Extremal Transitions in Gromov-Witten Theory

Duiliu-Emanuel Diaconescu

UCLA, June 6th 2003

[DFGi] D.-E. D., B. Florea and A. Grassi, "Geometric Transitions and Open String Instantons", Adv. Theor. Math. Phys. **6** (2002), 619, hep-th/0205234.
[DFGii] D.-E. D., B. Florea and A. Grassi, "Geometric Transitions, Del Pezzo Surfaces and Open String Instantons", Adv. Theor. Math. Phys. **6** (2002), 643, hep-th/0206163.
[DF] D.-E. D., B. Florea, "Large N Duality for Compact Calabi-Yau Threefolds", hep-th/0302076.

Related Work:
M. Aganagic, M. Marino and C. Vafa, "All Loop Topological Amplitudes from Chern-Simons Theory", hep-th/0206164.

Gromov-Witten potential:

$$F_X(g_s, q) = \sum_{g=0}^{\infty} g_s^{2g-2} \sum_{\beta \in H_2(X)} C_{g,\beta} q^\beta$$
$$q^{\beta+\beta'} = q^\beta q^{\beta'}$$

Hypersurfaces in Toric Varieties

$i : X \hookrightarrow \mathcal{Z}$ hypersurface in a (reflexive) toric variety $\mathcal{Z}$,

$$X \in |-K_Z| \text{ and } H_2(X) \simeq H_2(Z).$$

Convex obstruction bundle (Givental; Lian, Liu and Yau) :

$$C_{0,\beta} = \int_{[\overline{M}_{0,0}(Z,\beta)]^{vir}} e(\mathcal{V})$$
$$\mathcal{V}_{(\Sigma,f)} = H^0(\Sigma, f^* \mathcal{O}_Z(-K_Z))$$

Example :

$$\mathcal{Z} = \mathbb{P}\left(\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1) \oplus \mathcal{O} \rightarrow \mathbb{F}_0\right)$$

$$\sum_{a,b,c \geq 0, a+b+c=3} U^a V^b W^c f_{abc}(Z_i) = 0.$$

Toric Threefolds

i) $X \simeq$ Total space $(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1)$;

$$\begin{array}{ccc} & & X \\ & \overset{f}{\nearrow} & \uparrow \sigma \\ \Sigma & \xrightarrow{f'} & \mathbb{P}^1 \end{array} \quad \Rightarrow \quad \begin{array}{c} \overline{M}_{g,0}(X,\beta) \simeq \overline{M}_{g,0}(\mathbb{P}^1, d) \\ \beta = \sigma_*(d[\mathbb{P}^1]) \end{array}$$

Concave Obstruction Bundle (Givental; Lian, Liu and Yau) :

$$C_{g,\beta} = \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} 1 = \int_{[\overline{M}_{g,0}(\mathbb{P}^1,d)]^{vir}} e(\mathcal{V})$$
$$\mathcal{V}_{(\Sigma,f)} = H^1(\Sigma, f'^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)))$$

Manin (genus zero), Faber and Pandharipande (all g) :

$$F_X(g_s, q) = \sum_{d=1}^{\infty} \frac{q^d}{d\left(2\sin\frac{dg_s}{2}\right)^2}$$

ii) $X \simeq$ Total space $(\mathcal{O}(K_S)\rightarrow S)$, S toric del Pezzo surface;

$$\begin{array}{ccc} & & X \\ & \overset{f}{\nearrow} & \uparrow \sigma \\ \Sigma & \xrightarrow{f'} & S \end{array} \quad \Rightarrow \quad \begin{array}{c} \overline{M}_{g,0}(X,\beta) \simeq \overline{M}_{g,0}(S,\gamma) \\ \\ \beta = \sigma_*[\gamma], \ \gamma \in H_2(S) \end{array}$$

$$C_{g,\beta} = \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} 1 = \int_{[\overline{M}_{g,0}(S,\gamma)]^{vir}} e(\mathcal{V})$$

$$\mathcal{V}_{(\Sigma,f)} = H^1(\Sigma, f'^*\mathcal{O}(K_S))$$

2. Localization

One of the main tools for computing GW invariants: localization w.r.t a torus action (Kontsevich, Graber and Pandharipande.) Works for

i) Hypersurfaces $X \subset \mathcal{Z}$ for $g = 0$.

$$T \times \mathcal{Z}\rightarrow\mathcal{Z} \ \Rightarrow \begin{cases} T \times \overline{M}_{0,0}(\mathcal{Z},\beta)\rightarrow\overline{M}_{0,0}(\mathcal{Z},\beta) \\ \\ T \times \mathcal{V}\rightarrow\mathcal{V} \end{cases}$$

ii) Noncompact toric threefolds X for all $g \geq 0$.

$$T \times X\rightarrow X \ \Rightarrow \begin{cases} T \times \overline{M}_{g,0}(B,\beta)\rightarrow\overline{M}_{g,0}(B,\beta) \\ \\ T \times \mathcal{V}\rightarrow\mathcal{V} \end{cases}$$

$B = \mathbb{P}^1, S$ (as discussed above)

$[\overline{M}]^{vir}_T \in A^T_*(\overline{M})$ equivariant virtual cycle

$i_\Xi : \Xi \hookrightarrow \overline{M}$ connected components of the fixed locus in appropriate moduli space

$[\Xi]^{vir} = i^!_\Xi \left([\overline{M}]^{vir}_T\right)$ induced virtual cycle

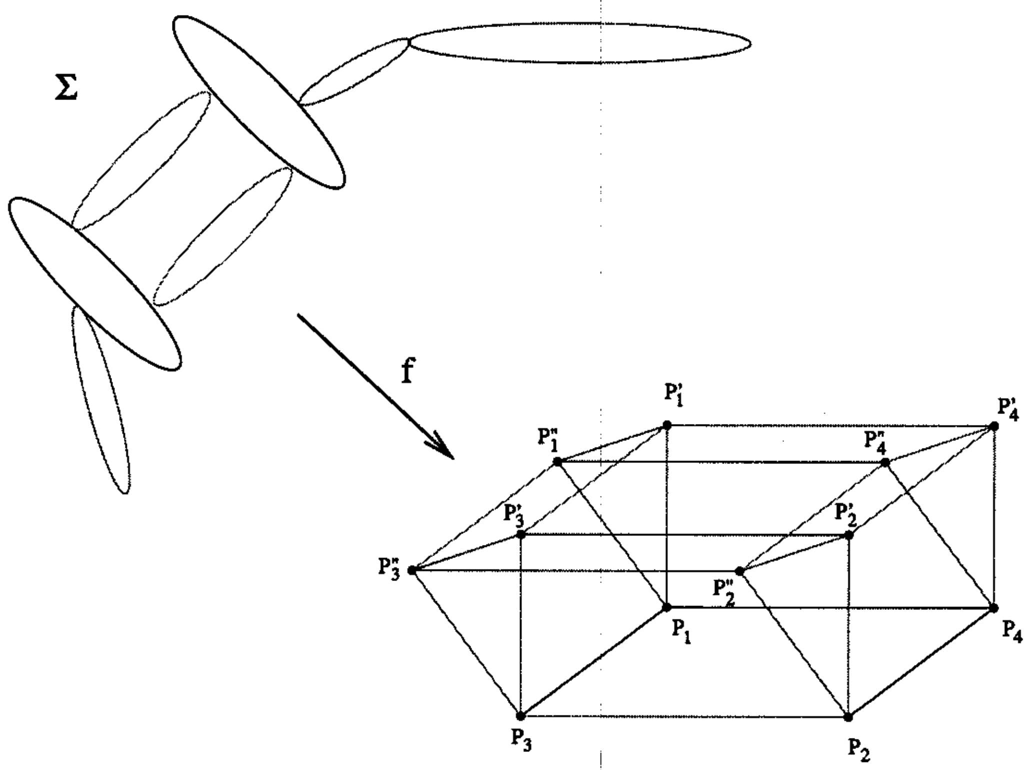
N^{vir}_Ξ virtual normal bundle

Atiyah-Bott localization on $\overline{M}$ (Graber and Pandharipande)

$$C_{(g,\beta)} = \sum_\Xi \int_{[\Xi]^{vir}} \frac{e_T(i^*_\Xi \mathcal{V})}{e_T(N^{vir}_\Xi)}$$

Example :

$$\mathcal{Z} = \mathbb{P}\left(\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1) \oplus \mathcal{O} \to \mathbb{F}_0\right)$$



3. The Problem

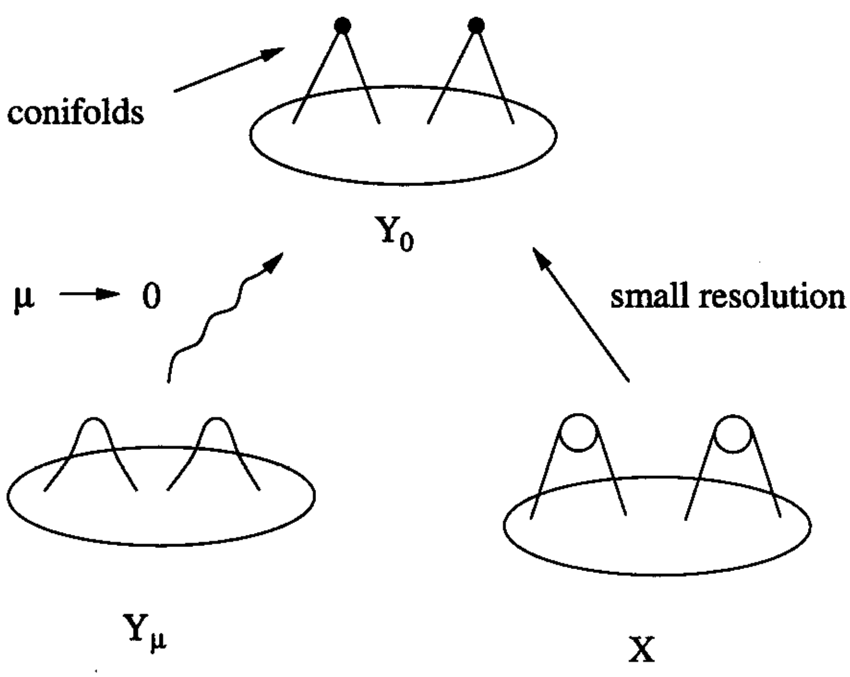
Extremal transitions (Clemens): 1-parameter family of CY threefolds Y_μ so that

i) Y_μ smooth for $\mu \neq 0$

ii) Y_0 has ordinary double points $P_1, \ldots, P_v$

iii) Y_0 admits a smooth crepant resolution $X \to Y_0$

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y_\mu & \rightsquigarrow & Y_0 \end{array}$$



Topology Change: $[L_1], [L_2], \ldots, [L_v] \in H_3(Y)$ vanishing cycles subject to r relations

$$h^{1,2}(X) = h^{1,2}(Y) - (v - r) \qquad h^{1,1}(X) = h^{1,1}(Y) + r$$

Note that in going from X to Y we are 'losing' curve classes and 'gaining' 3-cycles.

Gromov-Witten potentials

$$F_X(g_s, q_X) = \sum_{g=0}^{\infty} g_s^{2g-2} \sum_{\alpha \in H_2(X)} C_{g,\alpha}(X) q_X^{\alpha}$$

$$F_Y(g_s, q_Y) = \sum_{g=0}^{\infty} g_s^{2g-2} \sum_{\beta \in H_2(Y)} C_{g,\beta}(Y) q_Y^{\beta}$$

Imprecise Question : What is the relation (if any) between $F_X(g_s, q_X)$ and $F_Y(g_s, q_Y)$?

Using symplectic techniques, Li and Ruan proved the following result

$$C_{g,\beta}(Y) = \sum_{\alpha \in H_2(X), \phi(\alpha)=\beta} C_{g,\alpha}(X)$$

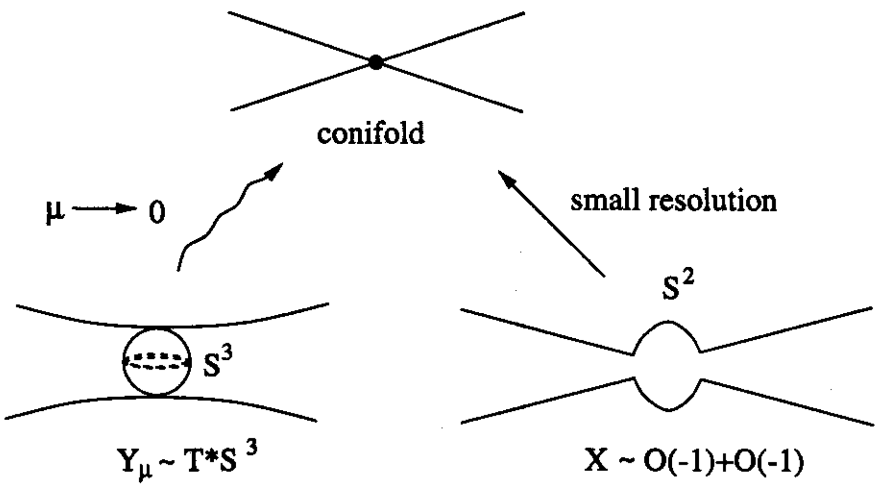
$$\phi : H_2(X) \rightarrow H_2(Y) \text{ (induced by symplectic cut)}$$

Large N duality suggests a different answer: the contributions of the "missing" curve classes to $\mathcal{F}_X(g_s, q_X)$ is encoded in a subtle way in the vanishing cycles $L_1, \ldots, L_v$. We have to regard them as $D-$ *branes* and *quantize* them.

4. Conifold Transition (Gopakumar and Vafa)

$X = \text{Total space}\left(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1\right)$

$Y_\mu \subset \mathbb{C}^4, \quad xy + zw = \mu$



$H_2(X) \simeq \mathbb{Z}$ generated by the 0-section $C = \sigma(\mathbb{P}^1)$ which is a $(-1,-1)$ curve on X; $H_3(X) = 0$

$H_3(Y) \simeq \mathbb{Z}$ generated by the vanishing cycle $[L]$; if $\mu \in \mathbb{R},\ \mu > 0$ can choose a lagrangian representative $L = Y \cap \{x = \overline{y},\ z = \overline{w}\} \simeq S^3$; $H_2(Y) = 0$

$$F_X(g_s, q) = \sum_{d=1}^{\infty} \frac{q^d}{d\left(2\sin\frac{dg_s}{2}\right)}$$

$$F_Y = 0$$

How can we recover $F_X(g_s, q)$ out of the data (Y, L)?

Gopakumar and Vafa: $U(N)$ Chern-Simons theory on $L = S^3$

$E \to L$ rank N complex vector bundle, $A \in \mathcal{A}(E)$ unitary connection

Physicists refer to (E, L) as 'N D-branes wrapped on L'; in fact L is 'promoted' to a K-homology cycle.

$$S_{CS}(A) = \frac{k}{2\pi} \int_L \mathrm{Tr}\left(AdA + \frac{2}{3} A^3 \right)$$

Witten, Reshetikhin-Turaev – quantum CS invariant which can be formally written

$$Z_{N,k}(L) = \int_{\mathcal{A}(E)/\mathcal{G}} e^{-S_{CS}(A)}$$

Large N expansion of CS theory: there is a formal series $F_{CS}(\kappa, \lambda)$ so that

$$F_{CS}\left(\kappa = \frac{2\pi}{k+N}, \lambda = e^{\sqrt{-1}N\kappa} \right) = \ln Z_{N,k}(L), \qquad \forall N, k$$

Geometric large N duality:

$$F_X(g_s, q) = F_{CS}(g_s, q)$$

The GW invariants of X are encoded in CS theory on L! (Quite striking if we remember the definition of $F_X(g_s, q)$)

5. Hypersurfaces in Toric Varieties

Extremal transition

$$\begin{array}{ccc} & & \widetilde{\mathcal{Z}} \supset X \\ & & \downarrow \\ \mathcal{Z} \supset Y_\mu & \rightsquigarrow & Y_0 \end{array}$$

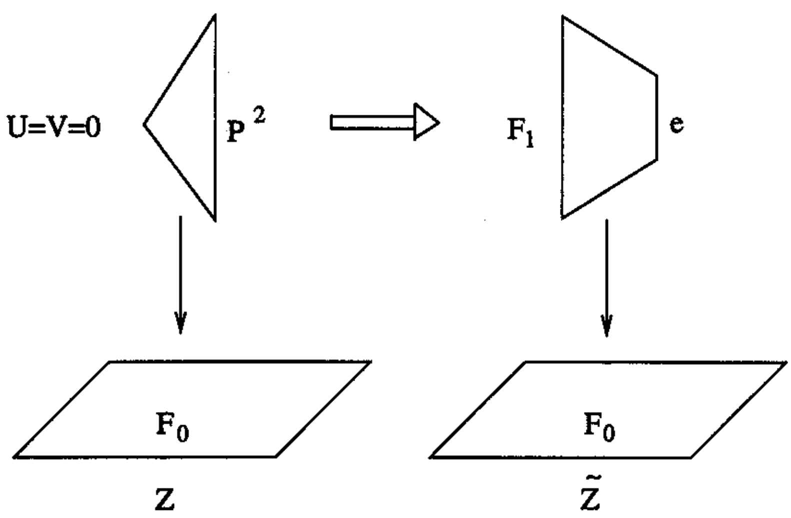
$Z, \widetilde{Z}$ smooth compact toric varieties

Assumption : The ODP's of Y_0 are fixed points of the torus action on $\mathcal{Z}$

$$\mathcal{Z} = \mathbb{P}\left(\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1) \oplus \mathcal{O} \rightarrow \mathbb{F}_0\right)$$

$$Y_\mu : \; (UZ_1Z_4 + VZ_2Z_3 - \mu W)W^2 + \sum_{a,b,c\geq 0, a+b+c=3}' U^a V^b W^c f_{abc}(Z_i) = 0$$

$$\mu \rightarrow 0 \;\Rightarrow\; \text{two ODP's} \begin{cases} P_1 = \{Z_1 = Z_3 = U = V = 0\} \\ P_2 = \{Z_2 = Z_4 = U = V = 0\} \end{cases}$$



$\widetilde{\mathcal{Z}} =$ (Toric) Blow-up of $\mathcal{Z}$ along $\{U = V = 0\}$

Topology change: $h^{1,1}(X) = h^{1,1}(Y) + 1$, $h^{1,2}(X) = h^{1,2}(Y) - 1$

Exceptional curves $e_1, e_2 \subset X$, $[e_1] - [e_2] = 0$ in $H_2(X)$

Vanishing cycles $L_1, L_2 \simeq S^3$, $[L_1] + [L_2] = 0$ in $H_3(Y)$

$$F_X^{(0)}(q_X) = \sum_{\beta \in H_2(X)} \sum_{n=1}^{\infty} \frac{N_{X,\beta}}{n^3} q_X^{n\beta}$$

$$F_Y^{(0)}(q_Y) = \sum_{\gamma \in H_2(Y)} \sum_{n=1}^{\infty} \frac{N_{Y,\gamma}}{n^3} q_Y^{n\beta}$$

How can we recover $F_X^{(0)}(q_X)$ using the data (Y, L_1, L_2) ?

Chern-Simons theory:

$$E_1 = L_1 \times \mathbb{C}^N \quad U(N) \text{ bundle on } L_1 = S^3$$

$$E_2 = L_2 \times \mathbb{C}^N \quad U(N) \text{ bundle on } L_2 = S^3$$

with levels $k_1, k_2 \Rightarrow$ formal series

$$F_{CS}(\kappa_1, \lambda_{1,2}) = \sum_{d=1}^{\infty} \left(\frac{\lambda_1^d}{d\left(2\sin\frac{d\kappa_1}{2}\right)} + \frac{\lambda_2^d}{d\left(2\sin\frac{d\kappa_2}{2}\right)} \right)$$

$$F_{CS}\left(\kappa_i = \frac{2\pi}{k_i + N}, \lambda_i = e^{\sqrt{-1}N_i\kappa_i}\right) = \ln Z_{N,k_1}(L_1) + \ln Z_{N,k_2}(L_2)$$

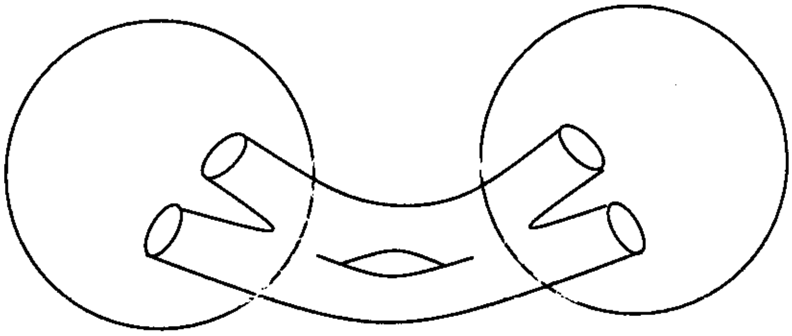
This ~~is~~ does not reproduce $F_X(g_s, q_X)$, but

$$F_{CS}(\kappa_1 = \kappa_2 = g_s, \lambda_1 = \lambda_2 = q_e)$$

represents the contribution of the two exceptional curves e_1, e_2 (plus multicovers) to $F_X(g_s, q_X)$.

What is Missing ?

There are holomorphic bordered Riemann surfaces in Y 'ending' on the lagrangian cycles L_1, L_2.



Witten (1991) – In the presence of such surfaces, the CS theory must be corrected by 'open string instanton effects'.

Suppose we have a single rigid disc D with boundary $\Gamma = \partial D$ on a lagrangian sphere L. Then the CS action should be corrected to

$$S(A) = \frac{k}{2\pi} \int_L \mathrm{Tr} \left(AdA + \frac{2}{3} A^3 \right) + \frac{1}{g_s} q_d \mathrm{Tr} V_\Gamma$$

where V_Γ is the holonomy of A about the knot Γ and

$$q_d = e^{-(\text{symplectic area of D})}$$

(in the following we will think of q_d as a formal variable.)

This yields a correction of the CS free energy

$$\ln Z = \ln Z_{CS} \ + \ln \langle e^{\frac{q_d}{g_s} \mathrm{Tr} V_\Gamma} \rangle$$

where $\langle\ \rangle$ denotes CS expectation value. This formula can be given a rigorous construction (as a formal series) as follows

$$\langle e^{\frac{q_d}{g_s}\mathrm{Tr}V_\Gamma}\rangle = \langle\sum_{n=0}^{\infty}\frac{q_d^n}{g_s^n n!}(\mathrm{Tr}V_\Gamma)^n\rangle$$
$$= \sum_{n=0}^{\infty}\frac{q_d^n}{g_s^n n!}\sum_R C_n(R)\langle\mathrm{Tr}_R V_\Gamma\rangle$$

where $\langle\mathrm{Tr}_R V_\Gamma\rangle$ is physicist notation for the $U(N)$ Jones polynomial of the knot Γ in the representation R.

Remarks:

i) In general D will not be rigid and isolated (can have families)

ii) Even if D is rigid, one has to take into account multicovers

iii) Higher genus bordered surfaces

How can we obtain numerical results taking into account all these aspects?

Solution

Concrete algorithm for building $S_{inst}(A)$ based on localization with respect to the torus action ([DFGi], [DFGii], [DF]).

Previous work:

- closed strings – Kontsevich, Graber and Pandharipande
- open strings – Katz and Liu, Li and Song, Graber and Zaslow (different context)

Let $L = L_1 \cup L_2$ and $\beta \in H_2(Y, L)$.

• $\overline{M}_{0,h}(Y,L;\beta)$ moduli 'space' of stable 'open string' maps $f:\Sigma_{0,h}\rightarrow Y$, $f(\partial\Sigma_{g,h})\subset L$. $\Sigma_{0,h}$ genus 0 bordered Riemann surface with h boundary components

• virtual 0-cycle $[\overline{M}_{0,h}(Y,L;\beta)]^{vir}$ and orientation

These are standard ingredients, but this is not a standard counting problem. Exotic aspects:

i) The result of the open string 'counting problem' has to be a series in holonomy variables rather than rational numbers.

ii) In the presence of families of maps $f:\Sigma_{0,h}\rightarrow(Y,L)$, how does one choose the right holonomy variables (the boundaries may move and deform in L) ?

The Plan (Ideal world) :

• Torus action $T\times Y\rightarrow Y$ preserving $L\Rightarrow T\times\overline{M}_{g,h}(Y,L;\beta)\rightarrow\overline{M}_{g,h}(Y,L;\beta)$

• Fixed configuration of invariant surfaces in Y with boundary on L (discs and/or cylinders) $\Rightarrow$ fixed set of holonomy variables V_i

• Localization formula:

$$[\overline{M}_{0,h}(Y,L,\beta)]^{vir}=\sum_{\Xi}i_{\Xi*}\left(\frac{i_{\Xi}^{!}[\overline{M}_{0,h}(Y,L,\beta)]^{vir}}{e_T(N_{\Xi}^{vir})}\right)$$

Then to each fixed locus $\xi \subset \overline{M}_{0,h}(Y,L)$ we assign

i) An instanton factor q_{op}^{beta}

ii) A monomial in holonomy variables $\prod_i (\mathrm{Tr} V_i^{m_i})^{k_i}$

iii) A local coefficient $C_{0,\beta}(\Xi) = \int_{[\Xi]^{vir}} \frac{1}{N_{\Xi}^{vir}}$

• The final step :

$$
\begin{aligned}
S(A) &= S_{CS}(A) + S_{inst}(A) \\
&= S_{CS}(A) + \sum_{h=1}^{\infty} g_s^{h-2} \sum_{\beta \in H_2(Y,L)} \sum_{\Xi} C_{0,\beta}(\Xi) \times (\text{inst factor}) \times (\text{hol factor})
\end{aligned}
$$

and evaluate

$$
\ln Z_{op}^{(0)} = \ln Z_{CS}^{(0)} + \ln \left\langle e^{S_{inst}(A)} \right\rangle
$$

Real life :

• For generic smooth Y, no T-action ! Must take a degenerate limit of $Y = Y_\mu$ of the form

$$
Y_\mu^{deg} : \ (UZ_1Z_4 + VZ_2Z_3 - \mu W)W^2 = 0
$$

Special representatives of vanishing cycles L_1, L_2

$$
xy + zw = \mu, \quad x = \overline{y}, \quad z = \overline{w}
$$

in affine toric coordinates (μ assumed real and positive.)

• Y_μ^{deg} is reducible and nonreduced $\Rightarrow \overline{M}_{0,h}(Y_\mu^{deg}, L, \beta)$ not well-behaved. Must work with the moduli space $\overline{M}_{0,h}(\mathcal{Z}, L, \beta)$ of open string maps

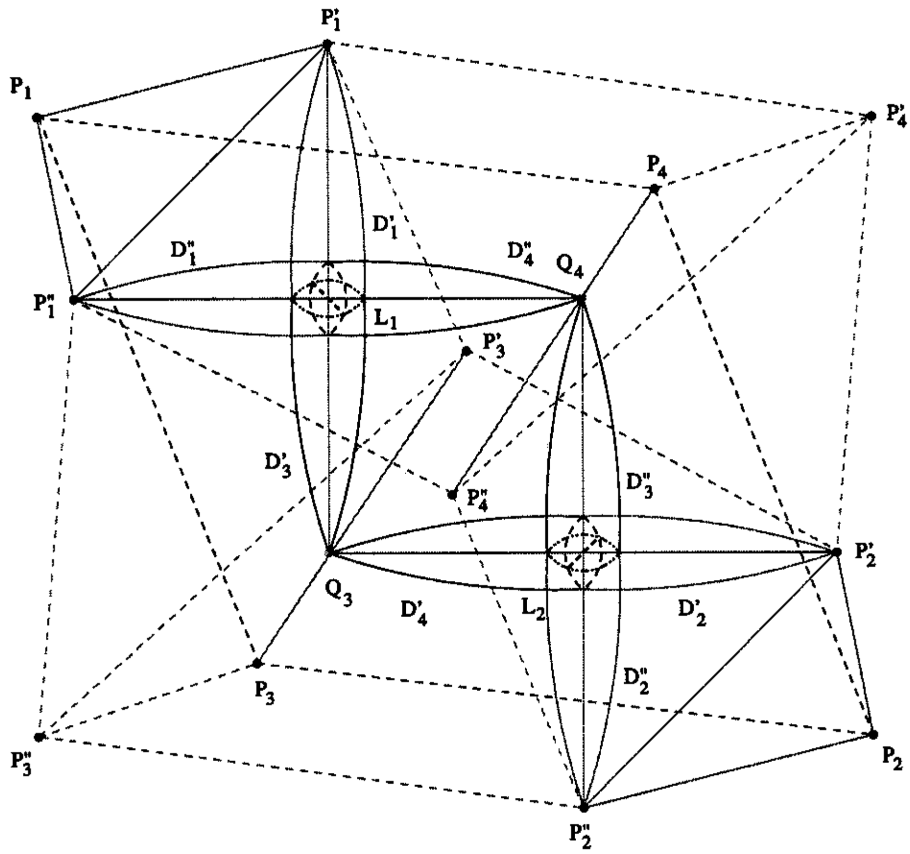
$$f : \Sigma_{0,h} \to \mathcal{Z}, \qquad f(\partial\Sigma_{0,h}) \subset L$$

• Convex obstruction bundle for open strings

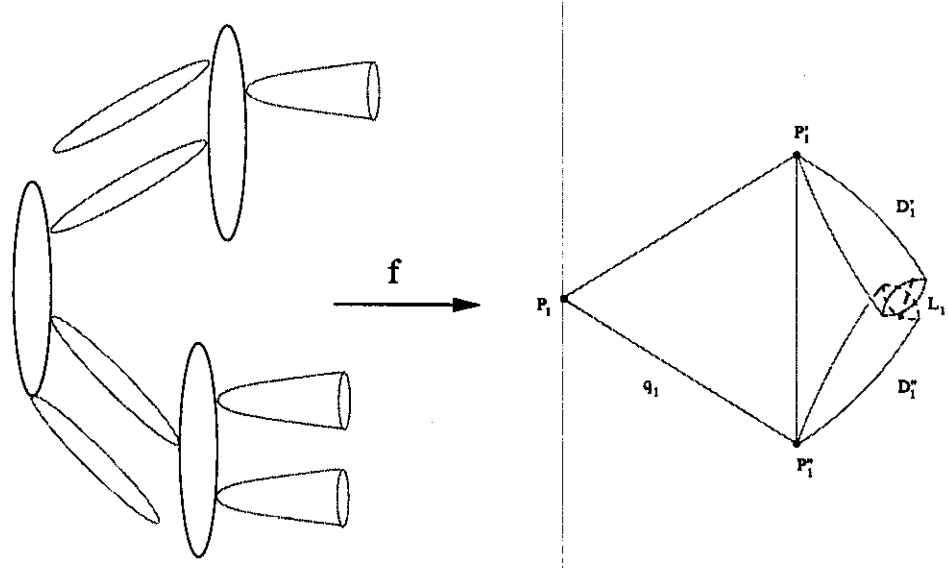
$$\mathcal{V}^{op} \to \overline{M}_{0,h}(\mathcal{Z}, L, \beta), \qquad \mathcal{V}^{op}_{(\Sigma_{0,h}, f)} = H^0(\Sigma_{0,h}, f^*\mathcal{O}(K_{\mathcal{Z}}))_\partial$$

Remark : No rigorous construction for $\overline{M}_{0,h}(\mathcal{Z}, L, \beta)$, $[\overline{M}_{0,h}(\mathcal{Z}, L, \beta)]^{vir}$ and $\mathcal{V}^{op}$. Can perform explicit computations by using only the fixed loci Ξ (Katz and Liu, Li and Song, Graber and Zaslow)

• Configuration of invariant discs in $\mathcal{Z}$ with boundary on L



Invariant open string maps



Sum over all such maps as explained above $\Rightarrow$

$$S_{inst}(A) = \sum_{h=1}^{\infty} g_s^{h-2} \sum_{\beta \in H_2(Y,L)} \sum_{\Xi} C_{0,\beta}(\Xi) \times (\text{inst factor}) \times (\text{hol factor})$$

$$\ln Z_{op}^{(0)} = \ln Z_{CS}^{(0)} + \ln \left\langle e^{S_{inst}(A)} \right\rangle$$

There is a unique formal power series $F_{op}^{(0)}(g_s, q_{op}, \kappa_{1,2}, \lambda_{1,2})$ so that

$$F_{op}^{(0)}\left(g_s, q_{op}, \kappa_{1,2} = \frac{2\pi}{k_{1,2}+N}, \lambda_{1,2} = e^{N\sqrt{-1}\kappa_{1,2}}\right) = \ln Z_{op}^{(0)}$$

<u>Large N duality conjecture</u>

There exists a duality map of the form

$$\kappa_1 = \kappa_2 = g_s \quad \lambda_1 = \lambda_2 = q_e \quad q_{op} = q_{op}(q_X) \quad q_Y = q_Y(q_X)$$

so that

$$\mathcal{F}_Y^{(0)}(q_Y) + g_s^2 \mathcal{F}_{op}^{(0)}(g_s, q_{op}, \kappa_{1,2}, \lambda_{1,2}) = \mathcal{F}_X^{(0)}(q_X)$$

Objections :

- By construction, $S_{inst}(A)$ is a formal series with coefficients in $\mathcal{R}_T$ (the fraction field of the representation ring of T) since $C_{0,\beta}(\Xi) \in \mathcal{R}_T$. $\mathcal{F}_X^{(0)}(g_s, q_X), \mathcal{F}_Y^{(0)}(g_s, q_Y)$ have coefficients in $\mathbb{Q}$!

- In order to evaluate knot and link invariants in CS theory, we have to specify the framing. A priori, there is no obvious choice, so is there a discrete ambiguity in $\mathcal{F}_{op}^{(0)}(g_s, q_{op}, \kappa_{1,2}, \lambda_{1,2})$?

Solution :

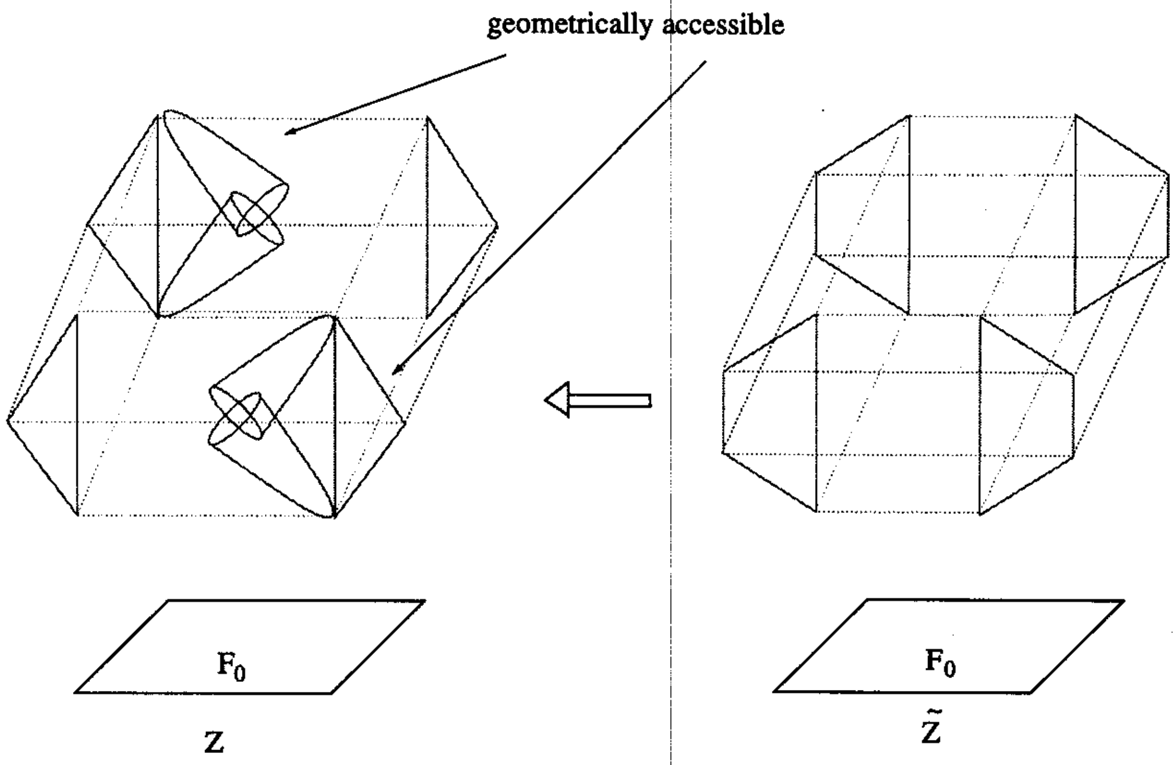
- There is a canonical choice of framing variables $p_i \in \mathcal{R}_T$ (!) depending on the weights of the T action on $\mathcal{Z}$.

- Refined duality conjecture

$$F_X^{(0)}(q_X) = \sum_{\beta \in H_2(X)} \widetilde{C}_{0,\beta} q_X^\beta, \quad \widetilde{C}_{0,\beta} = \sum_{\widetilde{\Xi} \subset \overline{M}_{0,0}(X,\beta)} \widetilde{C}_{0,\beta}(\widetilde{\Xi})$$

$$F_Y^{(0)}(q_Y) = \sum_{\gamma \in H_2(Y)} C_{0,\gamma} q_Y^\gamma, \quad C_{0,\gamma} = \sum_{\Xi \subset \overline{M}_{0,0}(Y,\gamma)} C_{0,\gamma}(\Xi)$$

For geometric reasons we may have to truncate the coefficients $\widetilde{C}_{0,\beta}$, $C_{0,\gamma}$ to a sum over *geometrically accessible* fixed loci



This yields $F_X^{(0)}(q_X)_{tr}$, $F_Y^{(0)}(q_Y)_{tr}$ with coefficients in $\mathcal{R}_T$. Then can check that the conjecture holds!

$$\mathcal{F}_Y^{(0)}(q_Y)_{tr} + g_s^2 \mathcal{F}_{op}^{(0)}(g_s, q_{op}, \kappa_{1,2}, \lambda_{1,2}) = \mathcal{F}_X^{(0)}(q_X)_{tr}$$

- For a "clever" choice of weights,

$$F_X^{(0)}(q_X)_{tr} = F_X^{(0)}(q_X), \qquad F_Y^{(0)}(q_Y)_{tr} = F_Y^{(0)}(q_Y)$$

Then the conjecture holds in the original form

$$\mathcal{F}_Y^{(0)}(g_s, q_Y) + \mathcal{F}_{op}^{(0)}(g_s, q_{op}, \kappa_{1,2}, \lambda_{1,2}) =$$

$$\begin{aligned}
&2(\widetilde{q}_1 + \widetilde{q}_2 + \widetilde{q}_3) + 36\widetilde{q}_4 - 2(\widetilde{q}_1\widetilde{q}_2 + \widetilde{q}_1\widetilde{q}_3 + \widetilde{q}_2\widetilde{q}_3) + 126(\widetilde{q}_1 + \widetilde{q}_2 + \widetilde{q}_3)\widetilde{q}_4 \\
&+ \frac{9}{2}\widetilde{q}_4^2 + \frac{1}{4}(\widetilde{q}_1^2 + \widetilde{q}_2^2 + \widetilde{q}_3^2) - \frac{1}{4}(\widetilde{q}_1^2\widetilde{q}_2^2 + \widetilde{q}_1^2\widetilde{q}_3^2) + \frac{4}{3}\widetilde{q}_4^3 + 126(\widetilde{q}_1 + \widetilde{q}_2 + \widetilde{q}_3)\widetilde{q}_4^2 \\
&+ 36(\widetilde{q}_1\widetilde{q}_2 + \widetilde{q}_1\widetilde{q}_3 + \widetilde{q}_2\widetilde{q}_3)\widetilde{q}_4 + 6\widetilde{q}_1\widetilde{q}_2\widetilde{q}_3 + \frac{2}{27}\widetilde{q}_1^3 + 2\widetilde{q}_1\widetilde{q}_4^3 + \frac{207}{4}\widetilde{q}_1^2\widetilde{q}_4^2 \\
&+ 2178(\widetilde{q}_1\widetilde{q}_2 + \widetilde{q}_1\widetilde{q}_3)\widetilde{q}_4^2 - 144\widetilde{q}_1\widetilde{q}_2\widetilde{q}_3\widetilde{q}_4 - 4\widetilde{q}_1^2\widetilde{q}_2\widetilde{q}_3 + 152\widetilde{q}_1^2\widetilde{q}_4^3 \\
&+ 126(\widetilde{q}_1^2\widetilde{q}_2 + \widetilde{q}_1^2\widetilde{q}_3)\widetilde{q}_4^2 + 108\widetilde{q}_1^2\widetilde{q}_2\widetilde{q}_3\widetilde{q}_4 + \frac{20}{3}\widetilde{q}_1^3\widetilde{q}_4^3
\end{aligned}$$

where $q_X = (\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3, \widetilde{q}_4)$, $\widetilde{q}_1 = q_e$.

Precise agreement with the genus zero GW expansion of X computed from mirror symmetry.

Concluding Remarks

• This is not a proof of the duality construction. The open string enumerative data $\overline{M}_{0,h}(\mathcal{Z}, L; \beta)$, $[\overline{M}_{0,h}(\mathcal{Z}, L; \beta)]^{vir}$ have not been rigorously constructed. Very likely need symplectic techniques (Fukaya, Oh). Recent progress made by C.-C. M. Liu.

• Interesting connection between localization and Chern-Simons theory. Should be better understood.

• Same techniques apply to extremal transitions between (noncompact) toric CY threefolds. In that case one can check duality for all genus amplitudes.

• Our approach is perhaps too "equivariant". Works only when the ODP's are fixed under torus action on ambient toric variety. However most transitions are not of this type $\Rightarrow$ must find an extension of the present formalism (symplectic geometry ?)

Extremal Transitions in Gromov-Witten Theory

Duiliu-Emanuel Diaconescu
UCLA, June 6th 2003

[DFGi] D.-E. D., B. Florea and A. Grassi, "Geometric Transitions and Open String Instantons", Adv. Theor. Math. Phys. **6** (2002), 619, hep-th/0205234.
[DFGii] D.-E. D., B. Florea and A. Grassi, "Geometric Transitions, Del Pezzo Surfaces and Open String Instantons", Adv. Theor. Math. Phys. **6** (2002), 643, hep-th/0206163.
[DF] D.-E. D., B. Florea, "Large N Duality for Compact Calabi-Yau Threefolds", hep-th/0302076.