGIAN L IN X

THE CENTRAL PROBLEM IS
HOW TO EVACUATE THE MAPS

1. LOCALISATION
2. MIRROR SYMMETRY
   - MAPS THE PROBLEM TO A SIMPLER ONE IN THE TOPOLOGICAL B-MODEL.
3. LARGE N DUALITY

ALL OF THESE HAVE LIMITATIONS

HERE: PROVIDE A NATURAL, EFFICIENT WAY OF CALCULATING A-MODEL AMPLITUDES

RECALL THE GEOMETRY OF TORIC THREEFOLDS:

X IS A $T^2 \times \mathbb{R}$ FIBRATION OVER $\mathbb{R}^3$,

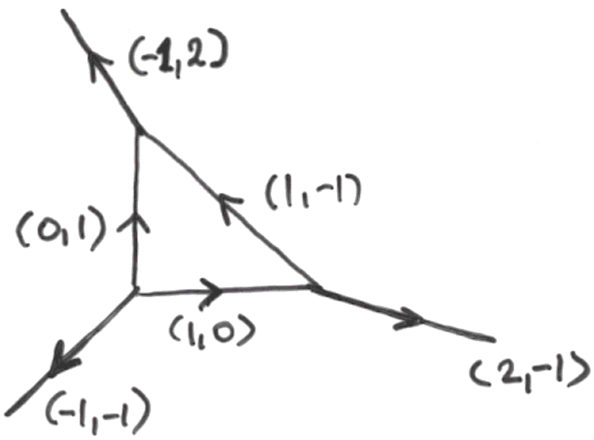
WHERE (p,q) 1-CYCLES OF T^2

DEGENERATE OVER LINES IN BASE $\mathbb{R}^3$

$\Rightarrow$ GEOMETRY OF X IS ENCODED IN

1-DIM. PLANAR GRAPH Γ:

$\mathcal{O}(-3) \to \mathbb{P}^2$

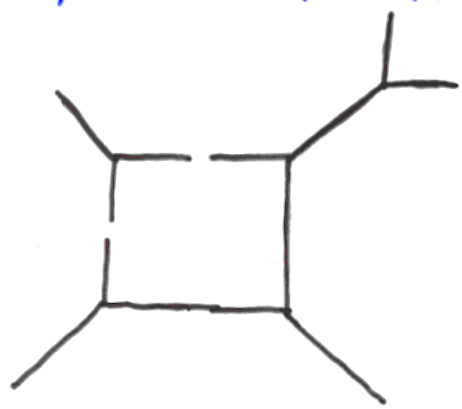


WHERE CALABI-YAU CONDITION $\Rightarrow$

$$\sum_i (p_i, q_i) = 0$$

AT EACH VERTEX.

NOTE THAT ALL VERTICES OF A GENERIC THREEFOLD ARE TRIVALENT:



WHERE THE 3-VALENT VERTEX ITSELF:

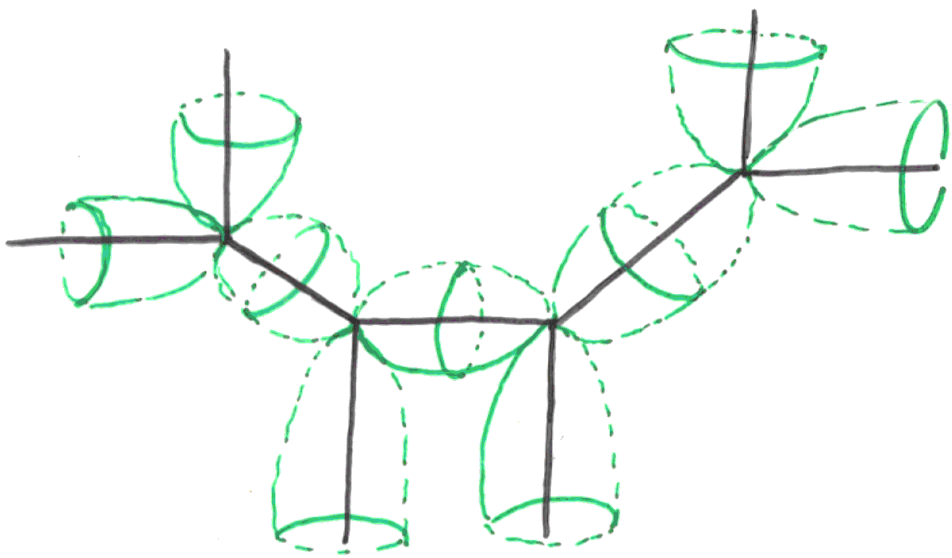
IS THE GRAPH OF $\mathbb{C}^3$, VIEWED AS TORIC MANIFOLD

$\rightarrow$ TORIC $\mathbb{C}$Y3 IS GLUED FROM $\mathbb{C}^3$ PATCHES.

THE GRAPH + FIBER OVER IT

$\Rightarrow$ CHAIN OF HOLOMORPHIC $\mathbb{P}^1$'s

INVARIANT UNDER T^2 ACTION

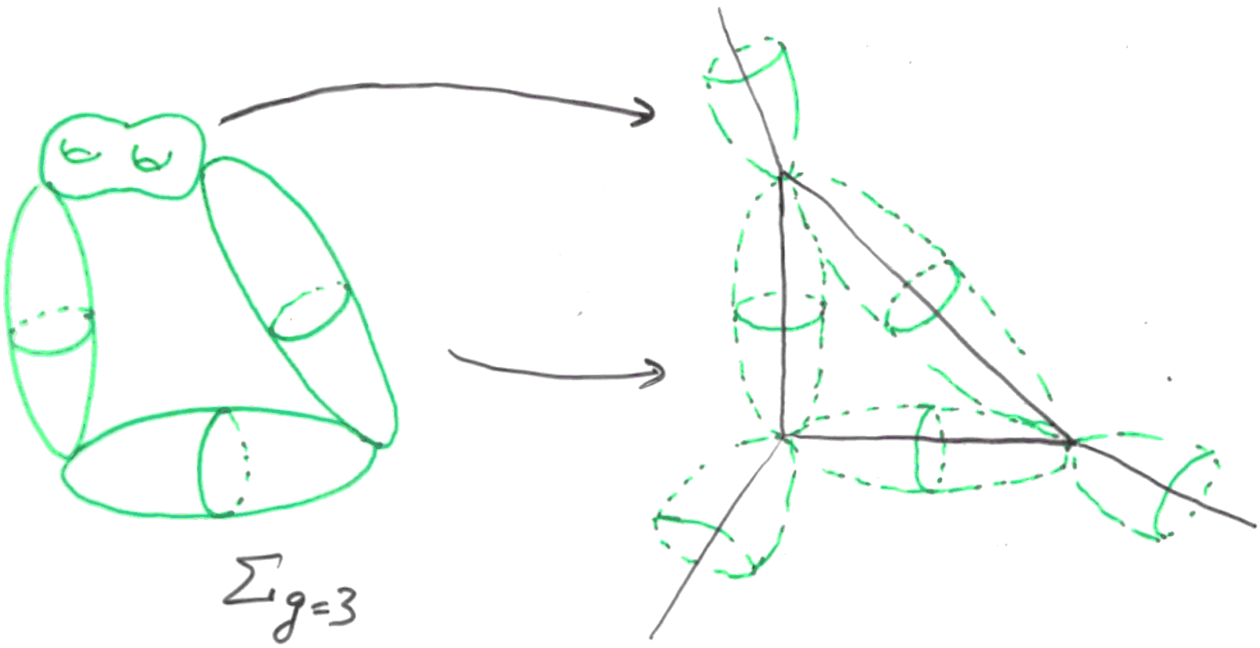


LOCALIZATION:

COUNTING OF HOLOMORPHIC MAPS

$$\Sigma_g \to X$$

REDUCES TO COUNTING OF MAPS INVARIANT UNDER TORUS ACTION,



GENUS BY GENUS

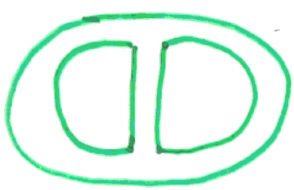
LARGE N CHERN-SIMONS DUALITIES
PROVIDE AN EFFICIENT WAY TO OBTAIN
ALL-GENUS AMPLITUDES FOR CERTAIN
CLASSES OF CY 3 MANIFOLDS
TORIC

A-MODEL OF N D-BRANES ON $S^3 \in T^*S^3$

IS $U(N)$ CHERN-SIMONS THEORY ON S^3

$$Z = \int \mathcal{D}A \; e^{k \int_{S^3} \mathrm{Tr}\, A dA + A^3}$$

WHERE RIBBON GRAPHS OF CHERN-SIMONS ARE A-MODEL OPEN STRING DIAGRAMS



$$Z = e^{\sum_{g,h} F_{g,h} N^h \lambda^{2g-2+h}}$$

$$\lambda = \frac{2\pi i}{k+N}$$

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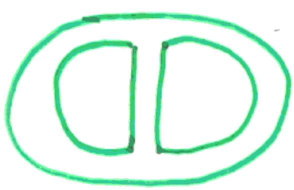
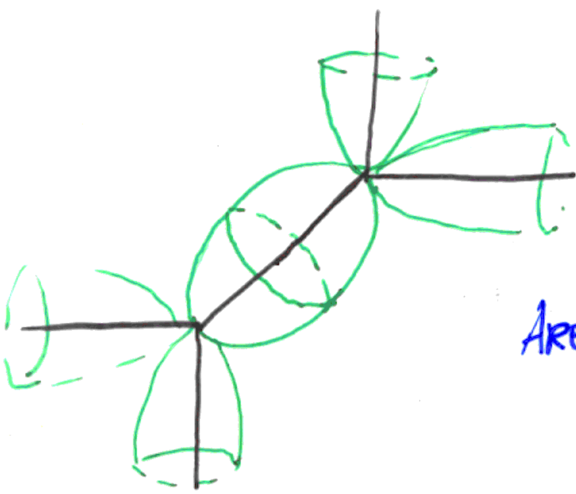


$$Z = e^{\sum_{g,h} F_{g,h} N^h \lambda^{2g-2+h}}$$

$$\lambda = \frac{2\pi i}{k+N}$$

THIS THEORY IS THE SAME AS THE A-MODEL ON $\mathcal{O}(-1)+\mathcal{O}(-1) \to \mathbb{P}^1$



AREA $(\mathbb{P}^1) = t$

$$Z_{closed} = \exp\left[\sum_g F_g^{closed}(t)\,\lambda^{2g-2}\right]$$

WHERE WE SUM OVER ALL HOLES IN OPEN STRING DIAGRAMS

$$F_{cs}(\lambda, N) = \sum_g \left[\sum_h F_{g,h}^{open}(\lambda N)^h \right] \lambda^{2g-2}$$

TO OBTAIN CLOSED STRING DIAGRAMS:

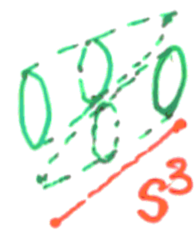
$$= \sum_g F_g^{closed}(\lambda N)\, \lambda^{2g-2}$$

$\Rightarrow$ SIZE OF THE $\mathbb{P}^1$ IS

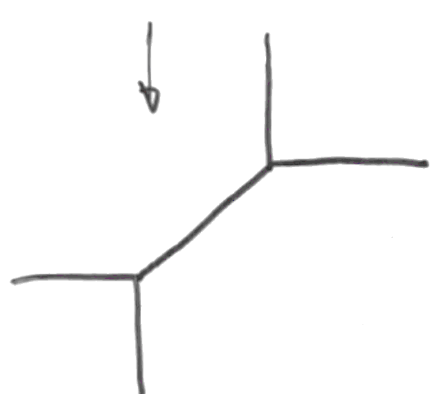
$$t = \lambda N$$

THE TWO CORRESPONDING GEOMETRIES ARE RELATED BY A GEOMETRIC TRANSITION:

T^*S^3:



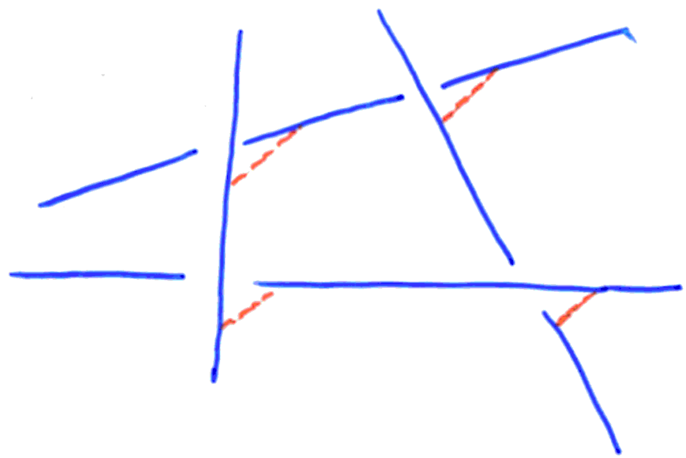
$\mathcal{O}(-1)\oplus\mathcal{O}(-1)$
$\downarrow$
$\mathbb{P}^1$:



THIS EASILY GENERALIZES

TO GEOMETRIES

WITH MANY S^3's

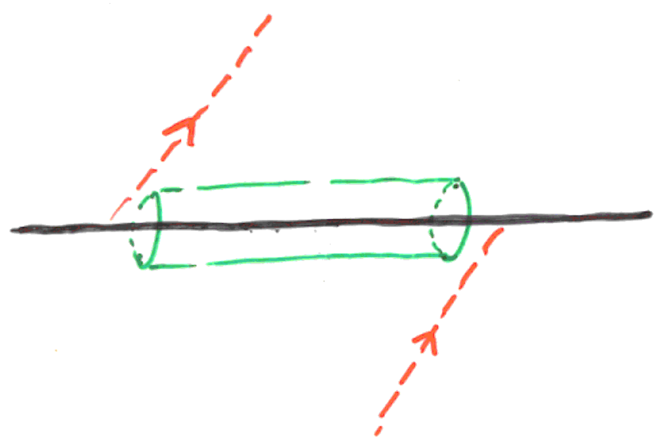


THE NEW INGREDIENT HERE ARE

KNOT INVARIANTS:

THERE ARE HOLOMORPHIC ANNULI

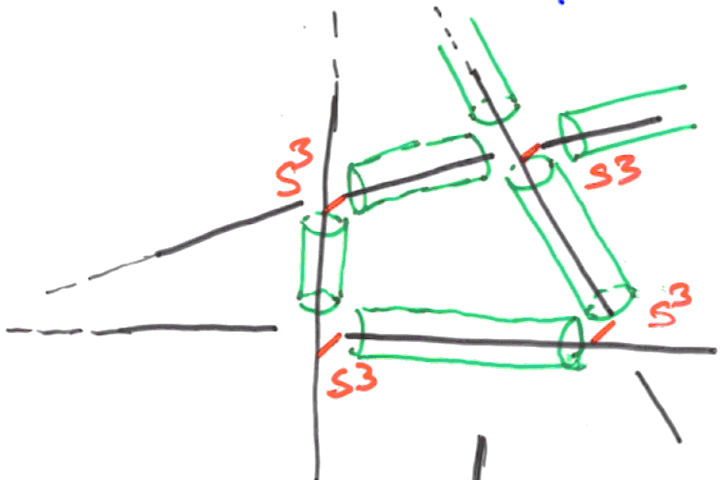
CONNECTING THE D-BRANES:

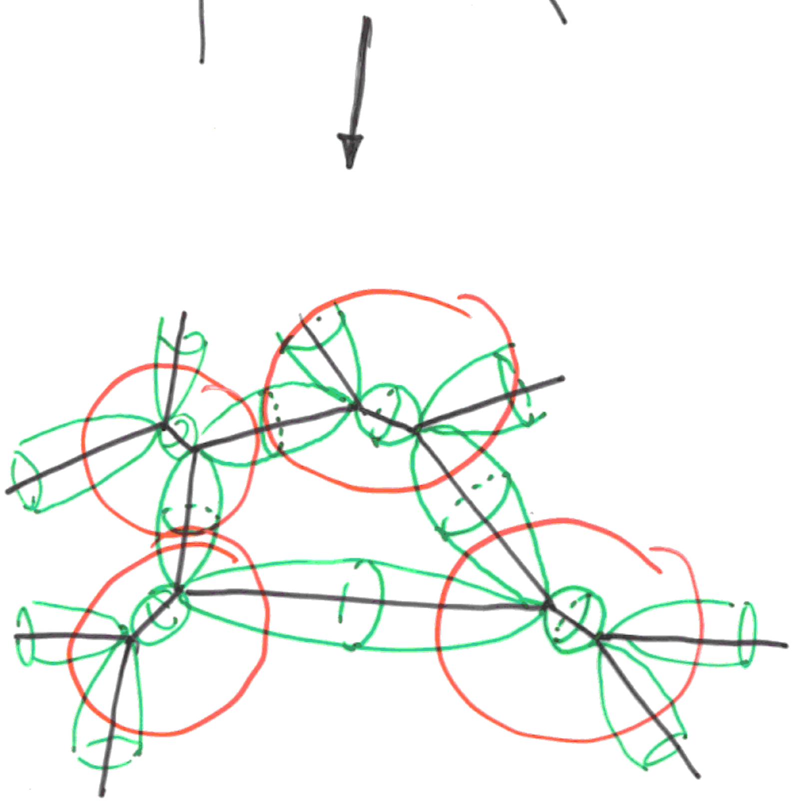


WHOSE BOUNDARIES FORM KNOTS

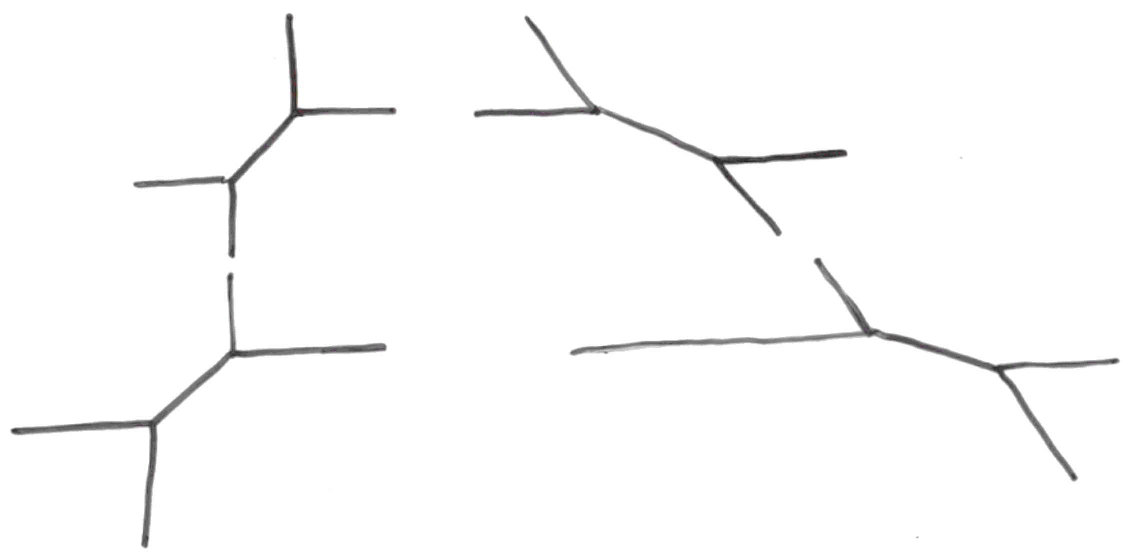
+ COUPLE TO $V = Pe^{\oint A}$

IN THE LARGE N LIMIT THE CHERN-SIMONS NODES ARE REPLACED BY $\mathbb{P}^1$'s AND BOUNDARIES OF ANNULI DISSAPEAR.





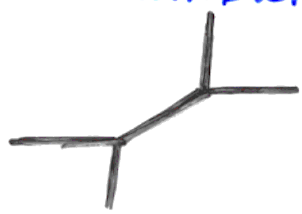
THE LARGE N DUALITY IS, ROUGHLY,
CONSTRUCTING GEOMETRY OUT OF
QUADRI-VALENT VERTICES



WHERE EACH VERTEX IS DESCRIBED,
TO ALL GENERA, BY CHERN-SIMONS
ON S^3 + SOME KNOTS IN THE
3-MANIFOLD.

IF THE GOAL IS COMPUTING CLOSED TOPOLOGICAL A-MODEL AMPLITUDES, THIS IS NOT PARTICULARLY NATURAL.

TORIC GEOMETRIES ARE GENERICALLY NOT BUILT OUT OF QUADRIVALENT VERTICES

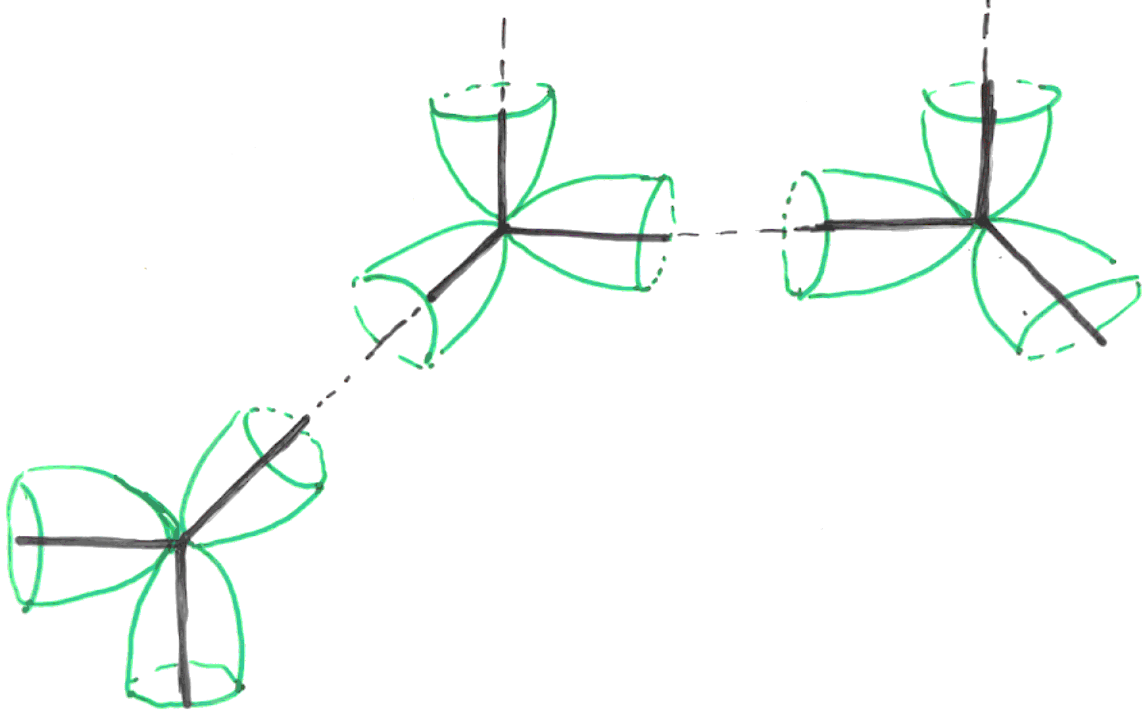


BUT OF TRIVALENT ONES:

MOREOVER, THE TRIVALENT VERTEX GIVES RISE TO THE NON-TRIVIAL PART OF THE A-MODEL AMPLITUDE.

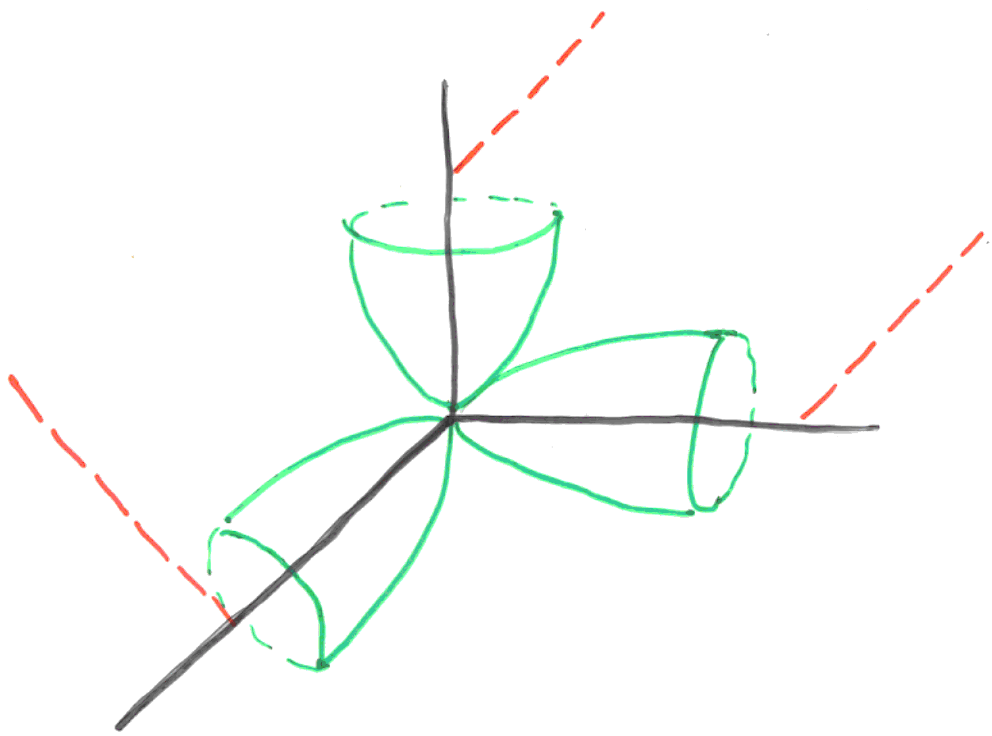
THE MORE NATURAL APPROACH WOULD BE TO CHOP UP THE GEOMETRY INTO 3-VALENT VERTICES CORRESPONDING TO $\mathbb{C}^3/\mathbb{Z}$,

WHILE CHOPPING UP THE MAPS AS WELL



THE FIRST IS SIMPLE, THIS IS HOW THE TORIC MANIFOLDS ARE CONSTRUCTED.

THE SECOND CAN BE DONE BY PLACING D-BRANES.



THE A-MODEL AMPLITUDES CORRESPONDING TO THE $\mathbb{C}^3$ WITH D-BRANES:

$$Z = \sum_{\vec{k}_i} C_{\vec{k}_1, \vec{k}_2, \vec{k}_3}(\lambda) \, \mathrm{Tr}_{\vec{k}_1} V_{(1)} \, \mathrm{Tr}_{\vec{k}_2} V_{(2)} \, \mathrm{Tr}_{\vec{k}_3} V_{(3)}$$

ALL GENUS PARTITION FN

CORRESPONDING TO MAPS WITH

$k^j_{(1)}$ BOUNDARIES

OF WINDING #j ON THE 1ST D-BRANE,

$$\mathrm{Tr}_{\vec{k}} V = \prod_j \left(\mathrm{Tr}\, V^j\right)^{k_j}$$

$$V = P e^{\oint A}$$

ONE HAS TO BE MORE PRECISE HERE AS THE A-MODEL AMPLITUDES ARE AMBIGUOUS.

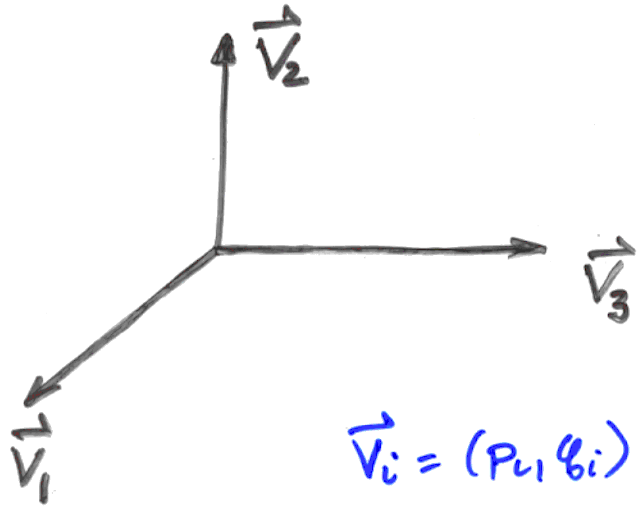
FOR EVERY STACK OF D-BRANES THERE IS AN INTEGER AMBIGUITY RELATED TO THE CHOICE OF THE FLAT COORDINATE (AND FRAMING OF KNOTS IN CHERN-SIMONS)

$\Rightarrow$ THE RELEVANT OBJECT IS THE FRAMED VERTEX

THE WAY TO PARAMETERIZE THIS IS AS FOLLOWS

CONSIDER THE VERTEX WITH

$\vec{v}_i \wedge \vec{v}_{i+1} = 1$



$\vec{v}_i = (p_i, q_i)$

$\Rightarrow$ FRAMING IS A CHOICE OF 3 2-VECTORS $\vec{f}_i$:

$$\vec{f}_i \wedge \vec{v}_i = 1$$

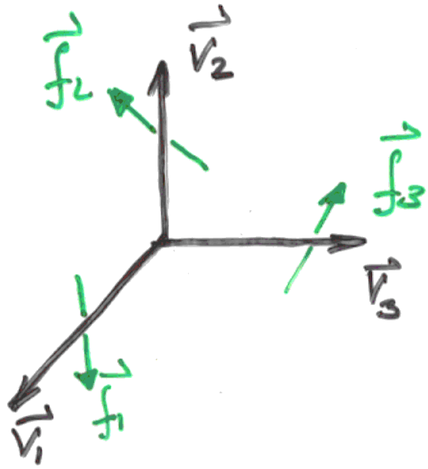
$$\Rightarrow \quad \vec{f}_i' - \vec{f}_i = n \vec{v}_i \qquad n \in \mathbb{Z}$$

THE THEORY HAS A SYMMETRY

CORRESPONDING TO $SL(2,\mathbb{Z})$

TRANSFORMATIONS OF THE T^2 FIBER

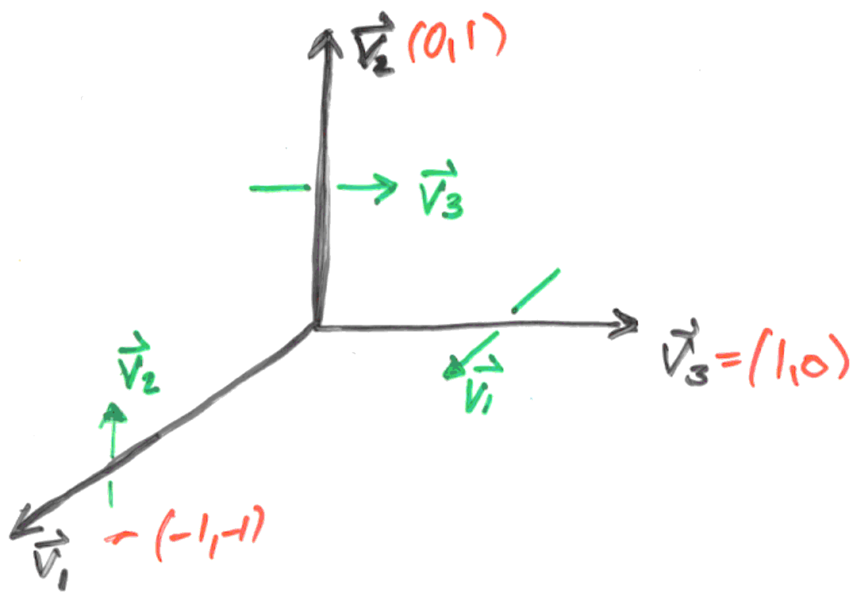
WHICH ACTS ON



BY

$$\vec{V}_i \to U\,\vec{V}_i \qquad U \in SL(2,\mathbb{Z})$$
$$\vec{f}_i \to U\,\vec{f}_i$$

THERE IS A PARTICULARLY NICE, $\mathbb{Z}_3 \in SL(2,\mathbb{Z})$ INVARIANT, CHOICE OF FRAMING



SINCE $\vec{V}_i \wedge \vec{V}_{i+1} = 1$

MOREOVER, IF WE KNOW THE VERTEX AMPLITUDE IN ANY ONE FRAMING,

$$C^{\vec{f}_1, \vec{f}_2, \vec{f}_3}$$

THE VERTEX AMPLITUDE IN ANY OTHER FRAMING

$$C^{\vec{f}_1{}', \vec{f}_2{}', \vec{f}_3{}'}$$

IS SIMPLY RELATED TO IT.

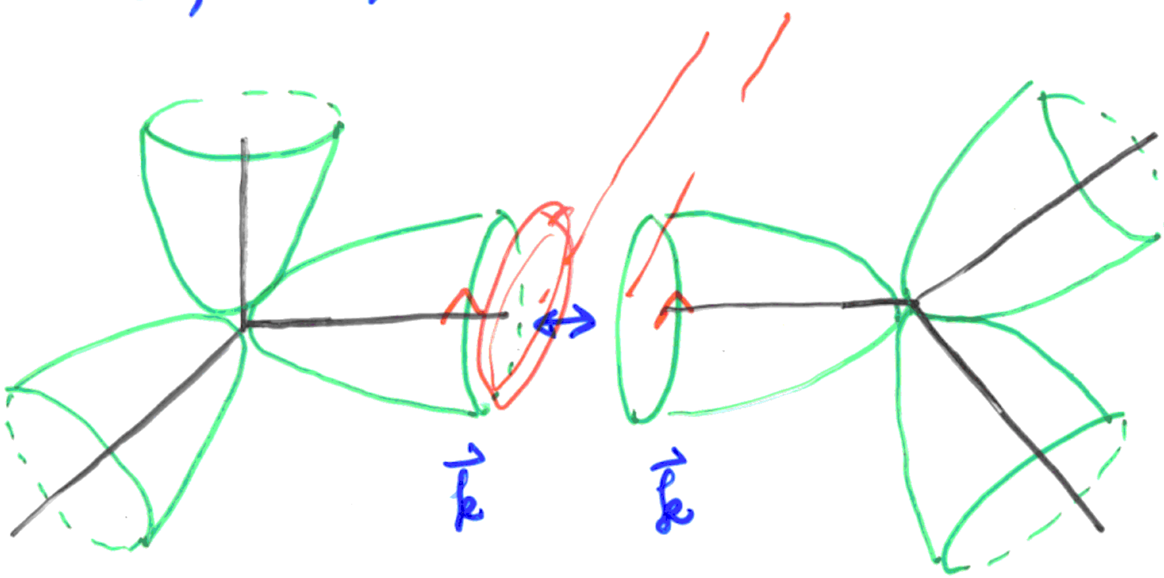
SUPPOSE, FOR A MOMENT,

WE KNEW THE

EXACT

VERTEX AMPLITUDE

THEN, BY GLUING:



$$\sum_{\vec{k}} C^{(\ldots,\ldots,\vec{f})}_{(\ldots,\ldots,\vec{k})} \frac{(-1)^h}{Z(\vec{k})} e^{-\omega(\vec{k})r} C^{(\vec{f},\ldots,\ldots)}_{(\vec{k},\ldots,\ldots)}$$

$r \to$ SIZE OF $\mathbb{P}^1$

$\omega(\vec{k})$ = TOTAL WINDING # $= \sum_j j k_j$

$Z(\vec{k})$ = NORMALIZATION FACTOR

$\sim \prod_j j^{k_j} k_j!$

h = # OF HOLES

THIS IS PARTICULARCY SIMPLE IN "REPRESENTATION" BASIS

$$Z = \sum_{R_i} C^{\vec{f}_1, \vec{f}_2, \vec{f}_3}_{R_1, R_2, R_3} \, Tr_{R_1} V^{(1)} Tr_{R_2} V^{(2)} Tr_{R_3} V^{(3)}$$

$$\prod_j (Tr V^j)^{k_j} = Tr_{\vec{k}} V = \sum_R X_R(C(\vec{k})) \, Tr_R V$$

$$C^{\vec{f}_1', \ldots}_{R_1, \ldots} = q^{n \frac{K(R_1)}{2}} \, C^{\vec{f}_1, \ldots}_{R_1, \ldots}$$

$$\vec{f}_1' = \vec{f}_1 + n \vec{V}_1 \qquad q = e^{\Lambda}$$

$$K(R) = \sum_i l_i (l_i - 2i + 1)$$

l_i = # OF BOXES IN i-th ROW

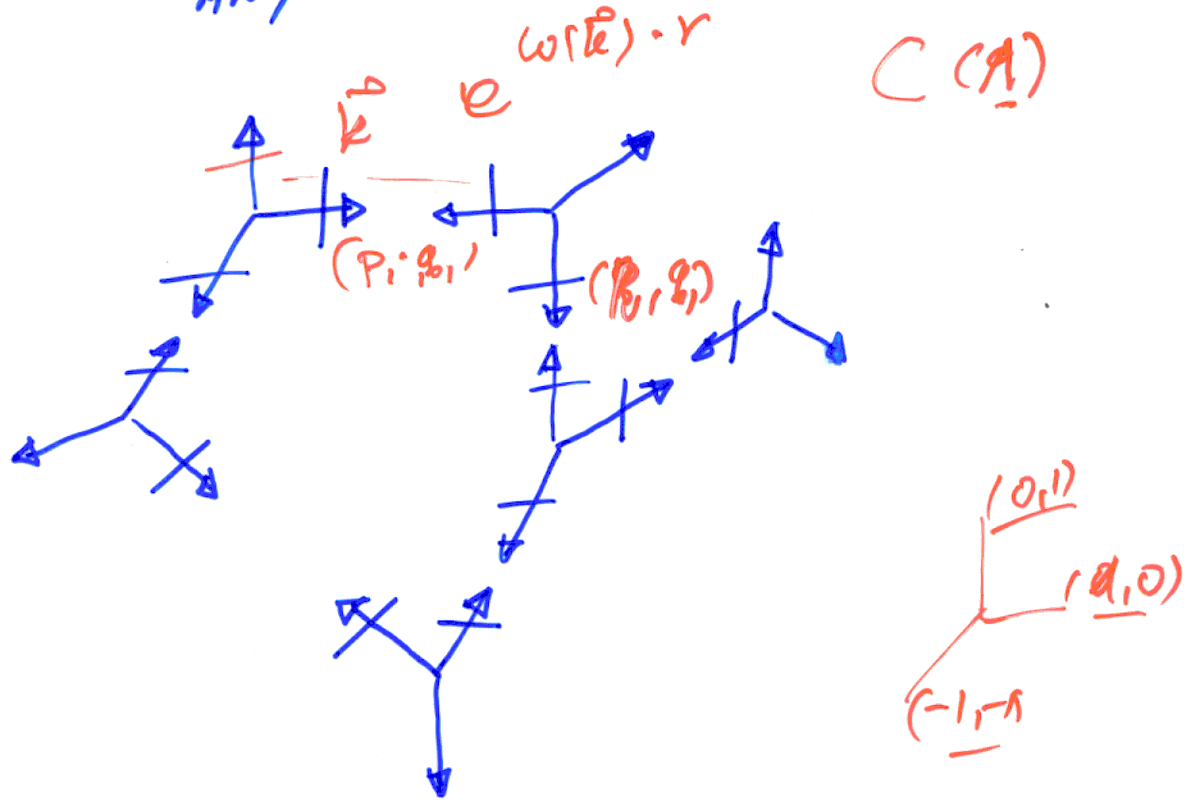
FROM THE VERTEX AMPLITUDE,

WE CAN GET EXACT CLOSED

OR OPEN TOPOLOGICAL

A-MODEL AMPLITUDES FOR

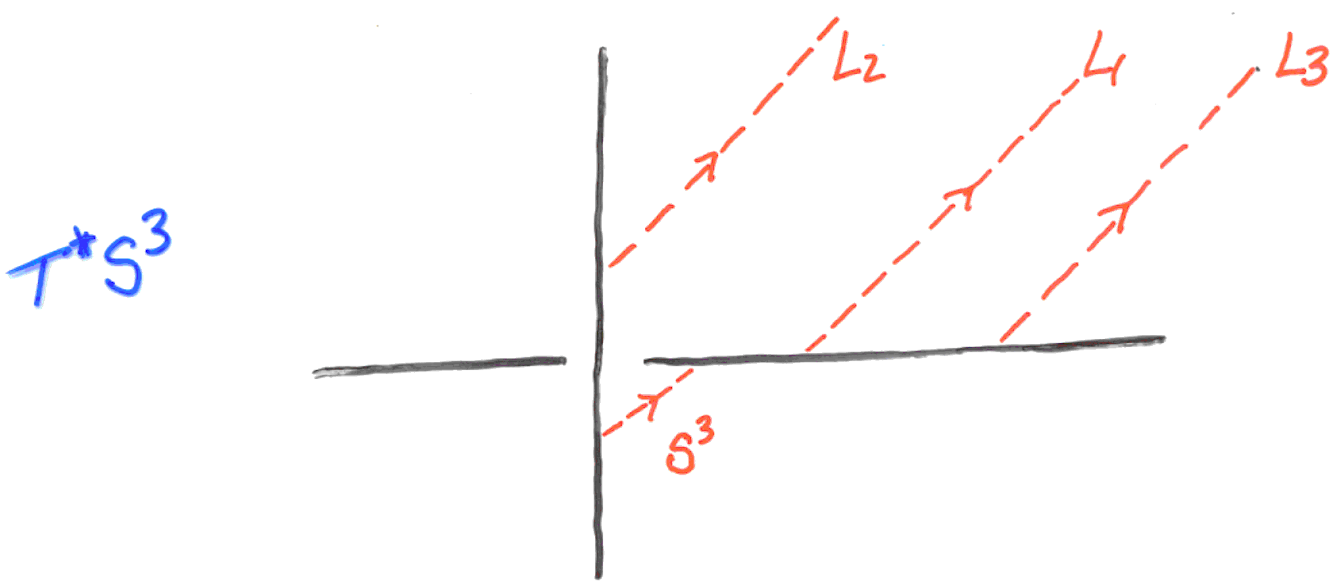
ANY TORIC CALABI YAU!



AT PRESENT, THE ONLY AVAILABLE DERIVATON OF THE VERTEX AMPLITUDE IS INDIRECT, INVOLVING LARGE N CHERN-SIMONS DUALITY.

HOWEVER, IT PROVIDES AN ALL-GENUS, EXACT ANSWER IN CLOSED FORM.

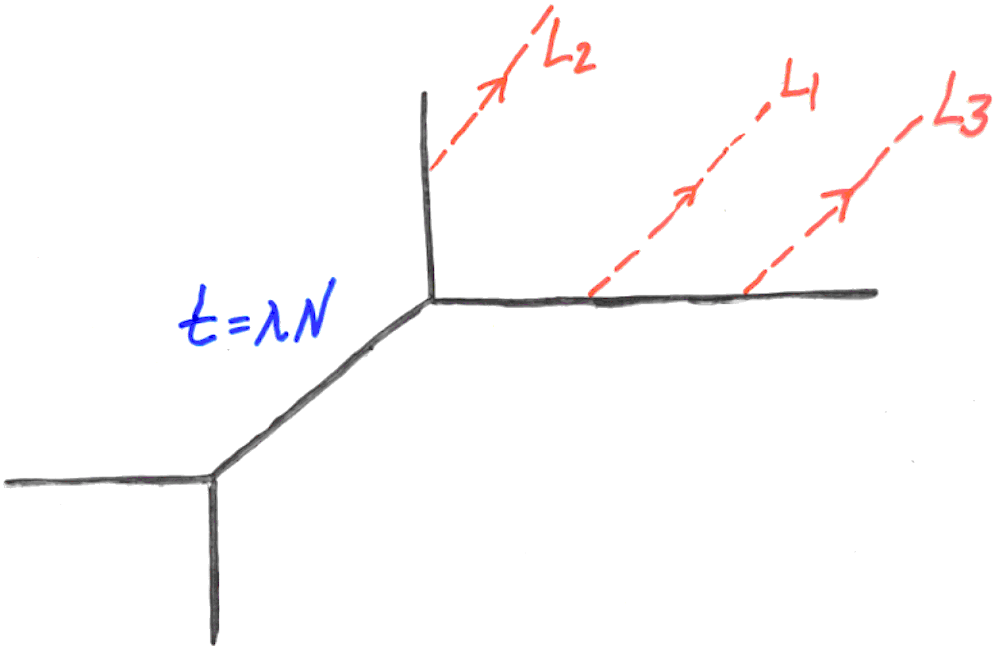
CONSIDER T^*S^3 WITH N D-BRANES ON S^3, AND 3 ADDITIONAL STACKS OF D-BRANE "SOURCES" ON WHICH HOLOMORPHIC MAPS CAN HAVE BOUNDARIES:



THE LARGE N DUAL OF THIS

IS $\mathcal{O}(-1)+\mathcal{O}(-1)\to\mathbb{P}^1$ WITH D-BRANES

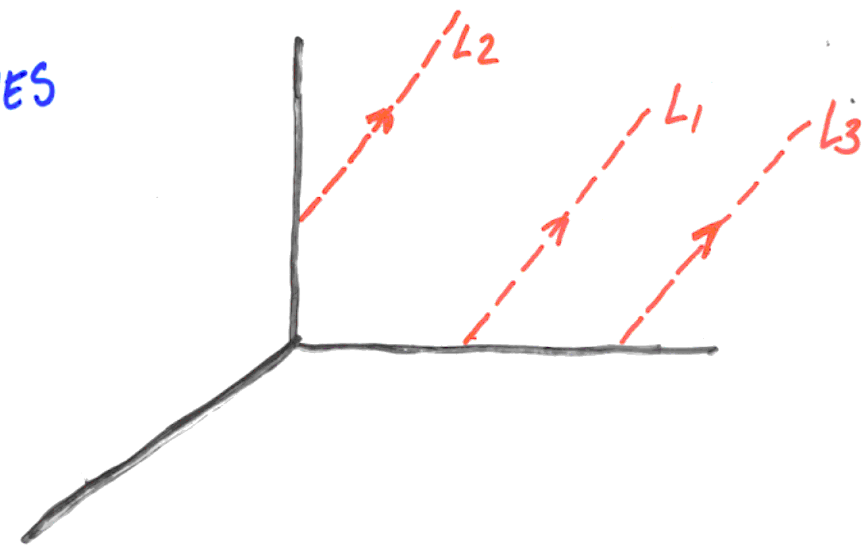
$\mathcal{O}(-1)\oplus\mathcal{O}(-1)$
$\downarrow$
$\mathbb{P}^1$:



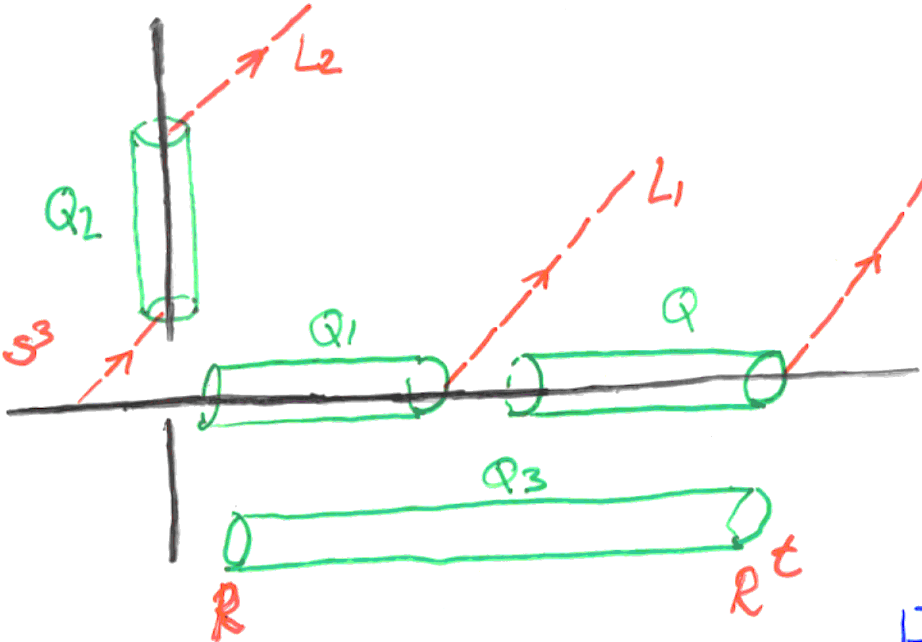
IN THE $t\to\infty$ LIMIT, GET $\mathbb{C}^3$

WITH BRANES

$\mathbb{C}^3$:



THIS IS SIMPLE TO EVALUATE, AS A LIMIT OF THE CHERN-SIMONS AMPLITUDE



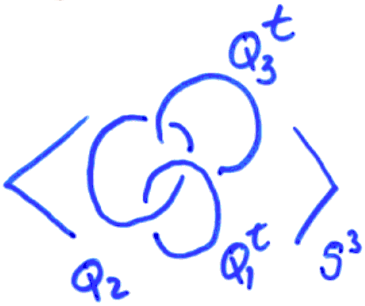
$\Rightarrow$ EACH ANNULUS

$$\sim \sum_R \mathrm{Tr}_R V^a \, \mathrm{Tr}_{R^t} V^b \, (-1)^{\ell(R)}$$

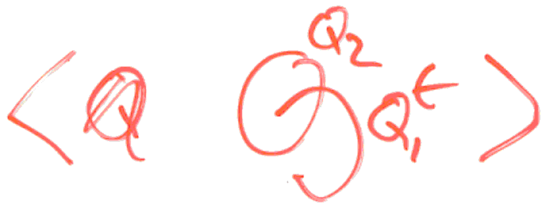
$$\Rightarrow Z \sim \sum_{Q_i, Q} \underline{\langle \mathrm{Tr}_{Q_2} V \, \mathrm{Tr}_{Q_1^t \otimes Q_2^t} V \rangle}_{S^3} \, \mathrm{Tr}_{Q_1} V^{(1)} \, \mathrm{Tr}_{Q_2} V^{(2)} \, \mathrm{Tr}_{Q_3} V^{(3)}$$

$$\times \, \mathrm{Tr}_{Q^t} V^{-(1)} \, \mathrm{Tr}_Q V^{(2)}$$

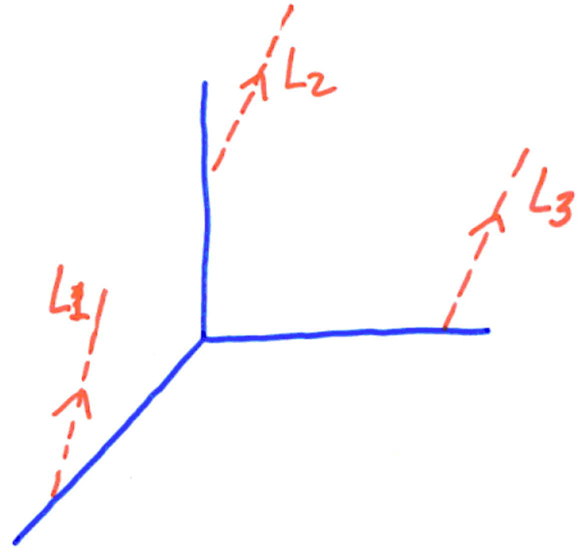
$N \to \infty$



$$= \frac{W_{Q_2 Q_1^t} \, W_{Q_2 Q_3^t}}{W_{Q_2 0}} (N, \lambda)$$

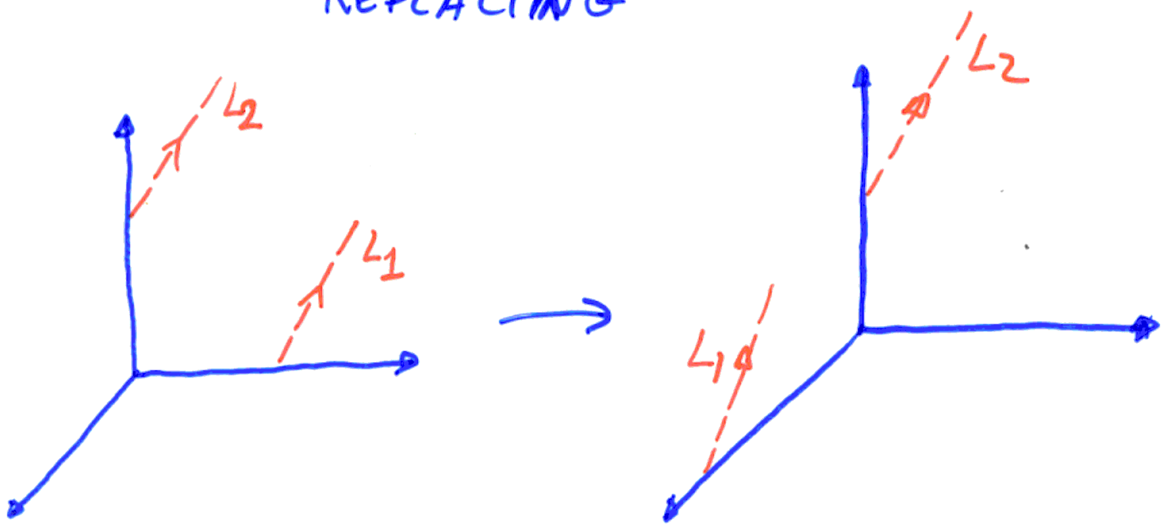


THE AMPLITUDE NEEDED:



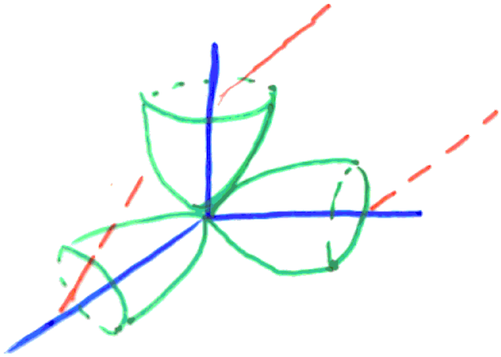
IS OBTAINED FROM THIS BY

REPLACING



$$W_{Q_2 Q_1^t}\, \mathrm{Tr}_{Q_1} V_1\, \mathrm{Tr}_{Q_2} V_2 \longrightarrow q^{\frac{\kappa(Q_2)}{2}}\, W_{Q_2^t Q_1}\, \mathrm{Tr}_{Q_1} V_1^{-1}\, \mathrm{Tr}_{Q_2} V_2$$

$$C^{0,0,-1}_{0 Q_2 Q_1} \longrightarrow C^{0,0,0}_{Q_1 Q_2 0}$$

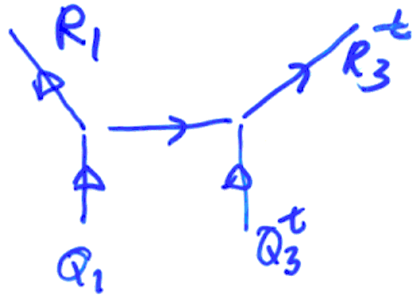


THE FINAL ANSWER IS

$$Z = \sum_{R_i} C_{R_1,R_2,R_3} \, Tr_{R_1} V^{(1)} \, Tr_{R_2} V^{(2)} \, Tr_{R_3} V^{(3)}$$

$$C_{R_1 R_2 R_3} = \sum_{Q_1, Q_3} N^{R_1 R_3^t}_{Q_1 Q_3^t} \, \frac{W_{R_2^t Q_1} \, W_{R_2 Q_3^t}}{W_{R_2 0}} \, q^{\frac{\kappa_{R_2} + \kappa_{R_3}}{2}}$$

$$N^{R_1 R_3^t}_{Q_1 Q_3^t} = \sum_{Q}$$



SOME VALUES OF THE VERTEX:

$$C_{\Box\Box\Box} = \frac{q^4 - q^3 + q^2 - q + 1}{q^{1/2}(q-1)^3},$$

$$C_{\genfrac{}{}{0pt}{}{\Box}{\Box}\genfrac{}{}{0pt}{}{\Box}{\Box}\Box} = \frac{q^8 - q^7 + q^5 - q^4 + q^2 - q + 1}{q^{3/2}(q+1)^2(q-1)^5},$$

$$C_{\substack{\Box\\\Box\\\Box}\Box\Box} = \frac{q^8 - q^7 + q^4 - q + 1}{(q+1)^2(1+q+q^2)(q-1)^5}, \quad \text{ETC.}$$

$$q = e^{\lambda}$$