Hori-Vafa formula, mirror principle and Quot-schemes

– Mirror Principle: its past, current, & future

(Chien-Hao Liu)

I. Mirror principle.

II. The Hori-Vafa formula.

III. Mirror principle proof of Hori-Vafa formula

IV. New directions of mirror principle.

based mainly on:

[Lian-K.Liu-Yau], "mirror principle I, II, III, IV", (1997 – 2000)
alg-geom/9712011, math.AG/9905006, math.AG/9912038

[Hori-Vafa] "Mirror symmetry", hep-th/0002222 (2000)

[Lian-C.H.Liu-K.Liu-Yau] "S^1-fixed points in Quot schemes and mirror principle computations", math.AG/0111256 (2001)

[Bertram-(Ciocan-Fontanine)-Kim] "Two proofs of a conjecture of Hori and Vafa", math.AG/0304403 (2003)

2d SQFT:

[Witten] "Phases of $N=2$ theories in two dimensions", hep-th/9301042 (1993)

[Morrison-Plesser] "Summing the instantons: quantum cohomology and mirror symmetry in toric varieties", hep-th/9412236 (1994)

I. ① a [mirror principle] & its stringy origin (1)

< stringy duality, mirror symmetry, mirror transformations >

d = 2 world-sheet aspects, d = 2 $N=(2,2)$

$(-\tilde{g}_L, \tilde{g}_R) \longleftrightarrow (\tilde{g}_L, \tilde{g}_R)$

$\widetilde{SCFT}$: (c,c) (a,c) — (a,c) (c,c)

d = 2. $N=(2,2)$ SCFT

$\tilde{\Phi}_{(-1,1)}$ $\tilde{\Phi}_{(1,1)}$ $\Phi_{(1,1)}$ $\Phi_{(-1,1)}$

CY $\tilde{M}$ — $\tilde{\Phi}_{(-1,1)}$, $\tilde{\Phi}_{(1,1)}$ — $\Phi_{(1,1)}$, $\Phi_{(-1,1)}$ — CY M

$\tilde{\mathcal{M}}_{cpx}$: complex structure deformation × $\tilde{\mathcal{M}}_{Kähler}$: Kähler deformation

$\mathcal{M}_{Kähler}$: Kähler deformation × $\mathcal{M}_{cpx}$: complex structure deformation

CY geometric realization

II B → II B [$\tilde{M}$] — moduli field at d = 4

Kaluza-Klein dimensional reduction of Type II superstrings

$\left(\tilde{h}^{2,1} = h^{1,1},\ \tilde{h}^{1,1} = h^{2,1} \right)$

II A → II A [M]

$\tilde{h}^{2,1}$ vector multiplets — $\tilde{h}^{1,1}+1$ hyper multiplets — $h^{1,1}$ vector multiplets — $h^{2,1}+1$ hyper multiplets

CY^3

d = 4 $N=2$ effective theory via superstring compactification

$\tilde{\Omega}(\tilde{t}) \in H^{3,0}(\tilde{M}(\tilde{t})) \subset H^3(\tilde{M}, \mathbb{C})$

$\{\tilde{A}_I, \tilde{B}_J\}$: symplectic basis for $H_3(\tilde{M}, \mathbb{Z})$

$$\tilde{F}(\tilde{t}) = \frac{1}{2} \sum_I \tilde{z}_I(\tilde{t})\, \tilde{\mathcal{G}}_I(\tilde{t}),$$

where $\tilde{z}_I(\tilde{t}) = \int_{\tilde{A}_I} \tilde{\Omega}(\tilde{t})$

$\tilde{\mathcal{G}}_I(\tilde{t}) = \int_{\tilde{B}_I} \tilde{\Omega}(\tilde{t})$

[Candelas - de la Ossa - Green - Parkes]

Kinetic term of vector multiplets in the d=4, N=2 SUGRA:

∫

special geometry on the related moduli space

∫

prepotential

$\tilde{F}(f(t)) = F(t)$

field redefinition = mirror transformation

$$F(t) = \sum_{ijk} d_{ijk}\, t_i t_j t_k + \text{const.} + \sum_{d} K_d\, e^{d\cdot t}$$

$d \in$ curve class in $H_2(M, \mathbb{Z})$

cubic form from topology on M

world-sheet loop correction

world-sheet instanton correction

>> (1): Cheat-sheet for mathematicians in the superworld.

- Supersymmetry: $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra
  - grade-0 part: bosonic component
  - grade-1 part: fermionic component

- Superspace: ringed space in the sense of Grothendieck.

  coordinates: $(x_1, x_2, \cdots ; \theta_1, \theta_2, \cdots)$

  = usual space (smooth-complex-manifold) $\Rightarrow (x_1, x_2, \cdots)$

  + anti-commuting elements in its structure sheaf $\Rightarrow (\theta_1, \theta_2, \cdots)$

- The N everywhere: # of minimal set of supercharges (spinor rep.) / fermionic generators

Eg. $d=4, N=2 \Rightarrow 8$ susy. chge

dim:	1	2	3	4	5	6	7	8	9	10	11	12
min:	1	1	2	4	8	8	16	16	16	16	32	64

- fermionic integration = taking coefficient

  Eg. $\int d\theta\,(a\theta + b) = a$

- Supermultiplets:

  = essential part / component fields from representation theory

  + derivatives of essential parts

  + auxiliary (i.e. non-dynamical) component fields

- chiral multiplet = fields to build up matter e.g. electrons, ... (= matter field)

  vector multiplet = gauge-field / connections, ...

- some differential operator on $d=2$, $N=(2,2)$ superspace

$$\{x^0, x^1 ; \underbrace{\theta^-, \bar\theta^-}_{(2,0)\text{-part}} ; \underbrace{\theta^+, \bar\theta^+}_{(0,2)\text{-part}}\} : \quad Q_\pm = \partial/\partial\theta^\pm + i\bar\theta^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$$

$$\bar{Q}_\pm = -\partial/\partial\bar\theta^\pm - i\theta^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$$

$$D_\pm = \partial/\partial\theta^\pm - i\bar\theta^\pm\left(\frac{\partial}{\partial x^0} \pm \partial/\partial x^1\right)$$

$$\bar{D}_\pm = -\partial/\partial\bar\theta^\pm + i\theta^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$$

[Freed]. [Manin]

[Wess-Bagger]

⟨gauged linear σ-models & A-twisted moduli space⟩

- gauged linear σ-model : a $d=2$, $N=(2,2)$ SQFT with abelian gauge sy[mmetry]
  - superfields : Φ_i : chiral superfields. $e^{i\alpha_a}\Phi_i = e^{iQ_{ia}\alpha_a}\Phi_i$ $\quad G = U(1)\times\cdots\times U(1)$
    V_a : vector superfields ∋ gauge field (v_μ), complex scalar σ_a
    a twisted chiral superfield ⟶ $\Sigma_a := \frac{1}{\sqrt{2}}\bar{D}_+ D_- V$:⇒ gauge field strength (= curvature)
  - Lagrangian density :

$$L = \int d^4\theta \Big(\underbrace{\sum_i \bar{\Phi}_i e^{2\sum_a Q_{ia} V_a} \Phi_i}_{\text{Kinetic term}} - \sum_a \underbrace{\frac{1}{2e_a^2}\bar{\Sigma}_a\Sigma_a}_{\text{Yang-Mills term}} \Big)$$

$t_a = r_a - i\theta_a$

$$- \frac{1}{2}\sum_a \left(\int d\theta^+ d\bar{\theta}^- \, t_a \Sigma_a \pm c.c. \right)$$ ⟵ Fayet-Iliopoulos term (Theta-term, topological term)

$$- \int d\theta^+ d\theta^- \, W(\Phi_i) \;-\; c.c.$$ ⟵ superpotential te[rm]

(+ perhaps twisted super-potential term $\widetilde{W}(\Sigma)\cdots$)

  - potential function of bosonic fields :

$$U_{(r_a, e_a)}(\phi_i, \sigma_a) = \sum_a \frac{e_a^2}{2}\Big(\sum_i Q_{ia}|\phi_i|^2 - r_a\Big)^2$$
$$+ \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2 + 2\sum_{a,b} \bar{\sigma}_a \sigma_b \sum_i Q_{ia} Q_{ib} |\phi_i|[^2]$$

  - Wilson's theory space, phase structure

• $U^{-1}(0)/G$: vacuum manifold
• effective field theory "expanded around vac[uum]"
  ⟶ nonlinear σ-model on vacuum m[anifold]
  ⟶ Landau-Ginzburg model
  ⋮

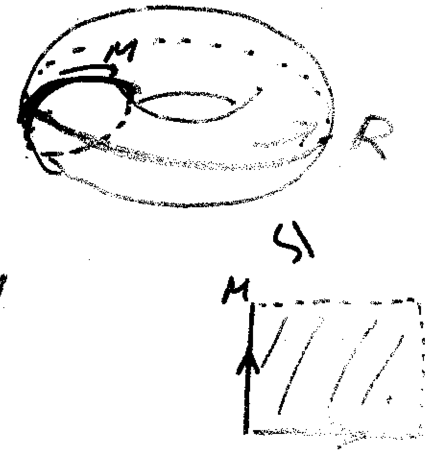
• family of 2d QFT descended from the gauged linear σ-model
• Nature of the 2d QFT ⇒ phase structure on the theory space.

⟵ Wilson's theory space = {coupling constants (t_a, e_a)} (+ renormalization pres[cription] e.g. energy cut-off …)

[② 6. [mirror principle] — ⟨its stringy origin (2)⟩

- Topological twists of $d=2$, $N=(2,2)$ SQFT.

(At the group level

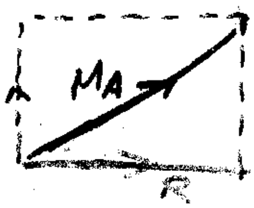


2-torus

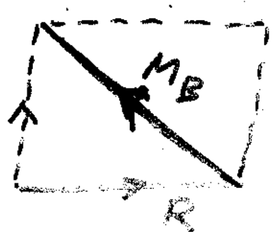
  - $d=2$, $N=(2,2)$ SUSY algebra $\supset$ 2d rotation M $\rightsquigarrow$ R-symmetries (automorphism of SUSY generators Q's)

  - "topological twist" = re-embedding of M by twisted along R.

A-twist / B-twist



(diagonal embedding)



(anti-diagonal embedding)

  - Effect to the original QFT:
    - no local difference, Lagrangian, ... all remain the same!
    - global meaning of fields modified (... now determined by either M_A or M_B)

  $\Longrightarrow$ Theory: 2d metric independent (i.e. "topological")
    - can be defined on general Riemann surface.

- A-twisted moduli space/stack from gauged linear σ-model

  gauged linear σ-model ($d=2$, $N=(2,2)$)

  $\xRightarrow{\text{A-twist}}$ residual SUSY generator: Q_-, $\overline{Q}_+$

  $\rightarrow$ realized as differential operators acting on fields

  $\Longrightarrow$ SUSY field configuration: $\{Q_-, \cdot\} = \{\overline{Q}_+, \cdot\} = 0$

  $\Longrightarrow$ $d=2$ vortex configuration

  $\Longrightarrow$ {global sections in $\oplus_i L_i$ on Riemann surface}/~

  $\xrightarrow{\text{Cox's functor for smooth toric vars}}$ Another compactification of space of maps into string target-space

[Morrison-Plesser], [C.H.L.-L.-Yau: math.AG/0212316]

> (2): Cheat-sheet for string theorists in the moduli-/stacky world.

- moduli problem: Want to parameterize geometric objects (e.g. stable maps, vector bundles, coherent sheaves, ...) by some "space" that has good universal property with respect to the moduli problem.
- Difficulty: geometric object usually has nontrivial automorphisms

$\Longrightarrow$ generalize the concept of local coordinate chart, gluing of local charts, ...

$\Longrightarrow$ stack! (cf. Grothendieck's descent.)

[Mumford]
[Laumon-Moret-Bailly]
[Kresch]

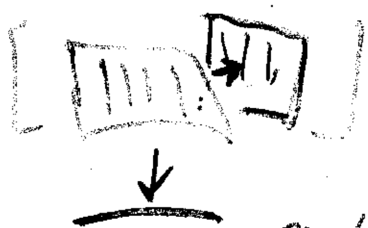
I. • Deligne-Mumford stack / Artin stack: has global atlas.

$\Longrightarrow$ Can DO the usual algebraic geometry if careful enough.

- [illegible] stable map $\Longrightarrow$ DM stack.
  coherent sheaf $\Longrightarrow$ Artin stack

III. "Self-correction" of stack associated to a moduli problem

moduli problem $\xrightarrow{\text{deformation theory}}$



E: obstruction complex $\in D$[illegible]

moduli stack $\mathcal{X}$

$\Longrightarrow$ $[\mathcal{X}]^{vir}$: virtual fundamental class

$N^{vir}_{\mathcal{Y}}\mathcal{X}$: virtual normal bundle / sheaf of a substack $\mathcal{Y}$ in stack $\mathcal{X}$.

IV. Deligne-Mumford stack is very close to the usual object in AG (i.e. variety, scheme). Complex manifold ... Essentially everything goes through if careful about the finite group factor

I ③ (i) [mirror principle] - < hoco mirror principle incorporates all >

- mirror principle diagram: [Lian-K. Liu-Yau: M.P I - IV]

  the moduli space, bundles, morphisms, group actions and special fixed-point locus in the setting of mirror principle

$$
\begin{tikzcd}
V, b \arrow[d] & U_d := \rho^* V_d \arrow[d] & V_d := \rho_* ev^* V \arrow[d] & \mathcal{U}_d := \pi^* V_d \arrow[d] & \\
X & \overline{M}_{0,1}(X,d) \arrow[l, "ev"'] \arrow[r, "\rho"] & \overline{M}_{0,0}(X,d) & M_d \arrow[l, "\pi"'] \arrow[r, "\varphi"] & W_d \\
 & & & F_0 \arrow[u, hook] \arrow[ul, "\rho"] \arrow[r, "ev^Y"] \arrow[d, "ev^X"] & Y_0 \arrow[u, hook] \arrow[d] \\
 & & & X \arrow[r, "\tau"] & Y
\end{tikzcd}
$$

(over $\mathbb{C}$)

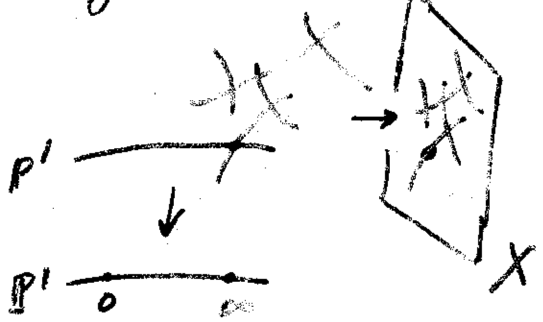
special S^1-fixed point locus

$S^1 \circlearrowright M_d$, $S^1 \circlearrowright W_d$

- V: bundle$/X$, b: multiplicative characteristic class.
- $\overline{M}_{0,1}(X,d)$, $\overline{M}_{0,0}(X,d)$, $M_d := \overline{M}_{0,0}(\mathbb{P}^1 \times X, (1,d))$: moduli space/stack of stable maps
- ev, ev^X, ev^Y : evaluation maps
  ρ : forgetful map
  π : contracting morphism
  φ : collapsing morphism

  W_d : linearized moduli space motivated/modified from the A-twisted moduli space of gauged linear σ-model.
- S^1-action on M_d, W_d: induced by the S^1-action on $\mathbb{P}^1$:

  $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$
  $z \longmapsto tz$,
  $t \in S^1 \subset \mathbb{C}^\times$
- F_0 :



$$
\begin{tikzcd}
F_0 \simeq \overline{M}_{0,1}(X,d) \arrow[dr, "\rho"'] \arrow[r, "\rho"] & \overline{M}_{0,0}(X,d)
\end{tikzcd}
$$

- $X \xrightarrow{\tau} Y$: $Y :=$ product of projective spaces
  τ : embedding such that $H^2(X) \simeq H^2(Y)$.
- $H^2(X)$: generated by pull-back hyperplane classes : $H_1, H_2, \cdots, H_m$

[3](a) [mirror principle] < how mirror principle incorporates all >

- Fundamental objects/concepts in the mirror principle setting:
  - The fundamental Euler series from (V, b):

$$A^{V,b}(t) = e^{-H\cdot t/\alpha} \sum_d A_d\, e^{d\cdot t}, \qquad \text{cf. Kähler moduli parameter}$$

where: $H := (H_1, \cdots, H_n)$, $t := (t_1, \cdots, t_n)$;

$d := (d_1, \cdots, d_n)$ ← (degree of world-sheet instanton)

$H\cdot t = \sum_i H_i t_i$; $d\cdot t = \sum_i d_i t_i$.

$\alpha = S^1\text{-weight factor} = c_1(\mathcal{O}_{BC^\times}(1))$, $BC^\times = \mathbb{C}P^\infty$.

$$A_d := ev_*^X\left(\frac{\rho^* b(V_d) \wedge [F_0]^{vir}}{e_{S^1}(N^{vir}(F_0/M_d))}\right) \in A_X^{S^1}(X).$$

  - From $A^{V,b}(t)$ to intersection numbers on $\overline{M}_{0,0}(X,d)$.

Define $K_d^{V,b} := \int_{[\overline{M}_{0,0}(X,d)]^{vir}} b(V_d)$ ← related to Gromov-Witten invariants, instanton number.

then $\int_X e^{-H\cdot t/\alpha} A_d = \alpha^3 (2 - d\cdot t) K_d$ [MP III, Thm 3.(ii)]

  - Three technical cores in [Mirror Principle I – IV]
    - general theory of Euler data, Euler series, linked Euler series on the linearized moduli space
    - Various identities relating push/pull and restriction/localization of equivariant (co)homology classes.
    - Construction of the collapsing morphism $\varphi: M_d \to W_d$

I ②-(3) [mirror principle] — < how mirror principle incorporates all >

- What is "mirror principle"?

Assume X: balloon manifold (e.g. toric variety, homogeneous space, ...),

$T^* \curvearrowright X$.

Then: • the above setting can be promoted to $S^1 \times T^*$-equivariant setting.

•

[use only b, splitting type of V, some T^*-action data,]

$$\begin{tikzcd}
(V, b) \arrow[d] \arrow[drr] & & \\
A^{V,b}(t) \arrow[rr, leftrightarrow, "\exists\, \tilde{t} = \tilde{t}(t)", "\text{such that}"'] & & B(\tilde{t})
\end{tikzcd}$$

$A^{V,b}(t) = B(\tilde{t}(t))$.

$B(\tilde{t})$: linked to $A^{V,b}(t)$

prepotential $\Downarrow$ $\Downarrow$ prepotential

$\mathcal{F}$ in $\Pi A[H]$ <- - - - - - - - - - - -> $\tilde{\mathcal{F}}$ in $\Pi B[H]$

MIRROR TRANSFORMATION

The principle says that,

For quite general class of $\begin{array}{c} V \\ \downarrow \\ X \end{array}$,

$A^{V,b}(t)$ (i.e. instanton #'s) can be computed via an Euler series $B(\tilde{t})$ readily obtainable & linked to $A^{V,b}(x)$ and a mirror transformation

$\tilde{t} = \tilde{t}(t)$.

(i.e. $A^{V,b}(t) = B(\tilde{t}(t))$.)

I ③ [mirror principle] ⟨the special role of $HG[\mathbb{1}]^X(t)$⟩

- Recall:

$$
\begin{tikzcd}
\cdots & M_d \arrow[l, "\pi"'] \arrow[r, "\varphi"] & W_d \\
& F_0 \arrow[u, hook] \arrow[r, "ev^Y"] \arrow[d, "ev^X"] & Y_0 \arrow[u, hook] \arrow[d, "\wr"] \\
& X \arrow[r] & Y
\end{tikzcd}
$$

For $b = 1$, the bundle V over X is irrelevant.

Define
$$\mathbb{1}_d := A_d^{V,1} = ev_*^X\left(\frac{[F_0]^{vir}}{e_{S^1}(N_{F_0}^{vir} M_d)}\right)$$

$$HG[\mathbb{1}]^X(t) := A^{V,1}(t)$$
$$= e^{-H\cdot t/\alpha} \sum_d \mathbb{1}_d \, e^{d\cdot t}.$$

- Properties of $HG[\mathbb{1}]^X(t)$.
  - independent of V, and hence a universal object associated to X.
  - can be employed to construct linked Euler series which in turn computes K_d for a general b.
  - related to period integral.
    $HG[\mathbb{1}]^X(t)$ enters the Hori-Vafa formula.
    for X grassmannian manifold.

II a ① [Hori and Vafa's derivation of the Hori-Vafa formula]

<mirror symmetry in the sense of Hori and Vafa>

- Mirror symmetry, 1991 [Greene-Plesser]

  "Two Calabi-Yau manifolds M and $\tilde{M}$ constitut
  a mirror pair if they correspond to the
  same conformal field theory, and the associat
  of geometric objects on the two manifolds t
  fields in the conformal field theory differs by
  a reversal of the charges under the
  left-moving $U(1)$ of all fields."

- mirror symmetry, 2000 [Hori-Vafa]

  "mirror symmetry"

  = an equivalence of two $d=2$, $N=(2,2)$ SQFT
  such that $Q_- \longleftrightarrow \bar{Q}_-$, $F_V \longleftrightarrow F_A$, $Z \longleftrightarrow \tilde{Z}$.

- Set $q_L = \frac{1}{2}(F_A - F_V)$, $q_R := -\frac{1}{2}(F_A + F_V)$,

  then Hori-Vafa $\Longleftarrow$ Greene-Plesser.

[Recall] $Q_-, \bar{Q}_-$: left supercharges
F_V, F_A : generators of R-symmetries
$Z, \tilde{Z}$: central charges
} in $d=2$ $N=(2,2)$ SUSY algebra

II a. (2) [Hori-Vafa's derivation of Hori-Vafa formula]

< mirror of a gauged linear σ-model >

Rk: ALL superfield in the sense of $d=2$, $N=(2,2)$ SQFT.

PERTURBATIVE LEVEL

$$L' = \int d^4\theta \left(e^{2QV+B} - \frac{1}{2}(Y+\bar{Y})B \right),$$

V : vector
B : real
Y : twisted chiral
$Q \in \mathbb{Z}$

$$Z := \int \mathcal{D}Y \mathcal{D}\bar{Y} \mathcal{D}V \mathcal{D}B \, e^{-\int d^2x \, L'}$$

$\int \mathcal{D}Y\mathcal{D}\bar{Y} \cdots$ + field redefinition

$\int \mathcal{D}B$ + field redefinition

$$Z = \int \mathcal{D}V \mathcal{D}\Phi \mathcal{D}\bar{\Phi} \, e^{-\int d^2x \, L}$$

$$L = \int d^4\theta \, \bar{\Phi} e^{2QV} \Phi$$

$$Z = \int \mathcal{D}Y \mathcal{D}\bar{Y} \mathcal{D}\Sigma \mathcal{D}\bar{\Sigma} \, e^{-\int d^2x \, \tilde{L}}$$

$$\tilde{L} = -\frac{1}{2}\int d^4\theta \, (Y+\bar{Y}) \log(Y+\bar{Y})$$
$$+ \frac{1}{2}\int d^2\tilde{\theta} \, \Sigma Q Y$$
$$+ \frac{1}{2}\int d^2\bar{\tilde{\theta}} \, \bar{\Sigma} Q \bar{Y}$$

QUANTUM

In full:
gauge kinetic term
topological term
$-\frac{1}{2}\int d^2\tilde{\theta} \, t\Sigma$

$\xleftrightarrow[\text{EXACT}]{\text{QUANTUM PERTURBATIVE}}$

$$\tilde{L} = \cdots + \frac{1}{2}\int d^2\tilde{\theta} \, \underbrace{\Sigma(QY - t)}_{\tilde{W}} + c.c.$$

NON-PERTURBATIVE CORRECTION

INSTANTON FIELD CONFIGURATION
$\sigma = 0$
$D_{\bar{z}}\phi = 0$
$F_{12} = e^2(|\phi|^2 - r_0)$
(vortex)

$\xleftrightarrow{\text{INDEX 1 VORTEX} \qquad \text{VERTEX OPERATOR}}$

$$\tilde{W} = \Sigma(QY - t) + e^{-Y}$$

GENERAL FORM: twisted superpotential

$$\tilde{W} = \sum_a \Sigma_a \left(\sum_i Q_{ia} Y_i - t_a \right) + \sum_i e^{-Y_i}$$

II a [Hori + Vafa's derivation of Hori-Vafa formula]

⟨Non-abelian generalization of GLSM⟩

- MAIN IDEA:

group data. $\begin{cases} G: & \text{gauge group} \\ \mathfrak{g}: & \text{Lie algebra of } G. \\ R: & \text{representation of } G \text{ (and hence } \mathfrak{g}\text{)} \end{cases}$

V : vector superfield

Φ : chiral superfield

- "Thicken V to $\mathfrak{g}$-valued $\mathbf{V}$
- "Thicken Φ to R-valued $\boldsymbol{\Phi}$
- "minimal work principle:
  ALL the rest kept the same except for necessity, e.g. twisting terms.

- Remark. In math. term:

  Thicken $= \otimes\,\cdot$

- Appropriate choice of superpotential in GLSM $\Longrightarrow$ vacuum manifold = homogeneous space or complete intersection therein.

II 9 ⑦ [Hori and Vafa's derivation of Hori–Vafa formula]

⟨To get $Gr_k(\mathbb{C}^n)$ as vacuum manifold⟩

- $\mathfrak{g} = \mathfrak{u}(k)$, $G = U(k)$, $\mathcal{R} = M_{k\times n} = \{k\times n$ matrix ($\mathbb{C}$

  $G \curvearrowright \mathcal{R}$: left matrix multiplication.

- one $\mathfrak{u}(k)$-valued vector multiplet $\mathbf{V}$

  one $M_{k\times n}$-valued chiral multiplet $\mathbf{\Phi}$

  $\mathbf{\Sigma}$: $\mathfrak{u}(k)$-valued twisted chiral superfield

  $$= \frac{1}{2}\{e^{-\mathbf{V}}\bar{D}_+ e^{\mathbf{V}}, e^{-\mathbf{V}} D_- e^{\mathbf{V}}\}$$

- The $U(k)$-gauged linear σ-model

$$L = \int d^4\theta \left(Tr(\mathbf{\Phi}^\dagger e^{2\mathbf{V}} \mathbf{\Phi}) - \frac{1}{2e^2} Tr(\mathbf{\Sigma}^\dagger \mathbf{\Sigma}) \right)$$

$$- \frac{1}{2}\left(\int d^2\tilde{\theta}\; t \cdot Tr\, \mathbf{\Sigma} + c.c. \right), \quad t = r - i\theta \in \mathbb{C}.$$

- phase structure.

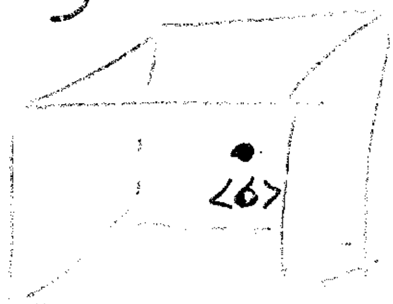
$$U(r) = \frac{e^2}{2} Tr\left[(\phi^\dagger\phi - r\mathbb{1}_{k\times k})^2 \right] + Tr\left[(\sigma\cdot\phi)^\dagger (\sigma\cdot\phi) \right]$$

Higgs phase ($\sigma \equiv 0$) ↙ ↘ Coulomb phase ($r = 0$, $\phi \equiv 0$)

$$U^{-1}(0)/U(k)$$

$$= \{\phi^\dagger\phi = r\mathbb{I}_{k\times k}\}/U(k)$$

$$= Gr_k(\mathbb{C}^n).$$



II a ⑤ [Hori and Vafa's derivation of Hori-Vafa formula]

⟨ Further analysis of Coulomb phase. ⟩

- The Coulomb phase of the $U(k)$-gauged linear σ-model.

  $r=0$, $\phi\equiv 0$, σ arbitrary

  - Idea of further analysis:

    choose σ CONSTANT; consider the field theory expanded around it.

<u>Result</u>:

For generic σ (i.e. $\sigma \in$ interior of Weyl chamber) up to a Weyl group action,

$\Longrightarrow$ $d=2$, $N=(2,2)$ gauged linear σ-model

Weyl $U(k)$ $= Sym_k$.

with gauge group $Stab(\sigma) = T^k = U(1)\times\cdots$ $\simeq$ maximal torus of ...

(cf. Analogue in Seiberg-Witten theory)

corresponds to T^k $\Big\{$ kn chiral superfields: ϕ_α, $\alpha=(j\ell)$, $j=1,\cdots,k$, $\ell=1,\cdots,n$

k vector superfields: V_i, $i=1,\cdots,k$

charges: $\boxed{Q_{\alpha i} = Q_{(j\ell),i} = \delta_{ij}.}$

Hori-Vafa $\Longrightarrow$ MIRROR TRANSFORM

$$\widetilde{L} = \cdots$$

$$\widetilde{W} = \sum_i \Sigma_i\Big(\sum_\alpha Q_{\alpha i} Y_\alpha - t\Big) + \sum_\alpha e^{-Y_\alpha}$$

$$\widetilde{W}_{ind} = \widetilde{W}/\!/ sym_k$$

$\Longrightarrow$ Period integral $= \int \prod_i d\Sigma_i \prod_\alpha dY^\alpha \prod_{i<j}(\Sigma_i-\Sigma_j)\, e^{-\widetilde{W}}$

$$= \prod_{i<j}\Big(\frac{\partial}{\partial t_i}-\frac{\partial}{\partial t_j}\Big)\Big|_{t_i=t_j=t} \int \prod_\alpha dY^\alpha \underbrace{\prod_i \delta\Big(\sum_\alpha Q_{\alpha i} Y_\alpha - t_i\Big)}\, e^{-\sum_\alpha e^{\cdots}}$$

$$\mathbb{P} := \prod_k \mathbb{P}^{n-1} = \Big\{\sum_\alpha Q_{\alpha i} Y_\alpha = t_i\Big\}\Big/\underbrace{U(1)\times\cdots\times U(1)}_{k\text{-times}}$$

[The mirror principle statement]

- In summary: $U(k)$-gauged linear σ-model

$$
\begin{array}{ccc}
 & U(k)\text{-gauged linear } \sigma\text{-model} & \\
\swarrow & & \searrow \\
Gr_k(n) & & d=2,\ N=(2,2) \text{ gauged linear } \sigma\text{-model} \\
\text{Higgs phase} & & \text{Coulomb phase}
\end{array}
$$

Higgs phase $\swarrow$; Coulomb phase $\big\downarrow$ mirror transform

$$Z_{NLSM(Gr_k(n))} \longleftarrow\text{-----}\longrightarrow \prod_{i<j}\left(\frac{\partial}{\partial t_i}-\frac{\partial}{\partial t_j}\right)\Bigg|_{t_i=t_j=t} \quad \left(\text{period integral [illegible] over } \underbrace{\mathbb{P}^{n-1}\times\cdots\times\mathbb{P}^{n-1}}_{k} \right)$$

$\mathbb{P}$!!

- Mirror principle statement:
  - Notation: [illegible] $\to Gr_k(\mathbb{C}^n)$ Chern roots of Q: $x_1, \cdots, x_k$
    - $A_*(Gr_k(\mathbb{C}^n)) = \langle \sigma_1, \cdots, \sigma_k \rangle$, σ_i: elementary symmetric poly of [illegible]
      σ_1: generator of divisor class of $Gr_k(\mathbb{C}^n)$.
    - hyperplane class of $\mathbb{P}$ from i-th factor: x_i

- Then:

$$HG[\mathbb{1}]^{Gr_k(\mathbb{C}^n)}(t)$$

$$= e^{(k-1)\pi\sqrt{-1}\,\sigma_1/\alpha}\,\frac{1}{\prod_{i<j}(x_i-x_j)}\prod_{i<j}\left(\alpha\frac{\partial}{\partial x_i}-\alpha\frac{\partial}{\partial x_j}\right)\Bigg|_{t_i=t+(k-1)\pi\sqrt{-1}} HG[\mathbb{1}]^{\mathbb{P}}(t_1,\cdots

II (a)① [mirror principle proof of Hori-Vafa formula]

- Let $X = Gr_r(\mathbb{C}^n)$. Recall $H_2(Gr_r(\mathbb{C}^n), \mathbb{Z}) \cong \mathbb{Z}$.

$$HG[\mathbb{1}]^X(t) = e^{-Ht/\alpha} \sum_{d=0}^{\infty} ev_*^X\left(\frac{[F_0]}{e_{S^1}(\nu_{F_0} M_d)}\right) \cdot e^{dt}$$

Thus :

$$\boxed{\text{Computation of } \quad ev_*^X\left(\frac{[F_0]}{e_{S^1}(\nu_{F_0} M_d)}\right)}$$

$$\Downarrow$$

$$\boxed{\text{proof of Hori-Vafa formula}}$$

- Main idea :

$$\boxed{\text{computation on } M_d = \overline{M}_{0,0}(\mathbb{C}P^1 \times X, (1,d))}$$

(moduli space of curves)

$$\Uparrow$$

$$\boxed{\text{computation on } W_d, \text{ linearized moduli space}}$$

$$\Uparrow$$

$$\boxed{\text{computation on } Quot_{(d)}, \text{ Quot-schemes}}$$

$$:= Quot_{P(t)=(n-r)t+d+(n-r)}\left(\mathcal{E}^n = \mathcal{O}_{\mathbb{P}^1}^{\oplus}\right.$$

II 9.② [mirror principle proof of Hori-Vafa formula]

Outline

- Extended mirror principle diagram
  (Since bundles are irrelevant for $b=1$, they're all omitted)

$$\begin{array}{ccccccc}
 & \overline{M}_{0,0}(\mathbb{CP}^1\times X, (1,d)) & & & \text{induced by Plücker embedding} & & \\
 & \| & & & & & \\
\cdots \to & M_d & \xrightarrow{\varphi} & W_d & \xleftarrow{\psi} & \mathrm{Quot}_{(d)} & \left\{ \begin{array}{l} S^1\text{-equivariant} \\ \text{diagram} \end{array} \right. \\
 & \cup & & \cup & & \cup & \\
\text{distinguished } S^1\text{-fixed-pt component} & F_0 & \xrightarrow{\omega^Y} & Y_0 & \xleftarrow{g} & E_0 = \bigcup_s E_{0s} & \\
 & \downarrow & & \| & & & \\
Gr_r(\mathbb{C}^n) = & X & \xrightarrow[\text{Plücker embedding}]{\tau} & Y \simeq \mathbb{P}^{\binom{n}{r}-1} & & &
\end{array}$$

- $\mathrm{Quot}_{(d)} = \mathrm{Quot}_{P(t)}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n})$
  $P(t) = (n-r)t + d + (n-r)$
  smooth, irreducible, rational, projective, $\dim = dn + r(n-r)$

- ψ : S^1-equivariant resolution of singularities of $\varphi(M_d)$.

  codim Sing $\varphi(M_d) \geq 2$

  $\Longrightarrow \quad \varphi_*[M_d] = \psi_*[\mathrm{Quot}_{(d)}]$ in $A_*(W_d)$.

- (To be shown.)

  All the S^1-fixed-point components in $\mathrm{Quot}_{(d)}$ are naturally fibered over $X = Gr_r(\mathbb{C}^n)$.

III.2.③ [Mirror principle proof of Hori-Vafa formula]

- A consequent further detail:

$$
\begin{tikzcd}
M_d \arrow[r, "\varphi"] & W_d & \mathrm{Quot}_{(d)} \arrow[l, "\psi"'] \\
F_0 \arrow[u, hook] \arrow[r, "ev^Y"] \arrow[dr, "ev^X"'] & Y_0 \arrow[u, hook] \arrow[d, hook, "\tau"] & E_0 = \bigcup_s E_{0s} \arrow[u, hook] \arrow[l, "q"'] \arrow[dl, "p"] \\
 & X = Gr_r(\mathbb{C}^n) &
\end{tikzcd}
$$

In particular:

$$
(*):\quad
\begin{tikzcd}
M_d \arrow[r, "\varphi"] & W_d & \mathrm{Quot}_{(d)} \arrow[l, "\psi"'] \\
F_0 \arrow[u, hook, "i"] \arrow[r, "ev^X"] & X \arrow[u, hook, "j"] & E_0 = \bigcup_s E_{0s} \arrow[u, hook, "k"] \arrow[l, "p"']
\end{tikzcd}
$$

- Functorial localization formula. [MP III. Sec. 2.1] to the left and the right square of (*)

$$
\Rightarrow\ ev^X_*\left(\frac{[F_0]}{e_{S^1}(\nu_{F_0} M_d)}\right) = \frac{j^*\varphi_*[M_d]}{e_{S^1}(\nu_X W_d)} = \frac{j^*\psi_*[\mathrm{Quot}_{(d)}]}{e_{S^1}(\nu_X W_d)}
$$

$$
= \sum_s p_*\left(\frac{[E_0]}{e_{S^1}(\nu_{E_{0s}} \mathrm{Quot}_{(d)})}\right) \quad \text{(Expression in equivariant Chow group}
$$

$$
\overset{P.D.}{\rightsquigarrow} \sum_s p_*\left(\frac{1}{e_{S^1}(\nu_{E_{0s}} \mathrm{Quot}_{(d)})}\right) \quad \text{Expression in equivariant cohomology}
$$

III a. (4) [Mirror principle proof of Hori-Vafa formula]

- Conclusion: (in equivariant cohomology form)

$$\begin{array}{ccc} E_0 = \bigcup_s E_{0s} & \hookrightarrow & \mathrm{Quot}_{(d)} \\ \downarrow & & \\ X = Gr_r(\mathbb{C}^n) & & \end{array}$$

$$HG[\mathbb{1}]^X(t) = e^{-Ht/\alpha} \sum_{d=0}^{\infty} \left(\sum_s p_* \left(\frac{1}{e_{S^1}(\nu_{E_{0s}} \mathrm{Quot}_{(d)})} \right) \right) e^{dt} .$$

- Technical details to be understood:
  - E_{0s} : distinguished S^1-fixed-point components in $\mathrm{Quot}_{(d)}$.
  - $e_{S^1}(\nu_{E_{0s}} \mathrm{Quot}_{(d)})$: equivariant Euler class of normal bundle.
  - p_* : the push-forward from $E_0 = \bigcup_s E_{0s}$ to X.

- Once these are understood enough, one can check the Hori-Vafa formula directly.

(1)(a)$_1$ [S^1-invariant subsheaf of $\mathcal{E}^n$]

- $\mathcal{E}^n$: rank-n trivial bundle over $\mathbb{P}^1$
  with trivialization $\mathcal{E}^n = \mathbb{P}^1 \times \mathbb{C}^n = \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^n$

- $S^1 \curvearrowright \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ $\Longrightarrow$ $S^1 \curvearrowright \{0 \to \underset{\text{rank } r}{\mathcal{V}} \hookrightarrow \mathcal{E}^n\} \simeq \{\mathcal{E}^n \to \mathcal{E}^n/\mathcal{V} \to 0\}$
  $z \longmapsto t \cdot z$
  $(\mathcal{E}^n/\mathcal{V} : \text{rk} = n-r,\ \text{degree} = d)_{P = P(t) = (n-r)t + d + (n-r)} =: \mathrm{Quot}_P(\mathcal{E}^n)$

- Characterization of S^1-invariant subsheaves: $\mathcal{V} \hookrightarrow \mathcal{E}^n$.
  - $\hat{\mathcal{V}}$:= saturation of $\mathcal{V}$ $\rightsquigarrow$ constant subsheaf
    w.r.t. the trivialization of $\mathcal{E}$
  - On affine charts :
    $U_0 = \mathbb{P}^1 - \{\infty\} = \mathrm{Spec}\,\mathbb{C}[z]$ (similar for $U_\infty = \mathbb{P}^1 - \{0\} = \mathrm{Spec}\,\mathbb{C}[w]$)
    - canonical form under constant re-trivialization of $\mathcal{E}$

$$\mathcal{V}|_{U_0} \simeq \Big(\bigoplus_{i=1}^{r} \mathbb{C}[z]\, z^{\alpha_i}\Big)^{\sim} \simeq \bigoplus_{i=1}^{r} \mathcal{I}_{\alpha_i \cdot (0)}|_{U_0} \quad (\text{ideal sheaf})$$

$$\hookrightarrow \hat{\mathcal{V}}|_{U_0} \simeq \Big(\bigoplus_{i=1}^{r} \mathbb{C}[z]\Big)^{\sim}$$

where $0 \le \alpha_1 \le \cdots \le \alpha_r$.

- Corresponding flag in $\mathbb{C}^n$:

$$[0 \le \alpha_1 \le \cdots \le \alpha_r] \Longleftrightarrow \begin{bmatrix} & 0 \le a_1 < \cdots < a_k \\ \text{multiplicity}: & m_1 \quad \cdots \quad m_k \end{bmatrix}$$

$$\mathcal{V}|_{U_0} \Longrightarrow \bigoplus_{i=1}^{r} \mathcal{I}_{a_i \cdot (0)}\Big|_{U_0} = \bigoplus_{I=1}^{k} \Big(\mathcal{I}_{a_I \cdot (0)}^{\oplus m_I}\Big)$$

$$\underset{(\text{saturate})}{\Longrightarrow} \bigoplus_{I=1}^{k} \Big(\text{constant subsheaf of rank } m_I \text{ in } \mathcal{E}^n|_{U_0}\Big)$$

$$\Longrightarrow \text{flag}: \ V^{m_1} \subset V^{m_1+m_2} \subset \cdots \subset V^r$$

- Topology of S^1-fixed-point component in $\mathrm{Quot}_P(\mathcal{E}^n)$

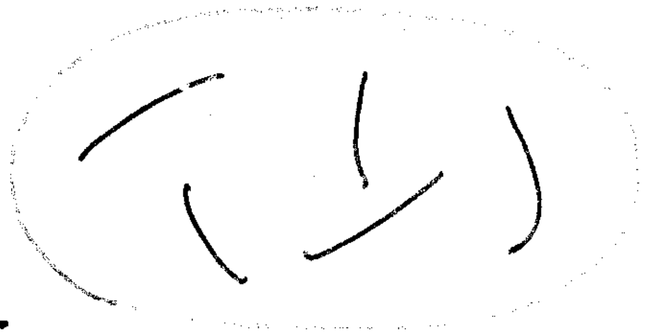
  - $$\left\{ \begin{array}{c} S^1\text{-invariant} \\ \text{subsheaf of } \mathcal{E}^n \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} S^1\text{-fixed-point} \\ \text{in } \mathrm{Quot}_P(\mathcal{E}^n) \end{array} \right\}$$

  - $$\begin{tikzcd}[column sep=small]
& & F_{\alpha_1,\cdots,\alpha_r;\,\beta_1,\cdots,\beta_r} \ (\text{fiber product}) \\[-20pt]
& & \mathcal{V}^\bullet \arrow[phantom]{u}[rotate=90]{\in} \arrow[dl, dash] \arrow[dr, dash] & & \\
Fl_{m_1,m_1+m_2,\cdots,r}(\mathbb{C}^n) \ni & \mathcal{V}|_{U_0} \arrow[dr, dash] & & \mathcal{V}|_{U_\infty} \arrow[dl, dash] & \in Fl_{n_1,n_1+n_2,\cdots,r}(\mathbb{C}^n) \\
& & \mathcal{V}^\bullet \hookrightarrow \mathcal{V} \in Gr_r(\mathbb{C}^n) & &
\end{tikzcd}$$

    where $0 \leq \alpha_1 \leq \cdots \leq \alpha_r \longleftrightarrow 0 \leq a_1 < \cdots < a_s$, multi: $m_1 \quad \cdots \quad m_s$

    $0 \leq \beta_1 \leq \cdots \leq \beta_r \longleftrightarrow 0 \leq b_1 < \cdots < b_t$, multi: $n_1 \quad \cdots \quad n_t$

    $$(\alpha_1 + \cdots + \alpha_r) + (\beta_1 + \cdots + \beta_r) = d$$

  - 

    $\mathrm{Quot}_P(\mathcal{E}^n)$

    $P = P(t) = (n-r)t + d + (n-r)$

    — : $F_{\alpha_1,\cdots,\alpha_r;\,\beta_1,\cdots,\beta_r}$.

(3)-(2)(a) $[\psi : Q_d \longrightarrow W_d]$ & $[E_0 = \bigcup_s E_{0s}]$

- Notation: $Q_d := Quot_P(\mathcal{E}^n)$, $P = P(t) = (n-r)t + d + (n-r)$
- The morphism:

$$Y_0 = \mathbb{P}\left(H^0(\mathcal{O}_{\mathbb{P}^1}(d\cdot(0))) \otimes \wedge^r \mathbb{C}^n\right)$$
$$\cap$$
$$\psi : Q_d \longrightarrow W_d = \mathbb{P}\left(H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \otimes \wedge^r \mathbb{C}^n\right)$$

- Another point of view of $\mathcal{S}^r \hookrightarrow \mathcal{E}^n$ (graded ring + graded module)

$\mathbb{P}^1 = \mathrm{Proj}\, \mathbb{C}[z_0, z_1]$

$\mathcal{E}^n \rightsquigarrow \mathcal{M} := \mathbb{C}[z_0, z_1]^{\oplus n}$

$\mathcal{S}^r \rightsquigarrow \mathcal{N}_{\mathcal{S}} = \mathbb{C}[z_0, z_1] e_1 + \cdots + \mathbb{C}[z_0, z_1] e_r$, $e_i \in \mathcal{M}$

(graded submodule of $\mathcal{M}$)

- The morphism ψ:

$$\mathcal{S}^r \rightsquigarrow \mathcal{N}_{\mathcal{S}} \rightsquigarrow [e_1, \cdots, e_r] = \underbrace{[f_{ij}(z_0, z_1)]_{n\times r}}_{=: A_{\mathcal{S}}}$$

$$\xrightarrow[\text{(Plücker embedding)}]{\psi} [\, r\times r\text{-minors of } A_{\mathcal{S}} \,]$$
$$\in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \otimes \wedge^r \mathbb{C}^n) = W_d$$

- Corollary:

(1) distinguished S^1-fixed-point components in Q_d:

$$E_0 = \coprod_{\substack{0 \le \alpha_1 \le \cdots \le \alpha_r \\ \alpha_1 + \cdots + \alpha_r = d}} \underline{F_{\alpha_1, \cdots, \alpha_r ; 0, \cdots, 0}}$$

$(= F_{\alpha_1, \alpha_1+\alpha_2, \cdots, \alpha_1+\cdots+\alpha_r} (\mathbb{C}^n))$

(2) pulled-back hyperplane class:

Recall:
$$\begin{array}{ccccc} F_0 & \xrightarrow{g} & Y_0 & \xleftarrow{h} & E_0 \\ i \downarrow & & j \downarrow & & z \downarrow \\ M_0 & \xrightarrow{f} & W_d & \xleftarrow{\psi} & Q_d \end{array}.$$

κ: hyperplane class on W_d

$$z^* \psi^* \kappa = g^* j^* \kappa$$
$$= -c_1(S_k) \quad S_k : \ldots$$

III d-ⓒ [computation of $e_{S^1}(\nu_{E_{0S}} \mathrm{Quot}_{(d)})$]

Method I :

- S^1-weight system analysis.
- homogeneous bundles & representations.

Method II :

- tautological / Euler-type sequences & their filtrations.
- representing a bundle by a combination of simpler objects in K-group.

§ (1)(b). [$T_{F_{d,\ldots,p,\ldots}}(\mathrm{Quot})$: S^1-weight & multiplicity] III

- Fact: $T_{[\mathcal{E}^n \to \mathcal{E}^n/\mathcal{V}]}\mathrm{Quot} \simeq \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V})$
  ($C = \mathbb{P}^1$)

- Given $\mathcal{V} \hookrightarrow \mathcal{E}^n$ S^1-invariant,

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V}) = \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, \hat{\mathcal{V}}/\mathcal{V} \oplus \mathcal{E}^n/\hat{\mathcal{V}})$$

($\hat{\mathcal{V}}$: saturation of $\mathcal{V}$)

$$= \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, F_0) \oplus \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, F_\infty) \oplus \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\hat{\mathcal{V}})$$

($\hat{\mathcal{V}}/\mathcal{V} = F_0 \oplus F_\infty$: torsion sheaf supported on $\{0, \infty\}$)

$$= (\mathrm{I}) \oplus (\mathrm{II}) \oplus (\mathrm{III})$$

$\Longrightarrow$
- (I), (II) : problem on affine scheme
- (III) : $(\cdot)|_{U_0}$, $(\cdot)|_{U_\infty}$ problems on affine scheme + matching condition for gluing

$\Longrightarrow$ Diagrammatic rule for S^1-weight system

$$Wt_1 + Wt_2 + Wt_3$$

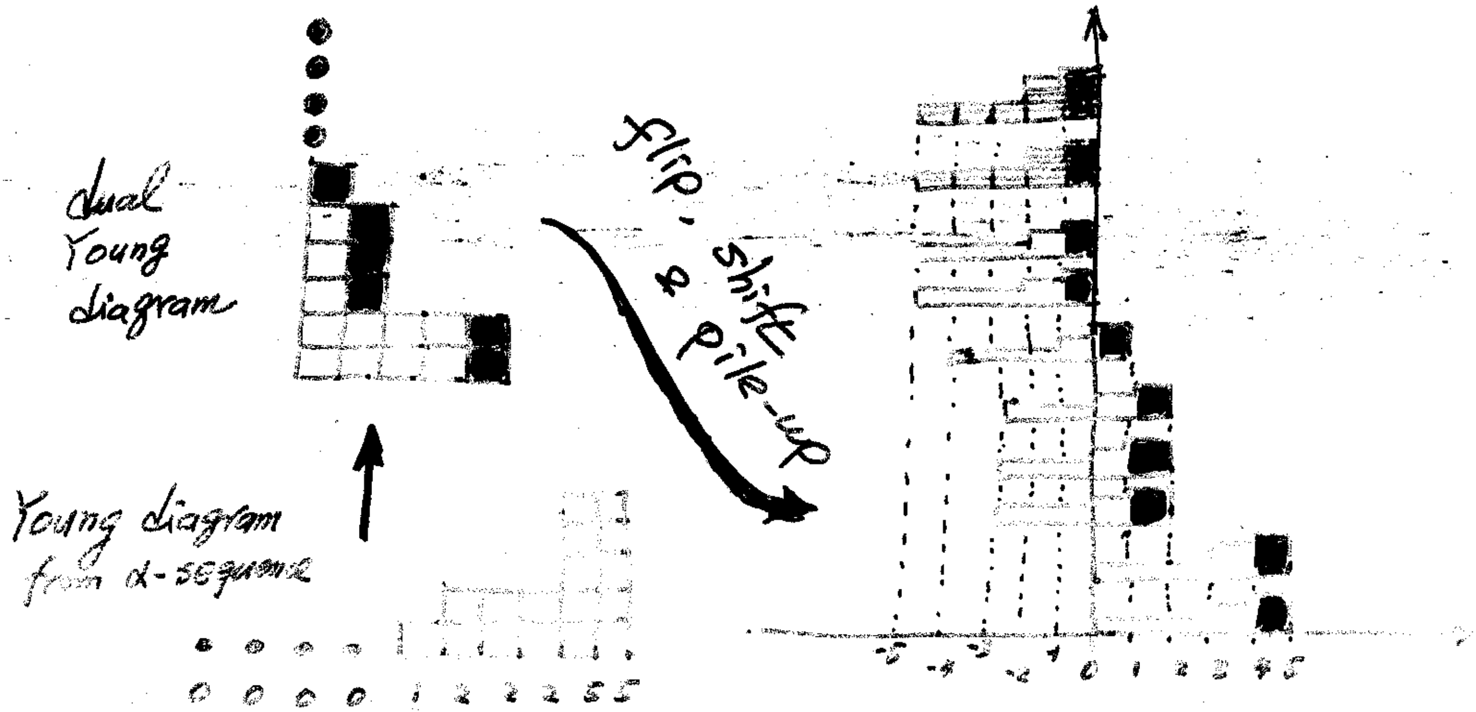
cf: (I) (II) (III)

$[\,T_{F_{\alpha_1\cdots;\beta_1\cdots}}\mathrm{Quot}:\ S^1\text{-weight \& multiplicity}\,]$

- <u>Diagrammatic rule</u>: (characteristic function of S^1-weights)

Wt_1 (Similarly for Wt_2)

Eg. $0 \le \alpha_1 \le \cdots \le \alpha_{10}$: 0, 0, 0, 0, 1, 2, 2, 2, 5, 5



$\square$

Wt_3 : $\quad Wt_3 = (n-r)\, Wt_3'$

Recall:

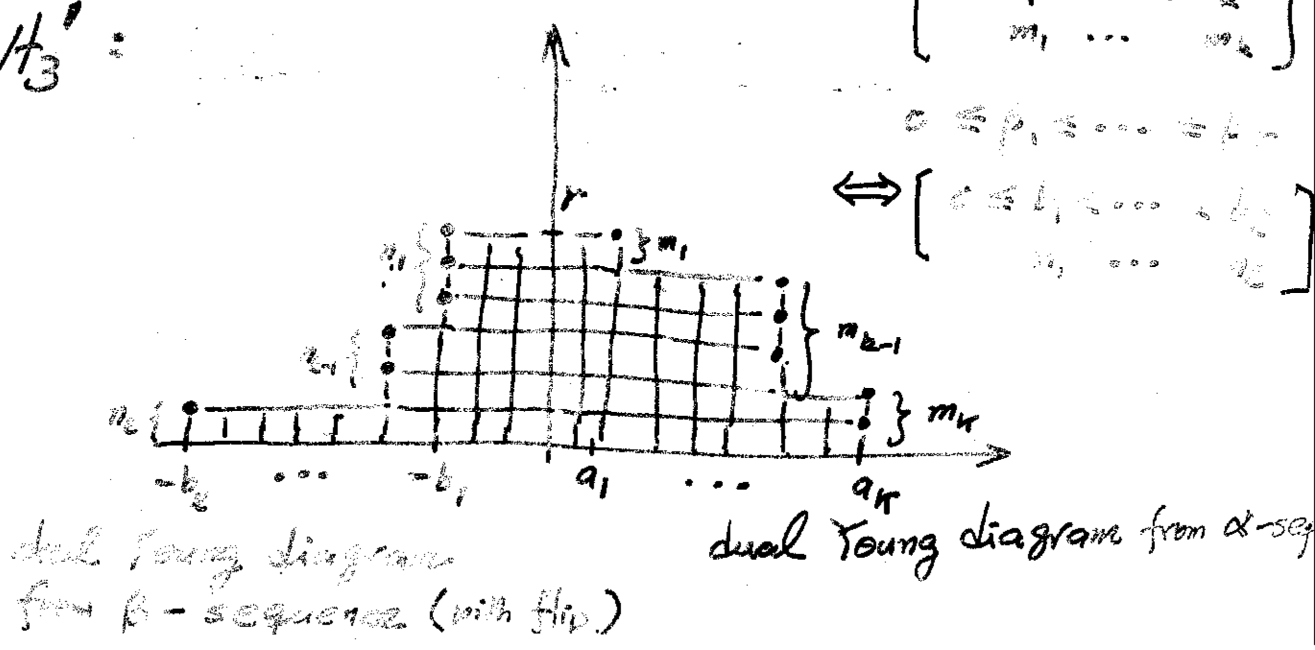
$0 \le \alpha_1 \le \cdots \le \alpha_r$

$\Leftrightarrow \begin{bmatrix} 0 \le a_1 < \cdots < a_k \\ m_1 \ \cdots \ m_k \end{bmatrix}$

$0 \le \beta_1 \le \cdots \le \beta_{n-r}$

$\Leftrightarrow \begin{bmatrix} 0 \le b_1 < \cdots < b_\ell \\ n_1 \ \cdots \ n_\ell \end{bmatrix}$

Wt_3' :



③-1(a)(b). [Equivariant Euler class $e_{S^1}(\nu_{E_{0S}} Q_d)$]

- Understanding $\nu_{E_{0S}} Q_d$:

  - $\mathcal{E}^n = \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^n \curvearrowleft GL(n;\mathbb{C}) \supset U(n)$ commutes with S^1-action

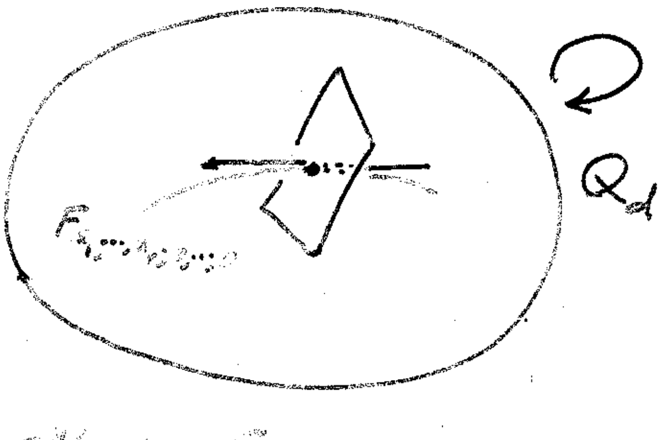
    $\Longrightarrow$ $U(n) \curvearrowright Q_d$, leaving $F_{d_1,\cdots,d_r;\beta_1,\cdots,\beta_r}$ invariant

    $\Longrightarrow$ $T_* Q_d \big|_{F_{d_1,\cdots,d_r;0,\cdots,0}}$ as homogeneous $U(n)$-bundle

    $\Longrightarrow$ determined by (Recall: $0 \le d_1 \le \cdots \le d_r$ ...)

    $$P_0 := \mathrm{Stab}_{[\mathcal{E}^n \to \mathcal{E}^n/\mathcal{V}]} \simeq U(m_1)\times\cdots\times U(m_k)\times U(n-r)$$

    $$\rightsquigarrow T_{[\mathcal{E}^n \to \mathcal{E}^n/\mathcal{V}]} Q_d,$$



[Problem of $U(n)$-action on $\mathbb{C}[z]$-modules

  - For $[\mathcal{E}^n \to \mathcal{E}^n/\mathcal{V}] \in F_{d_1,\cdots,d_r;0,\cdots,0}$

    $$T_{[\mathcal{E}^n \to \mathcal{E}^n/\mathcal{V}]} Q_d = \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V}) \circlearrowleft P_0 \text{ conjugation}$$

    $$\xrightarrow{(\cdot)|_{\mathbb{C}}} \mathrm{Hom}_{\mathbb{C}[z]}(\mathbb{C}[z]\cdot z^{d_1} \oplus \cdots \oplus \mathbb{C}[z]\cdot z^{d_r},\ \mathbb{C}[z]\cdot \bar{e}_1 \oplus \cdots \oplus \mathbb{C}[z]\,\bar{e}_r \oplus \mathbb{C}[z]^{\oplus}$$

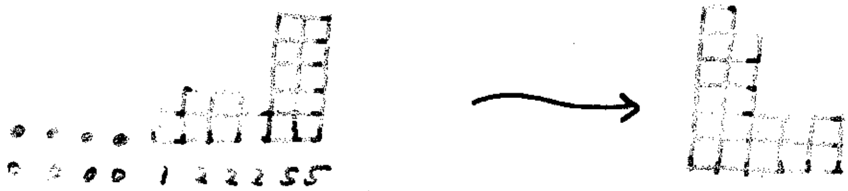
  - Diagrammatic rule for $(S^1, \underbrace{U(m_1)\times\cdots\times U(m_k)\times U(n-r)}_{\shortparallel})$-...

3-(a)-(ii) [Equivariant Euler class $e_{S^1}(\nu_{E_0 s} Q_d)$]

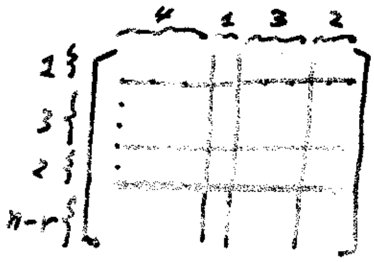
- Diagrammatic rule for (S^1, $U(m_1)\times\cdots\times U(m_k)\times U(n-r)$) weights & multiplicity

Eg: $0 \le q_1 \le \cdots \le q_n$

$0,0,0,0,1,2,2,2,5,5 \longleftrightarrow$ q_i: 0, 1, 2, 5 ; m_i: 4 1 3 2

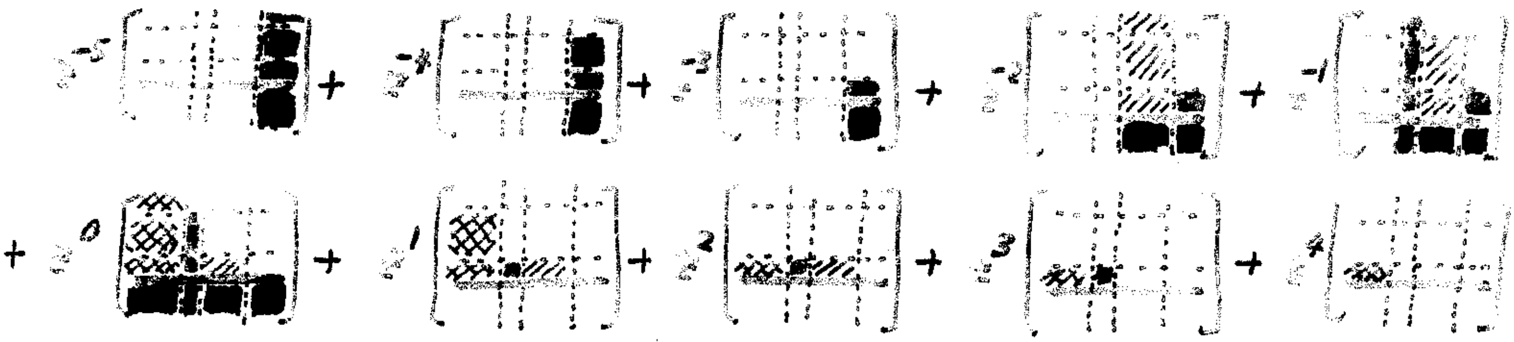


(dual Young diagram + flip)



block form determined by Young diagram (partition of d)

- Filling the block form:



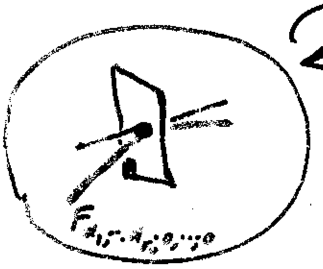
where S^1-weight : exponent of z
multiplicity: dim of filled blocks of the associated matrix

S^1-invariant P_0-module : each block
P_0 representation on each block : $\rho_{m_i} \otimes \bar{\rho}_{m_j}$
where ρ_{m_i} : defining rep'n of $U(m_i)$
$\bar{\rho}_{m_j}$: complex conjugate of ρ_{m_j}.
weight : $\amalg_{m_i m_j} \{ \lambda_i - \lambda_j' \mid 1 \le i \le m_i, 1 \le j \le m_j \}$
(with diagonal ... index)

 [Equivariant Euler class $e_{S^1}(\nu_{E_{0S}} Q_d)$]

- From (weight, multiplicity) to equivariant Euler class:

  - Recall:



$$\circlearrowleft^{GL(n,\mathbb{C})} \quad \overset{\cup}{U(n)} \supset P_0 = \mathrm{Stab}(\mathbb{E}^n \to \mathbb{E}^n/_{\mathbb{Z}\mathbb{B}}) \simeq U(m_1) \times \cdots \times U(m_k) \times U(n-r)$$

Q_d

$\bullet = [\mathbb{E}^n \to \mathbb{E}^n/_{\mathbb{Z}\mathbb{B}}]$

  - $\{P_0\text{-module / reps}\} \longleftrightarrow \left\{\begin{array}{l}\text{homogeneous } [\ldots]\text{-bundle} \\ \mathcal{F} \text{ on } F_{\lambda_1,\ldots}\end{array}\right\}$

    $\left\{\begin{array}{l} P_0\text{-weights} \\ \quad + \text{multiplicities}\end{array}\right\} \Longrightarrow$ total Chern class $c(\mathcal{F})$
    [Borel-Hirzebruch]

  - $\left.\begin{array}{c}(\text{[illegible]}) \\ + \\ (\text{[Borel-Hirzebruch]})\end{array}\right\} \Longrightarrow \left.\begin{array}{l} c(\nu_{E_{0S}} Q_d) \\ \quad + (S^1\text{-weight + multiplicity})\end{array}\right\}$

    $\Longrightarrow e_{S^1}(\nu_{E_{0S}} Q_d)$

  - Conclusion:

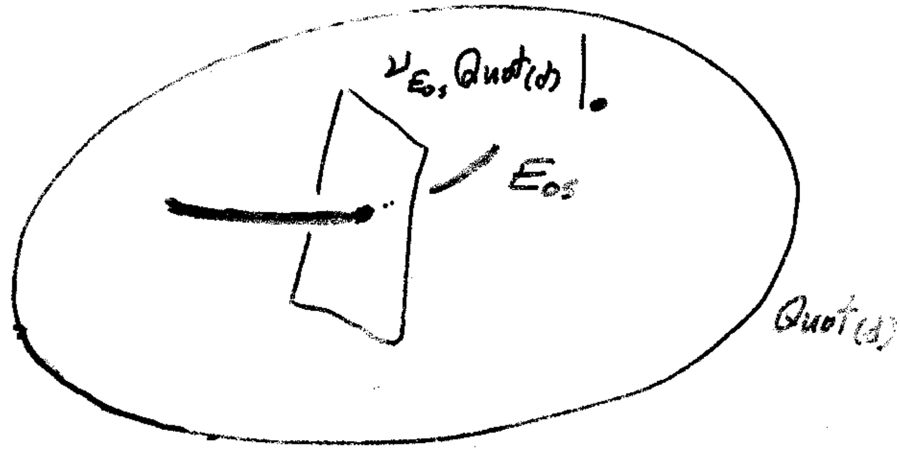
$$e_{S^1}(\nu_{E_{0S}} Q_d) = \prod_{\substack{\text{block from } \Delta_\omega \\ \text{ass. [illegible]}}} \quad \prod_{\substack{\text{block } \square \\ \in \Delta_\omega}} \quad \prod_{(i,j) \in \square} (-\omega\alpha + y_{\ldots} - y_{\ldots}$$

where $\alpha := c_1(\mathcal{O}_{\mathbb{C}P^\infty}(1))$

$y_i :=$ Chern roots $= c_1\left(\begin{array}{l}\text{quotients of} \\ \text{tautological bundles} \\ \quad \text{over complete flag manifold}\end{array}\right.$

III d_2 ① [Equivariant Euler class $e_{S^1}(\nu_{E_{0S}}\mathrm{Quot}_{(d)})$]

Outline:



- Both $\mathrm{Quot}_{(d)}$ and E_{0S} are smooth.

$$\Longrightarrow \quad 0 \longrightarrow T_* E_{0S} \longrightarrow T_* \mathrm{Quot}_{(d)}\big|_{E_{0S}} \longrightarrow \nu_{E_{0S}}\mathrm{Quot}_{(d)} \longrightarrow 0$$

- multiplicativity of e_{S^1}

$$\Longrightarrow \quad e_{S^1}(\nu_{E_{0S}}\mathrm{Quot}_{(d)}) = \frac{e_{S^1}(T_*\mathrm{Quot}_{(d)}|_{E_{0S}})}{e_{S^1}(T_* E_{0S})}$$

on $B\mathbb{C}^\times \times E_{0S}$ $(= \mathbb{CP}^\infty \times E_{0S})$

Rk I am not distinguishing S^1-action and $\mathbb{C}^\times$-action

- Represent $T_*\mathrm{Quot}_{(d)}\big|_{E_{0S}}$ and $T_* E_{0S}$ in the Grothendieck group $K(E_{0S})$ of E making the exact computation possible.

□

III d_2 ② [Equivariant Euler class $e_{S^1}(\nu_{E_{0S}}\,\mathrm{Quot}_{(d)})$]

⟨ Presentation of $T_* \mathrm{Quot}_{(d)}|_{E_{0S}}$ ⟩

- Euler sequence for Quot-scheme.
  - Fact [Huybrechts-Lehn : Prop. 2.2.7]

    X : k-scheme, $\mathcal{H}$: coherent $\mathcal{O}_X$-module, P : polynomial

    $q_0 = [\mathcal{H} \to F \to 0] \in \mathrm{Quot}_P(\mathcal{H})$.

    $K := \ker(\mathcal{H} \to F)$

    Then $T_{q_0}\mathrm{Quot}_P(\mathcal{H}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(K, F) = H^0(X, K^\vee \otimes F)$ □

  - Construction : (π : $\mathrm{Quot}_{(d)} \times \mathbb{P}^1 \longrightarrow \mathrm{Quot}_{(d)}$)

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\mathrm{Quot}_{(d)} \times \mathbb{P}^1} \otimes \mathbb{C}^n \longrightarrow \mathcal{Q} \longrightarrow 0$$

(universal quotient sheaf on $\mathrm{Quot}_{(d)} \times \mathbb{P}^1$)

($\mathcal{K}^\vee \otimes (\cdot)$: exact functor

$$0 \to \mathcal{K}^\vee \otimes \mathcal{K} \to \mathcal{K}^\vee \otimes \mathbb{C}^n \longrightarrow \mathcal{K}^\vee \otimes \mathcal{Q} \longrightarrow 0$$

(π_* : left-exact functor

$T_* \mathrm{Quot}_{(d)}$.

on $\mathrm{Quot}_{(d)}$

$$0 \longrightarrow \pi_*(\mathcal{K}^\vee \otimes \mathcal{K}) \longrightarrow \pi_*(\mathcal{K}^\vee \otimes \mathbb{C}^n) \longrightarrow \pi_*(\mathcal{K}^\vee \otimes \mathcal{Q})$$

$$\longrightarrow R^1\pi_*(\mathcal{K}^\vee \otimes \mathcal{K}) \to R^1\pi_*(\mathcal{K}^\vee \otimes \mathbb{C}^n) = 0 \quad \text{by Grauert's th}$$

$\Rightarrow$ Euler sequence for $T_* \mathrm{Quot}_{(d)}$:

$$0 \to \pi_*(\mathcal{K}^\vee \otimes \mathcal{K}) \longrightarrow \pi_*(\mathcal{K}^\vee \otimes \mathbb{C}^n) \longrightarrow T_* \mathrm{Quot}_{(d)} \longrightarrow R^1\pi_*(\mathcal{K}^\vee \otimes \mathcal{K}) \to 0$$

III d_2 ③ [Equivariant Euler class $e_{S^1}(\nu_{E_{0S}}\mathrm{Quot}_{(d)})$]

⟨ presentation of $T_*\mathrm{Quot}_{(d)}|_{E_{0S}}$ ⟩

- Restriction to E_{0S}.

- Recall: $E_{0S} =: Fl := F_{\alpha_1,\alpha_2,\cdots;r;d,n,r}$

$$\simeq Fl_{m_1,\, m_1+m_2,\, \cdots,\, m_1+\cdots+m_{k-1},\, r}(\mathbb{C}^n)$$

where: $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$

rewritten as $0 \leq a_1 < \cdots < a_k$

$m_1 \quad \cdots \quad m_k$ (multiplicity)

- Recall: (universal quotient sheaf on $\mathrm{Quot}_{(d)}\times\mathbb{P}^1$)

  - $0 \to \mathcal{K} \to \mathcal{O}_{\mathrm{Quot}\times\mathbb{P}^1}\otimes\mathbb{C}^n \to \mathcal{Q} \to 0$

$0 \to S \to \mathcal{O}\otimes\mathbb{C}^n \to Q \to 0$

universal flag bundle

  - $E_{0S} = Fl \xrightarrow{\ \rho\ } Gr_r(\mathbb{C}^n)$

$0 \to S_1 \hookrightarrow \cdots \hookrightarrow S_{k-1} \hookrightarrow S_k = \rho^*S$

$E_{0S}\times\mathbb{P}^1 \xrightarrow{\qquad \pi \qquad} E_{0S}$

- $\boxed{\text{geometric meaning of elements in } E_{0S} \text{ as subsheaves in } \mathcal{E}^n}$ + $\boxed{\text{the way of identification } E_{0S} \xrightarrow{\sim} \text{flag manifold (discussed earlier)}}$

$\Rightarrow$ • $\mathcal{K}|_{E_{0S}\times\mathbb{P}^1} = \pi^*\rho^*S = \pi^*S_k$

• $\mathcal{K}|_{E_{0S}\times\mathbb{P}^1}$ admits a filtration:

$$0 \to \mathcal{K}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{K}_{k-1} \hookrightarrow \mathcal{K}_k = \mathcal{K}$$

such that $\mathcal{K}_i/\mathcal{K}_{i-1} \simeq \pi^*\left(S_i/S_{i-1}\right)(-a_i z)$,

where $\mathbb{P}^1 = \mathbb{C}\cup\{\infty\}$ parameterized by z

(z) : divisor on $\mathbb{P}^1$ supported at 0 then pulled-backed to $E_{0S}\times\mathbb{P}^1$

III d₂ ④ [Equivariant Euler class $e_{S^1}(\nu_{E_{0S}} \mathrm{Quot}_{(d)})$]

⟨ presentation of $T_* \mathrm{Quot}_{(d)}|_{E_{0S}}$ ⟩

- Conclusion :

In the Grothendieck group $K(E_{0S} \times \mathbb{P}^1)$,

$$[\mathcal{K}] = \sum_{i=1}^{k} \left[\pi^*(S_i/S_{i-1})(-q_i) \right]$$

$$[\mathcal{K}^\vee \otimes \mathcal{K}] = \sum_{i,j=1}^{k} \left[\pi^*\left((S_i/S_{i-1})^\vee \otimes (S_j/S_{j-1}) \right)(q_i - q_j) \right]$$

where $\pi : E_{0S} \times \mathbb{P}^1 \longrightarrow E_{0S}$

cf.

$$\begin{array}{ccccc} & & \wr & & \wr \\ \longrightarrow & & \pi_*(\mathcal{K}^\vee \otimes \mathcal{K}) & \longrightarrow & \pi_*(\mathcal{K}^\vee \otimes \mathbb{C}^n) \end{array}$$

$$\longrightarrow T_* \mathrm{Quot}_{(d)}|_{E_{0S}} \longrightarrow R^1\pi_*(\mathcal{K}^\vee \otimes \mathcal{K}) \longrightarrow$$

III d_2 (5) [Equivariant Euler class $e_{S^1}(\nu_{E_{0S}}\,\mathrm{Quot}(d))$]

⟨presentation of $T_* E_{0S}$⟩

- Recall: $E_{0S} = Fl = Fl_{m_1,\, m_1+m_2,\, \dots,\, m_1+\dots+m_{k-1},\, r}(\mathbb{C}^n)$

  $\rho : E_{0S} \longrightarrow Gr_r(\mathbb{C}^n)$;

  $$0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_{k-1} \hookrightarrow S_k \hookrightarrow \underset{\substack{\| \\ \mathcal{E}^n}}{\mathcal{O}_{Fl}\otimes\mathbb{C}^n} \quad \text{on } F$$

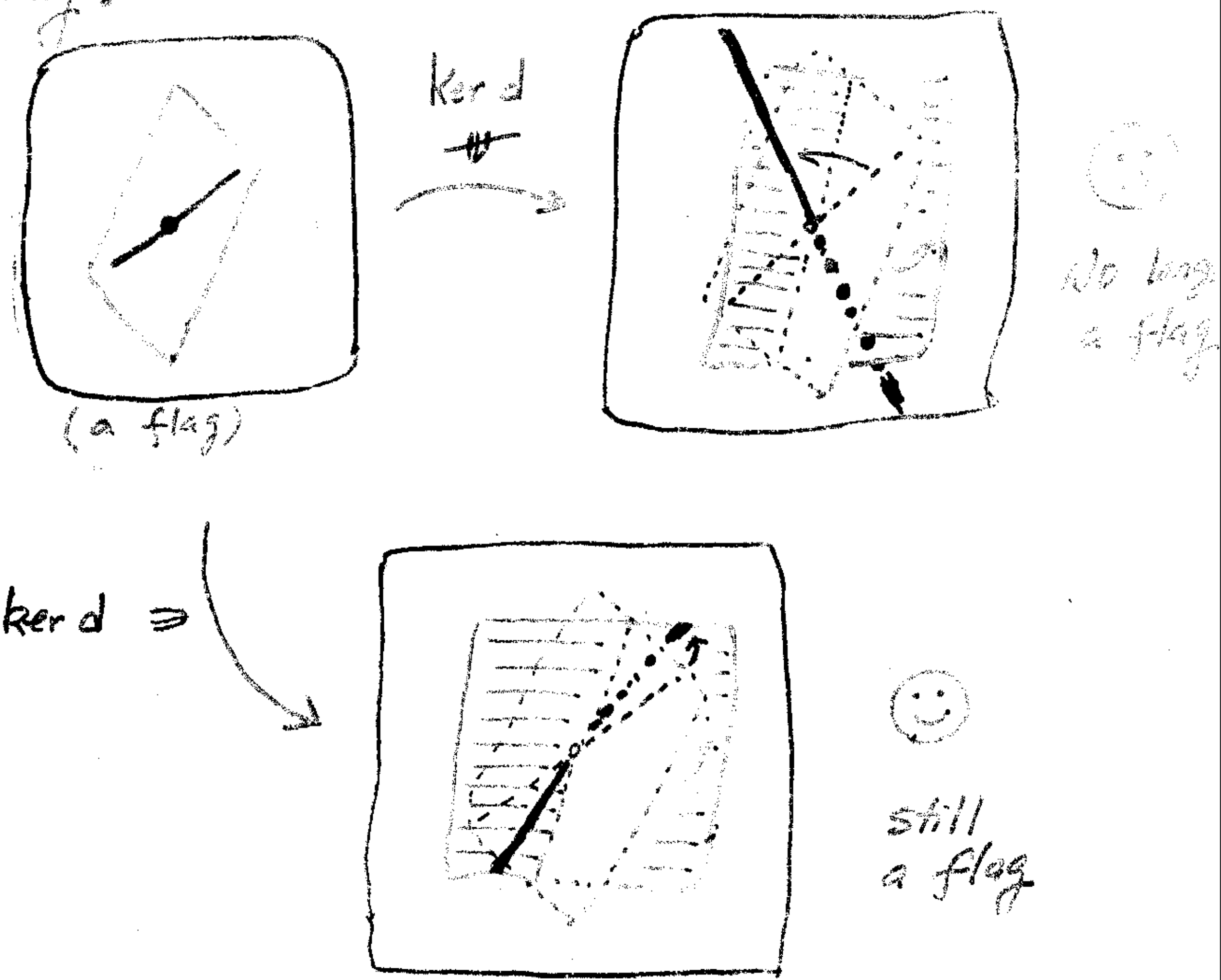
- A tautological exact sequence:

$$0 \longrightarrow T_* Fl \longrightarrow \underset{\substack{\| \\ \mathcal{H}om(S_i,\ \mathcal{E}^n/S_i)}}{\bigoplus_{i=1}^{k} S_i^\vee \otimes (\mathcal{E}^n/S_i)} \xrightarrow{\ d\ } \bigoplus_{i=1}^{k-1} S_i^\vee \otimes (\mathcal{E}^n/S_{i+1})$$

(infinitesimal deformation of each piece in the flag)

d: compatibility conditions (details omitted)

- meaning:



III. d_2 ② [Equivariant Euler class $e_{S^1}(\nu_{E_{0S}} \mathrm{Quot}_{(d)})$]

⟨presentation of $T_* E_{0S}$⟩

- From the tautological sequence to the Euler sequence.

  - $$0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots S_{k-1} \hookrightarrow \underset{= \rho^* S}{S_k} \hookrightarrow \mathcal{E}^n, \quad \rho : E_{0S} \to \mathrm{Gr}(k, \mathcal{E})$$

    dual

    $$\Rightarrow \mathcal{E}^n \to \rho^* S^\vee \to S_{k-1}^\vee \to \cdots \to S_2^\vee \to S_1^\vee \to 0$$

  - $$\left.\begin{array}{l} \rho^* S^\vee \longrightarrow S_i^\vee \\ \mathcal{E}^n \longrightarrow \mathcal{E}^n / S_i \end{array}\right\} \Rightarrow \rho^* S^\vee \otimes \mathbb{C}^n \longrightarrow S_i^\vee \otimes (\mathcal{E}^n / S_i), \ \forall i$$

    $$\Rightarrow \rho^* S^\vee \otimes \mathbb{C}^n \xrightarrow{\ \xi\ } \bigoplus_{i=1}^{k} S_i^\vee \otimes (\mathcal{E}^n / S_i)$$

    check: $\mathrm{Im}(\cdot) = \ker d = T_* F\ell = T_* E_{0S}$.

    $$\Rightarrow \boxed{\rho^* S^\vee \otimes \mathbb{C}^n \longrightarrow T_* F\ell \longrightarrow 0}$$

  - $$K := \mathrm{Ker}(\rho^* S^\vee \otimes \mathbb{C}^n \to T_* F\ell)$$

    Then can show: K admits a filtration

    $$0 \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K_k = K$$

    Not unique, determined essentially by any decomposition of $\mathcal{E}^n$ transversal to flags

    such that $K_i / K_{i-1} \simeq (S_i / S_{i-1})^\vee \otimes S_{i-1}$

    Together with filtration of S_i: $0 \to S_1 \hookrightarrow \cdots \hookrightarrow S_{i-1} \hookrightarrow S_i$

    $\Rightarrow$ filtration of K_i / K_{i-1}: $(S_i / S_{i-1})^\vee \otimes S_1 \hookrightarrow \cdots \hookrightarrow (S_i / S_{i-1})^\vee \otimes S_i \simeq K_i / K_{i-1}$

- Conclusion: in $K(E_{0S})$:

  $$\boxed{[T_* E_{0S}] = n \cdot [\rho^* S^\vee] - \sum_{1 \le j < i \le k} \left[(S_i / S_{i-1})^\vee \otimes (S_j / S_{j-1}) \right]}$$

III d_2 (?) [Equivariant Euler class $e_{S^1}(\mathcal{V}_{E_{0S}} \mathrm{Quot}_{(d)})$]

<final expression>

- Introducing Chern roots: $x_1, \cdots, x_r$ for $(S_i/S_{i-1})^\vee$, $i=1$
- Facts from Borel–Hirzebruch
- Basic identities for Chern roots under tensor product & dual
- functorial nature $\Rightarrow$ compatible with S^1-actions

$$e_{S^1}(\mathcal{V}_{E_{0S}} \mathrm{Quot}_{(d)})$$

$$= \frac{\prod_{i=1}^{k} \prod_{s=1}^{m_i} \prod_{\ell=1}^{a_i} (x_{m_1+\cdots+m_{i-1}+s} - \ell\alpha)^n}{\prod_{1\le i<j\le k} (-1)^{m_i m_j (a_j - a_i - 1)} \prod_{s=1}^{m_i} \prod_{t=1}^{m_j} \left(-x_{m_1+\cdots+m_{i-1}+s} + x_{m_1+\cdots+m_{j-1}} + (a_i - a_j)\alpha \right)}$$

Thus:

$$HG[\mathbb{1}]^X(t) = e^{-Ht/\alpha} \sum_{d=0}^{\infty} e^{dt} \sum_{\substack{0\le d_1\le\cdots\le d_r \\ d_1+\cdots+d_r=d}} p_*\left(\frac{1}{(\cdots)}\right),$$

$$p:\ E_{0S} \longrightarrow X = Gr_r(\mathbb{C}^n).$$

III. [illegible] [continuing proof of Hori-Vafa formula]

- Last step:
  - formula of [M. Brion, 1996: "The push-forward and Todd class of flag bundles".]
  - simplification

$$\Longrightarrow \quad HG[\mathbb{1}]^X(t)$$

$$(*) \quad = e^{-Ht/\alpha} \sum_{d=0}^{\infty} e^{dt} (-1)^{(r-1)d} \cdot \sum_{\substack{0 \le \alpha_1 \le \cdots \le \alpha_r \\ \alpha_1 + \cdots + \alpha_r = d}} \frac{\prod_{1 \le i < j \le r} (x_i - x_j - (\alpha_i - \alpha_j)\alpha)}{\prod_{1 \le i < j \le r} (x_i - x_j) \prod_{i=1}^{r} \prod_{l=1}^{\alpha_i} (x_i - l\alpha)^n}$$

([illegible])

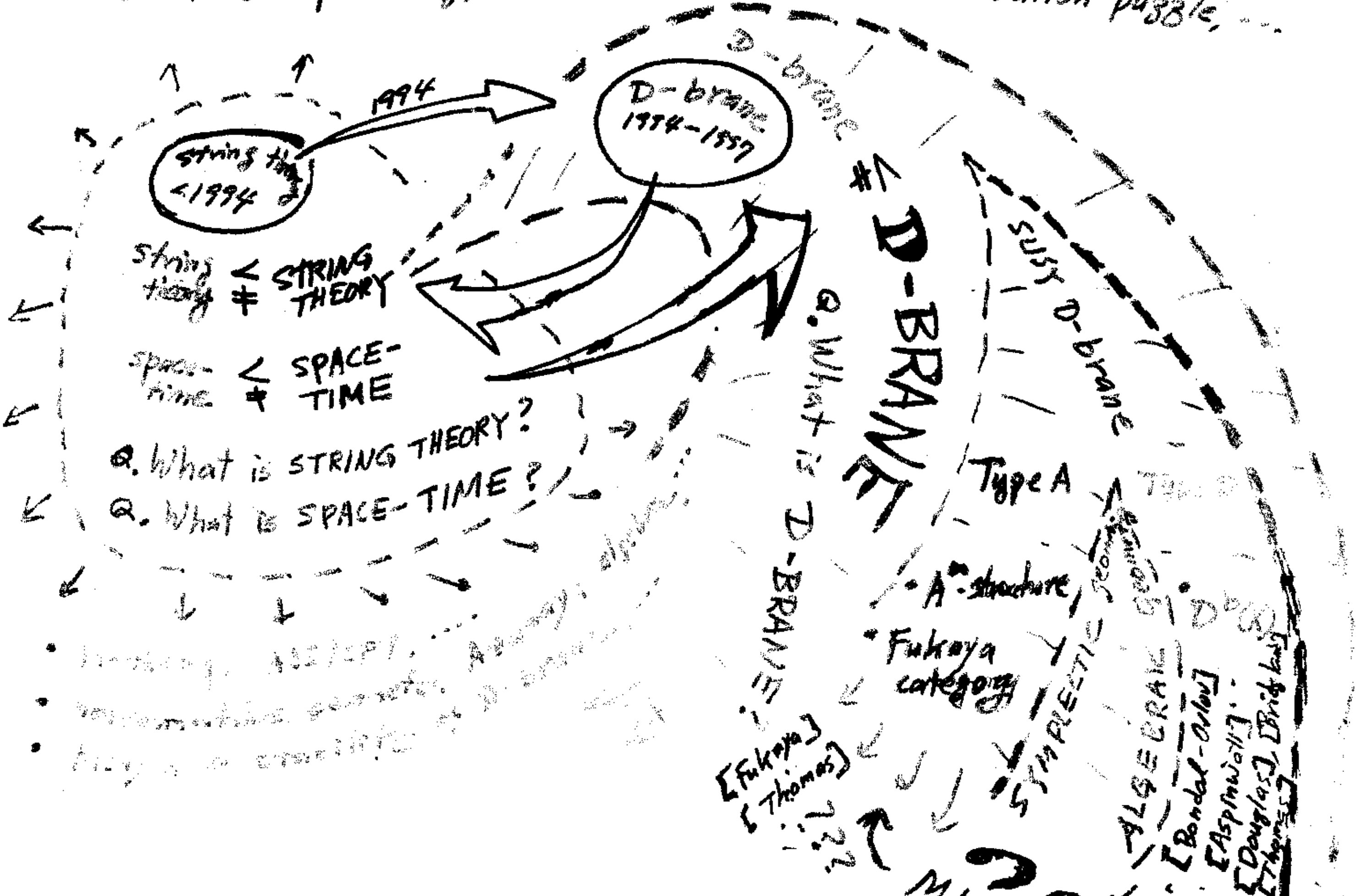
- Check the Hori-Vafa formula
  - $\mathbb{P}^{n-1} = Gr_1(\mathbb{C}^n)$, $\quad \mathbb{P} := \mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}$ (r times)
  - $HG[\mathbb{1}]^{\mathbb{P}}(t_1, \cdots, t_r) = \prod_{i=1}^{r} HG[\mathbb{1}]^{\mathbb{P}^{n-1}}(t_i)$
  - Apply $(*)$ to $X = Gr_1(\mathbb{C}^n)$ and $X = Gr_r(\mathbb{C}^n)$

    $\Longrightarrow$ Hori-Vafa formula. $\square$

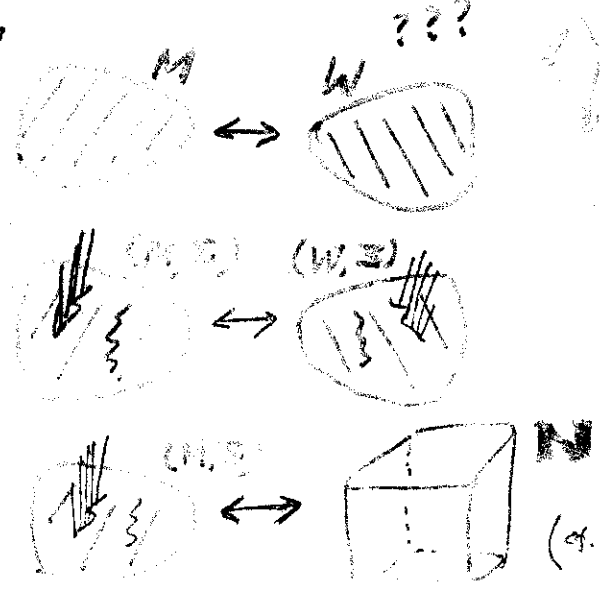
- Generalization:
  flag manifold, complete intersection therein, homogeneous bundles in A-series

IV ① [New directions of mirror principle] in string theory

- 1994 : a **S**trongly-re-**W**arding, **F**antastic, **D**eterminant year
  [Seiberg-Witten theory], [Het/F-theory [Morrison-Vafa]], [D-branes [Polchinski]]
- Impact : open string. Dirichlet b.c. momentum conservation puzzle, ...



- Open string mirror symmetry.
  - closed string mirror symmetry
    - inner geometry / fibration
  - open string mirror symmetry:
    - (··) + external geometry structure
  - open - closed duality :
    (Lerche-Mayr) : • inner + external ⇒ bigger inner



(cf. Het/

V② [New directions of mirror principle]

Q. What should be the new open string mirror principle

Q. What should be the new open mirror principle diagram

- Current answer:

Open gauged linear σ-models $\quad\cdots\quad$ D-branes $\quad\cdots\quad$ geometric transitions

$$\begin{tikzcd} V \arrow[d, "+???"] & ??? \arrow[d] & ??? \arrow[d] & ??? \arrow[d] & \\ X & ??? \arrow[l] \arrow[r] & ??? & ??? \arrow[l] \arrow[r] & ??? \end{tikzcd}$$

or completely another form/pattern?

boundary (S)CFT $\quad$ open Gromov-Witten theory $\quad$ $N=1$ special geometry, i.e. multi-pre-potentials

Chern-Simons theory $\quad$ Variation of Hodge structure with decorations

(S)CFT with gauge symmetry $\quad\cdots\quad$??

Comparison:

$$\begin{tikzcd} V \arrow[d] & U_d \arrow[d] & V_d \arrow[d] & \mathcal{U}_d \arrow[d] & \\ X & \overline{M}_{0,1}(X,d) \arrow[l] \arrow[r] & \overline{M}_{0,0}(X,d) & M_d \arrow[l] \arrow[r] & W_d \end{tikzcd}$$

(closed mirror principle diagram)