Geometric transitions

$$X_\mu \rightsquigarrow X_0 \longleftarrow X$$

X_μ : family of (complex strs. on) CY's
X_0 : a singular CY in the family
X : its small resolution, still CY, contains some exceptional 2 cycles $\mathbb{P}^1 \approx S^2$.

Large N duality :

closed strings on X_μ $\Longleftrightarrow$ open strings on X

Dualities:

$$
\begin{tikzcd}
\text{IIB closed } X_\mu \arrow[r, squiggly] \arrow[d, leftrightarrow, "\text{MS}"'] \arrow[rr, bend left, leftrightarrow, "\text{large } N"] & X_0 & X \text{ IIB open} \arrow[l] \arrow[d, leftrightarrow, "\text{OMS}"] \\
\text{IIA closed } Y \arrow[r] \arrow[rr, bend right, leftrightarrow, "\text{large } N"'] & Y_0 & Y_\nu \text{ IIA open} \arrow[l, squiggly]
\end{tikzcd}
$$

LHS: Mirror symmetry

IIB on X_μ period integrals on cycles in X_μ	$\Longleftrightarrow$	IIA on Y closed string GW invariants on Y

RHS: Open Mirror Symmetry (MS w/ branes)

integrals on chains in X bounding the vanishing cycles	$\Longleftrightarrow$	Open string GW invariants count holo maps from R.S. w/ boundary to Y, boundary $\to L_\nu$

Mirror of the exceptional 2-cycles $\mathbb{P}^1 \approx S^2 \subset X$ are SLAG vanishing 3-cycles $L_\nu \subset Y_\nu$.

Large N duality interchanges left & right. Sometimes, allows "calculation" of open GW invariants.

[D+V] relate B-model topological strings on local CYs, via large N duality, to <u>matrix models.</u>

Typical picture:

$$\begin{array}{ccc} X_a = X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \mathbb{C}_x & \xrightarrow{W'} & \mathbb{C}_z \end{array}$$

$\boxed{Z \subset \mathbb{C}^4 : -y^2 + uv + z^2 = 0}$

$\boxed{X \subset \mathbb{C}^4 : -y^2 + uv + W'(x)^2 = 0}$

$W = W_a(x) = \sum_{i=0}^{n+1} a_i x^i =$ superpotential

X : CY3, singular at n points $\| \; u = v = y = 0, \; W'(x) = 0.$

$\Omega_X = \frac{du\,dx\,dy}{u} = \ldots$ holo 3 form.

When $a = 0$, X is singular along a curve
$$C \simeq \mathbb{C}_x : \{u = v = y = 0\}.$$

When $a \neq 0$, have transversally holomorphic family of (non-holomorphic) S^2's

Integrate Ω_X on these S^2's $\Rightarrow$ holo 1-form ω on $\mathbb{C}_x$.

$W(x) := \int_{x_0}^{x} \omega$: <u>classical superpotential.</u>

[DV] picture:

Combine superpotential deformations, X_a with smoothing deformations, X_μ:

$$\boxed{X_{a,\mu} : -y^2 + uv + W'(x)^2 + f_\mu(x) = 0}$$

$f_\mu(x) = \Sigma_{i=0}^{n-1} \mu_i x^i$

From matrix models, they get a quantized superpotential $W = W_{a,\mu}(\tilde{x})$

$\tilde{x}$: coordinate on the hyperelliptic curve

$$\boxed{\tilde{C}_{a,\mu} : y^2 = W'(x)^2 + f_\mu(x)}$$

Coefficients of W w.r.t. special coordinates $\Rightarrow$ open string GW invariants on mirror Y_v.

Our goal:

* understand [DV] geometrically
(no matrix model)

* extend to include compact CYs
(as well as the local ones)

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A simple <u>compact</u> example: (cf. [KMP])

$X_{a,\mu} = Q \cap R$ $Q =$ quadric $\subset \mathbb{P}^5$, $R =$ quartic $\subset \mathbb{P}^5$

$a = \mu = 0 \Leftrightarrow \operatorname{rank}(Q) = 3 \Leftrightarrow \operatorname{Sing}(Q) = \mathbb{P}^2 \subset \mathbb{P}^5$

$\Rightarrow \operatorname{Sing}(X_{a,\mu}) =$ plane quartic $g=3$ $C \subset \mathbb{P}^2$ $\Rightarrow X$ has 1-param family of $\mathbb{P}^1$'s

$\mu = 0 \Leftrightarrow \operatorname{rank}(Q) \leq 4 \Leftrightarrow \operatorname{Sing}(Q) \supset \mathbb{P}^1 \subset \mathbb{P}^5$

$\Rightarrow \operatorname{Sing}(X_{a,\mu}) \supset 4$ points $\Leftrightarrow a \in H^0(C, K_C)$.

$\operatorname{rank}(Q) = 5, 6 \Rightarrow X$ (generically) n.s.

rank(Q)	# param's Q	$h^{2,1}$
3	14	107
4	17	110
6	20	113

$$\underset{107}{S} \subset \underset{110}{\mathcal{M}} \subset \underset{113}{\mathcal{L}}$$

a-parameters $= 110 - 107 = 17 - 14 = 3$

μ-parameters $= 113 - 110 = 20 - 17 = 3$.

Large N duality $\Rightarrow$ expansion in μ (actually, in special coordinates equiv't to μ)

0-th order:

$$\boxed{\lim_{\mu\to 0} \int_{\Gamma_\mu} \Omega_{X_\mu} = \int_{\Gamma} \Omega_X}$$

Γ: 3-chain in X, $\partial\Gamma = \Sigma$ (exceptional $\mathbb{P}^1$'s)
Γ_0: its image in X_0, a 3-cycle.
Γ_μ: its deformation to 3-cycle in X_μ.

<u>Natural interpretation of W:</u>

it is a "normal function", i.e. a section of a family of <u>intermediate</u> <u>Jacobians</u> $J(X_\mu)$.

Want: behavior of $J(X_\mu)$ near the transition, $\mu \to 0$.

a la [DV], we'll study this for $J(X_{a,\mu})$ near $a = \mu = 0$.

<u>Torus</u>: $T = \mathbb{R}^n/\Lambda, \quad \Lambda \simeq \mathbb{Z}^n$
$ \simeq (S^1)^n$

<u>Integrable system</u>:

$$\begin{tikzcd} T_b \arrow[r, hook] \arrow[d] & X \arrow[d, "\pi"] \\ b \arrow[r, phantom, "\in"] & B \end{tikzcd}$$

(X, ω) <u>symplectic</u>

π <u>Lagrangian</u>: $\omega|_{T_b} = 0$

$\dim T = \dim B = \frac{1}{2} \dim X$

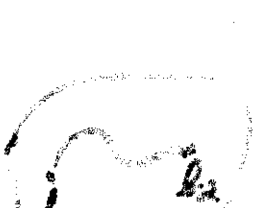
$\Rightarrow$ action-angle coordinates on X:

"<u>angles</u>" on T

"action" on B: flat structure / local trivialization

$T_B \overset{\omega}{\underset{\approx}{\longrightarrow}} T^*(X/B)$, fiber at $b \in B$ is $H^1(T_b, \mathbb{R})$

a path in B from b_1 to b_2 $\Rightarrow$



$H^1(T_{b_1}, \mathbb{R}) \longrightarrow H^1(T_{b_2}, \mathbb{R})$.

A function H ("<u>Hamiltonian</u>") on $B \Rightarrow$ straight line flows on tori T_b.

complex torus: $T = \mathbb{C}^n/\Lambda$, $\Lambda \approx \mathbb{Z}^{2n}$

$\approx (S^1)^{2n}$ with complex structure

most classical integrable systems a[re]

$\boxed{\begin{array}{l}\left.\begin{array}{l}\text{analytically}\\ \text{algebraically}\end{array}\right\} \text{integrable systems.}\end{array}}$

$T \to X$, everything is $\left\{\begin{array}{l}\text{analytic}\\ \text{algebraic}\end{array}\right.$

$\downarrow$

B (σ is holomorphic (2,0) form)

new features:

* Trivialization of B is <u>non-unique</u>
* Complex tori have <u>moduli</u>.

$\alpha_1, \ldots, \alpha_n,$
$\beta_1, \ldots, \beta_n$ symplectic basis of $H_1(T, \mathbb{Z})$

$dz_1, \ldots, dz_n$ holomorphic 1-forms,

$$\int_{\alpha_i} dz_j = \delta_{ij}$$

$\Rightarrow$ $\mathbb{P} = \left(\int_{\beta_i} dz_j\right)$ = <u>period matrix</u>.

T is <u>polarized</u> if $\mathbb{P}$ is symmetric

" " <u>algebraic</u> " " " " and $\mathbb{P} > 0$.

Example 1:

X compact Riemann surface

Hodge: $H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$

$\Rightarrow$ Jacobian $J(X) = H^1(X, \mathbb{C}) / (H^{1,0} + H^1(X, \mathbb{Z}))$

is an algebraic torus.

Example 2:

X compact Kahler 3fold

Hodge: $H^3(X, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$

Intermediate Jacobian:

$$J(X) = H^3(X, \mathbb{C}) / (H^{3,0} + H^{2,1} + H^3(X, \mathbb{Z}))$$

is a polarized, non-algebraic torus.

E.g. X = Calabi-Yau 3fold,

(i.e. $\Omega^3_X \approx \mathcal{O}$, $H^1(X, \mathbb{C}) = 0$)

$\Rightarrow$ signature of period is $(1, h^{21})$.

Q: When is a family of complex tori Lagrangian?

$B \underset{\text{open}}{\subset} V^*$, $\quad V \approx \mathbb{C}^g$.

$\pi : X \to B$ family of complex tori, with <u>period</u> <u>map</u>:

$$p : B \longrightarrow (\mathrm{Sym}^2 V)_{n.d}$$

<u>The cubic condition</u> [D, Markman]

TFAE:

① $\exists$ complex symplectic σ on X s.t.
$\pi : (X, \sigma) \to B$ is Lagrangian
σ induces identity: $\begin{array}{ccc} T_{X/B} & \longrightarrow & \pi^* T^*_B \\ \wr\wr & & \wr\wr \\ \pi^* V & & \pi^* V \end{array}$

② $p : B \to \mathrm{Sym}^2 V$ is (locally in B) the Hessian of a holomorphic function on B: "prepotential."

③ $dp_{\#} \in \mathrm{Hom}(T_B, \mathrm{Sym}^2 V) \approx V \otimes \mathrm{Sym}^2 V$ actually lives in: $\mathrm{Sym}^3 V$.

$\boxed{\text{Calabi-Yau integrable system}}$:

X : CY3, $\Omega^3_X = 0$

$\mathcal{M} =$ **moduli** space $= \{$complex structures on $X\}/$isom

$$T_{[X]}\mathcal{M} = H^1(T_X) \approx H^1(\Omega^2_X) = H^{2,1}$$

(Bogomolov, Tian, Todorov : unobstructed)

$\widetilde{\mathcal{M}} \to \mathcal{M}$: natural $\mathbb{C}^*$-bundle

(choose : holo. volume form ω)

$\mathcal{J} \to \mathcal{M}$: universal int. Jacobian

$\widetilde{\mathcal{J}} \to \widetilde{\mathcal{M}}$: pullback.

$$\begin{array}{ccc} \widetilde{\mathcal{J}} & \to & \mathcal{J} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}} & \to & \mathcal{M} \end{array}$$

[D+M, '94] : $\widetilde{\mathcal{J}} \to \widetilde{\mathcal{M}}$ is an analytically integrable system.

- fibers $\mathcal{J}(X)$ are Lagrangian
- the image of any Abel-Jacobi map is isotropic.

The cubic = Yukawa's :

$$\otimes^3 H^1(T_X) \to H^3(\Lambda^3 T_X) = H^3(\Omega^{-3}_X) \xrightarrow{\cdot\omega^2} H^3(\Omega^3_X) \xrightarrow{\int} \mathbb{C}$$

$\mathfrak{X} \to B$ family of CY3's

$\forall b \in B$, $C_b = C_b^+ - C_b^-$: a 1-cycle in X_b, homologous to 0.

$\Rightarrow$ $\boxed{\text{Abel-Jacobi map}}$

$$AJ : B \to J(\mathfrak{X}/B)$$
$$b \mapsto \int_{\Gamma_b} \in H^3(X_b, \mathbb{C}) / \ldots\ldots = J(X_b)$$

where Γ_b is a 3-chain in X_b, $\partial \Gamma_b = C_b$.

* Independent of choices.

Various extensions:

* C not null-homologous: replace $J(X)$ by <u>Deligne cohomology group</u>.

* Special case $\mathfrak{X} = X \times B$:

$$AJ : \mathcal{H}ilb^{\circ}(X) \longrightarrow J(X)$$

* $\exists$ "<u>transversally holomorphic</u>" version: C is a real surface (non holomorphic) but it "varies holomorphically".

$\boxed{\text{Other examples}}$ (from algebraic geom.)

S: ~~complex~~ symplectic surface

$C \subset S$: a holo. curve

$\Rightarrow$ short exact sequence

$$(*) \quad \boxed{0 \to T_C \to T_S|_C \to N_{C/S} \to 0}$$

S symplectic $\Rightarrow N_C = \omega_C$ = canonical bundle

The SES $(*)$ determines an extension class in:

$$\mathrm{Ext}^1(N_{C/S}, T_C) = H^1(N_{C/S}^{-1} \otimes T_C)$$

$$= H^1(T_C^{\otimes 2})$$

$$= H^0(\omega_C^{\otimes 3})^* \longrightarrow \mathrm{Sym}^3 H^0(C, \omega_C)^*$$

$\Rightarrow$ A.I.S.

Base $= H^0(C, \omega_C) = H^0(C, N_{C/S}) \sim$ deformations of C in S

Fiber over C is $\mathcal{J}(C)$.

E.g. $S = K3$ (or T^4): $\boxed{\text{Mukai's I.S.}}$

Related to: symplectic structure on moduli spaces of vector bundles or coherent sheaves on K3.

Another example:

$B =$ curve (= compact R.S.)

$S := T^*B$, holomorphically symplectic

$$\begin{array}{ccc} T^*B & \supset & C = \text{"spectral curve"} \\ \searrow & & \swarrow \\ & B & \begin{pmatrix}\text{n-sheeted} \\ \text{branched cover}\end{pmatrix} \end{array}$$

$\Rightarrow$ $\boxed{\text{Hitchin's I.S.}}$

Base $= \{C\} = H^0(S, \mathcal{O}(C)) \approx \bigoplus_{i=1}^{n} H^0(B, K_B^{\otimes i})$

Fiber over $C = \mathcal{J}(C)$.

Total space = {Higgs bundles (V, φ) on B}

V: rank n vector bundle on B

$\varphi : V \to V \otimes \omega_B$: Higgs field

Variants:

* meromorphic Higgs bundles $\rightsquigarrow$ Markman's Poisson I.S.

  ($\varphi : V \to V \otimes \omega_B(D)$ for fixed D)
* Replace the LB by a $\mathbb{C}^*$-bundle $\Rightarrow$ Sklyanin's.
* Replace the LB by an elliptic fibration $\Rightarrow$ moduli spaces of bundles on elliptic fibr'ns
* Replace vector bundles by principal G-bundles

. . . . . .

Our setup:

$X_{0,0}$: CY3 with curve C of singularities (say, of type G, e.g. simplest: A_1.)

$X_{a,0}$: CY3's with finite number n of singularities which can be resolved: $\widetilde{X}_{a,0} \to X_{a,0}$.

$X_{a,\mu}$: smoothing of $X_{a,0}$.

$$\begin{tikzcd}[column sep=small] & \widetilde{M} \arrow[dr] & & \\ S \arrow[ur, hook] & \subset & M & \subset & L \end{tikzcd}$$

107 110 113

We want to understand CYIS(L) near $a = \mu = 0$.

Claim: to first order,

$$\boxed{\mathrm{CYIS}(L) \approx \mathrm{CYIS}(S) \times \mathrm{Hitchin}(C,G).}$$

In fact: $\exists$ family of IS's parametrized by $t \in \mathbb{C}$, s.t. for $t \neq 0$ get CYIS(L), for $t=0$ get CYIS$(S) \times$ Hitchin(C,G).

2D analogue: [D, Ein, Lazarsfeld]

S = K3 surface
$\cup$
C = curve, $D \in |nC|$

$\boxed{\text{Mukai's I.S. for line bundles on } D \subset S \text{ degenerates to: Hitchin's I.S. for } C, \text{ group } G = SL(n, \mathbb{C}).}$

nilpotent cone in Mukai = sheaves supported on original C

is an affine twist of:
nilpotent cone in Hitchin $= \{(V, \varphi) / \varphi \text{ is nilpo}\}$

Data for the affine twist $\Leftrightarrow$ extension class encoded in n-th order neighborhood of C in S.

Idea: $\exists$ degeneration of S to $N_{C/S} = T^*C$ induces the degeneration of Mukai to Hitchin.

The deformation to normal cone for CY_3's:

$$\begin{array}{ccc} X & \longrightarrow & X_{0,0} \\ \cup & & \cup \\ F & \longrightarrow & C \end{array} \quad \text{ruled surface}$$

$N_{F \backslash X} = \mathrm{Tot}(K_F)$

$H^0(C, K_C) \simeq H^1(F, K_F) \to H^{2,1}(X).$

$$\begin{array}{ccccc} \tilde{S} & \subset & \widetilde{\mathcal{M}} & & \\ \| & & \downarrow & & \\ S & \subset & \mathcal{M} & \subset & L \end{array}$$

$$\begin{array}{ll} N_{S|\widetilde{\mathcal{M}}} & = H^0(C, K_C) \otimes (\text{weights of } G) \\ \downarrow & \\ N_{S|L} & = \oplus_{i=2}^{n} H^0(C, K_C^{\otimes i}) = \text{Hitchin base} \end{array}$$

The map is non-linear.

Hitchin base $\leftrightarrow$ spectral covers $\tilde{C} \xrightarrow{n} C$

Image of $N_{S|\widetilde{\mathcal{M}}}$ $\leftrightarrow$ completely reducible covers, $\tilde{C} = \cup_{i=1}^{n} C_i$.

after the deformation to normal bundle, $CY/S(l)$ becomes $CY/S(S) \times$
$\times$ Hitchin.