Noncontractible periodic orbits, Gromov invariants, and Floer-theoretic torsions

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I. Background and history

Let (M, ω) be a **symplectic manifold**.
ω: (*symplectic form*) a nowhere vanishing closed 2-form.
X: (*symplectic vector field*), generates diffeomorphisms preserving ω. (*symplectomorphisms* $\phi^*\omega = \omega$).

> *Problem: Find periodic orbits of X, or more generally, given a 1-periodic path X_t, find 1-periodic orbits.* [redacted] $\dot{x}(t) = X_t(x(t))$

Motivation from **classical mechanics**:
$M = T^*L$, the 'phase space' (the space of all possible positions and momenta).
L is the'configuration space' (the space of all possible positions).
$\omega = d\theta$, where $\theta = \sum_i p^i dq_i$ in local coordinates.

A **Hamiltonian** H is a function on M; $dH = \iota(X_H)\omega$ determines a symplectic vector field X_H.

> *The equation of motion may be recast as the flow equation of the Hamiltonian vector field X_H.*

Periodic orbits are particularly abundant in Hamiltonian or symplectic dynamics.

Famous examples:

- **Poincaré recurrence theorem**: The flow of a symplectic vector field is volume preserving, therefore strongly recurrent.
- **Pugh-Robinson's closing lemma**: For a generic C^2 H, P.O. are dense on compact, regular energy hypersurfaces (level surfaces of H).

General existence results:

- **Poincaré-Birkhoff theorem**: a volume preserving diffeomorphism (i.e. symplectomorphism in the 2D case) of an annulus twisting the two boundary circles in different directions must have at least two fixed points.

- **Arnold's Conjecture** (dynamical version): *Let* (M, ω) *be a compact symplectic manifold and let* $\{H_t\}$ $t \in [0,1]$ *be a 1-periodic path of Hamiltonian functions. If all P.O. are nondegenerate (as fixed points), then*

$$\#(P.O.) \geq \sum_i b_i(M).$$

Note:

$$\text{1-periodic orbits of } X_t \longleftrightarrow \text{fixed points of } \phi_1,$$

ϕ_t being the path of symplectomorphisms generated by X_t.

Periodic orbits are also significant because of its relation with the *rigidity* phenomenon of symplectic topology.

Symplectic rigidity (Eliashberg-Gromov): *the group of* C^∞ *symplectomorphisms is* C^0 *closed in the group of* C^∞ *diffeomorphisms.*

(aka 'Existence of symplectic topology' : otherwise the C^0 closure of $\mathrm{Symp}(M)$ is the group of volume-preserving diffeomorphisms (Gromov), and symplectic topology wouldn't be very different from differential topology.)

Contrast Arnold's conjecture with the Lefschetz fixed points formula: The bound $|\sum_i (-1)^i b_i(M)|$ is much weaker. In fact, in good cases (e.g. M is simply connected), ϕ is *diffeotopic* to one with $|\sum_i (-1)^i b_i|$ fixed points. Arnold's conjecture says that this is impossible for symplectic isotopy.

Also, Hofer-Zehnder used existence results of periodic orbits to define a *symplectic capacity*, which implies symplectic rigidity.

Tools for proving such existence results:

1. *Variational methods.* Periodic orbits as critical points of the action functional on loop space. (Least action principle). Results tend to be isolated to special cases.

   E.g. $H = K + V$ in classical mechanics,

   $$\left\{\text{periodic orbits}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{closed geodesics in } L \text{ of} \\ \text{the Jacobi metric.}\end{array}\right\}$$

2. *Gromov's pseudo-holomorphic curves.* (Inv. Math. 1985) J: almost complex structure on M *compactible* with ω. (i.e. $\omega(\cdot, J\cdot)$ is a metric).

3. Floer combined two approaches, and introduced **Floer homology**.

Floer's theorem: (JDG 1988, CMP 1990) *Arnold's conjecture holds for compact, monotone symplectic manifolds.*

monotonicity: $c_1(TM)\Big|_{\pi_2(M)} = \alpha[\omega]\Big|_{\pi_2(M)}$ for $\alpha > 0$.

The technical assumption of monotonicity has subsequently been removed by the effort of many people. (Fukaya-Ono, Liu-Tian, Hofer-Salamon, Ruan, Siebert..., around 1996)

Floer's idea: do Morse theory formally on the (unit-length, contractible) loop space:

- action functional $\leftrightarrow$ Morse function;
  $\mathcal{A}_H(\gamma) := \int_{D_\gamma} \omega - \int_0^1 H_t(\gamma(t))dt$
- periodic orbits $\longleftrightarrow$ critical point (approach 1);
- (perturbed) p-hol cylinders $\longleftrightarrow$ gradient flow lines (approach 2).
  $\partial_s u + J_t(u)\partial_t u + \nabla H_t(u) = 0$—an elliptic PDE. J gives a metric on $\mathcal{L}$.
- Floer complex $\longleftrightarrow$ Morse complex: $C_i = \bigoplus_{p=\text{ind i crit pt}} \mathbb{Z}.p$; $\langle q, \partial p\rangle = \sum \pm\#\{\text{flows p} \to \text{q}\}$, where $\text{ind}(p) = i$, $\text{ind}(q) = i-1$.

Technical details: for the relevant moduli spaces to have the desired properties (smoothness, compactness), need intricate transversality and compactness arguments.

Key point: *Floer homology is invariant under hamiltonian isotopies.* (i.e. $HF_*(M; J, H)$ is independent of J, H).

One may thus reduce the computation of HF to small, t-independent H, where periodic orbits and p-hol curves are both t-independent, i.e just critical points and gradient flow lines of H. So

$$HF_*(M) = H_*(M).$$

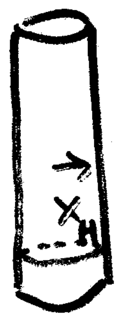
There are many variants of Floer theories, in which the Floer homology may vanish.

Example (1) *Floer theory of noncontractible loop space.*

$HF = 0$ always in this version. (At small X, all periodic orbits are contractible).

(How to find *noncontractible* periodic orbits? Results are rare and isolated. E.g. classical case—closed geodesics in L. Recently, Gatien-Lalonde (Duke 2000) using Gromov's method for $M = T^*L$, L being the mapping torus of an involution of a flat manifold).

$M = T^*S^1$



$\uparrow z$

$H = z$

∞-many noncontractible P.O.

Example (2) *Floer theory on symplectic manifolds of contact-type boundaries.* (Viterbo, [CFHW]). $HF(D^2) = 0$ according to Viterbo.

Example (3) *Floer theory of lagrangian intersections.* periodic orbit problems may be formulated in this version, but this is harder to define. Simplest case: compact L in $\mathbb{C}^n$. Can move L away from itself by hamiltonian isotopies. ϕ_H $HF(L, L) = HF(L, \phi_H(L)) = 0$

Example (4) *Twisted versions of Floer theory.* One often needs to consider twisted versions of Floer theory modelling on the Morse-Novikov theory of closed 1-forms (instead of functions), because the action functional is often not globally defined. E.g. when X_t are symplectic but not Hamiltonian, or M is not monotone.

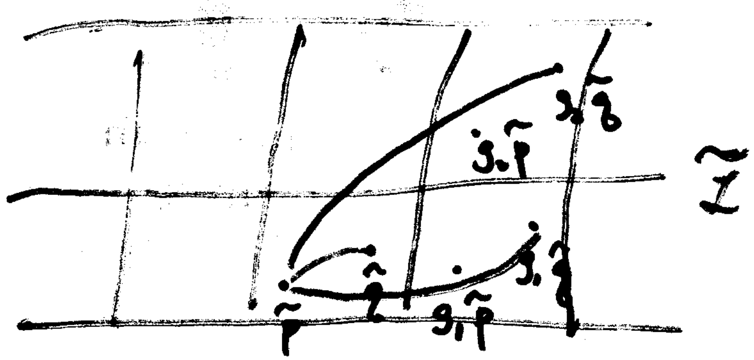
$$d\tilde{A}_X(\xi) = \int_0^1 [\iota(\xi(t))\omega](\gamma)\, dt - \int_0^1 (\iota(X_t)\omega)(\xi(t))\, dt \quad \text{for } \xi \in T_\gamma \mathcal{L}$$

Morse theory of a non-exact closed 1-form: In this case there might be ∞-many flow lines between p, q.

Idea: count with homology classes.

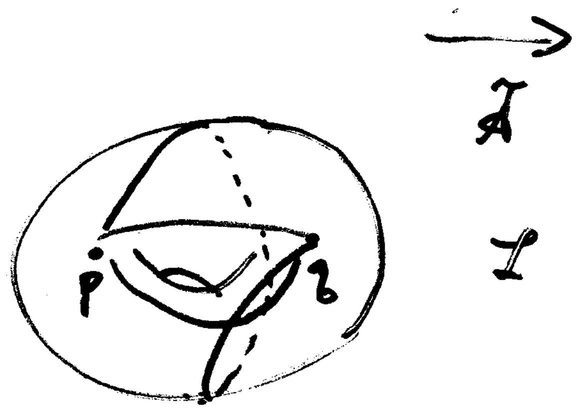
Coefficient ring: The Novikov ring is a completion of $\mathbb{Z}[H_1]$ along the direction of $[\theta]$.

Simplest e.g. $\theta = d\tau$ on S^1: $\mathrm{Nov} = \mathbb{Z}((t))$.



universal abelian covering

$$\begin{tikzcd} \tilde{\mathcal{L}} \arrow[d] & H_1(\mathcal{L}) \arrow[l] \\ \mathcal{L} & \end{tikzcd}$$



$\langle \tilde{q}, \partial \tilde{p} \rangle = \pm 1 \pm g_1 \pm g_2 \pm \dots$

$\in \mathrm{Nov}(H_1, \mathbb{Z}; -d\tilde{\mathcal{A}})$

$\downarrow$

possibly ∞ terms

finitely many flows with

$\Delta \tilde{\mathcal{A}} \le C$

By adapting Floer's arguments, Le-Ono defined a twisted version of Floer homology according to the above recipe, and proved:

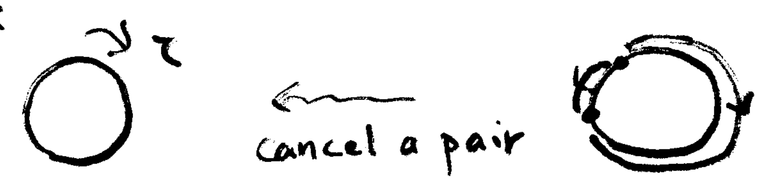
Theorem. (Le-Ono) *Suppose M is monotone, and $\{X_t \,|\, t \in [0,1]\}$ is a 1-periodic path of symplectic vector fields on M, then the number of 1-periodic orbits of X_t is at least the total betti number of the Novikov homology $HN(M, \theta_X)$ (twisted version of Morse homology), $\theta_X := \int_0^1 \iota(X_t)\omega\, dt$ is the "flux" or "Calabi invariant".* (generalization of H)

(Le-Ono actually had a slightly weaker assumption on M, and they worked with a smaller covering group.)

However, *twisting typically reduces the rank of homology.*

E.g. The twisted homology of S^1 is 0. in fact, $\theta = d\tau$ has no critical points. When there is a cancelling pair, $\partial = 1 - t$ is an isomorphism of $\mathbb{Z}((t))$. So are a lot of other mapping tori $\Sigma_\varphi = \Sigma \times [0,1]/(x,0) = (\varphi(x), 1)$.

$\theta = d\tau$
no crit pt



$\partial = 1-t$

II. Reidemeister torsion in Floer theories

What does one do in these cases?

Idea: *consider more refined invariants of the Floer complex.*

- Filtrated versions of Floer homology (Biran-Polterovich-Salamon, to appear) "relative symplectic homology, $M = T^*T^n$",
- *Other homotopy invariants:* Fukaya proposed using the A^∞ categories to capture the rational homotopy type of the Floer complex.
- *Non-homotopy invariants:* Historically the Reidemeister torsion is the first such defined.

Given a chain complex (C, ∂), the Reidemeister torsion is roughly the (multiplicative) difference between two natural bases of $\bigotimes_i (\det C_i)^{(-1)^i}$. (Assume field coefficients F for simplicity).

- *One* from the canonical isomorphism

$$\bigotimes_i \det(C_i)^{(-1)^i} \to \bigotimes_i \det(H_i)^{(-1)^i} = F$$

when (C_*, ∂_*) is acyclic. standard short exact sequences $0 \to Z_i \to C_i \to B_{i-1} \to 0$; $0 \to B_i \to Z_i \to H_i \to 0$.

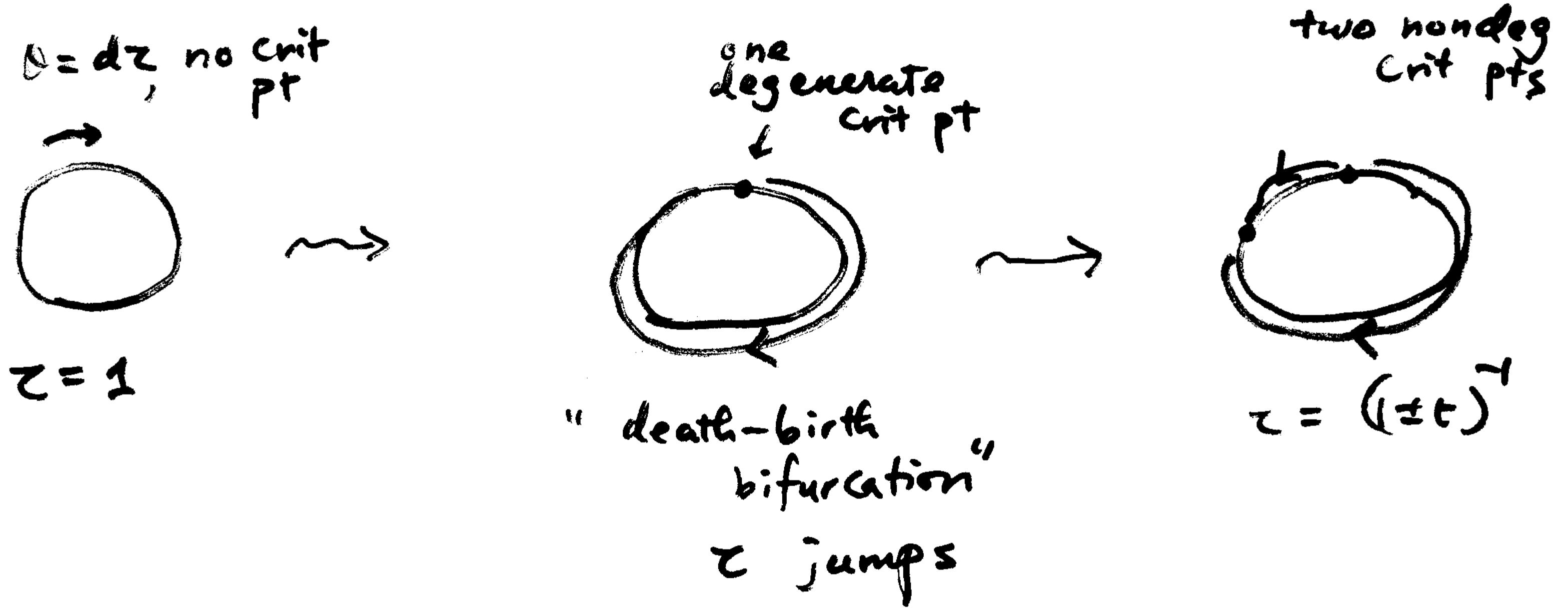
- *The other* from taking the critical points or cells as basis elements of (C, ∂). Our version of Reidemeister torsion takes value in $Q(\mathrm{Nov})/\pm H_1$.

Example. (C, ∂) = twisted Morse or Floer complex. e.g. S^1 with two cancelling critical points. $\tau = (\det \partial)^{-1} = (1-t)^{-1}$.

Example. Given a cell decomposition of M, let $(C(\tilde{M}), \partial) =$ the equivariant cell chain complex. The Reidemeister torsion of M is $\tau(M) = \tau(C(\tilde{M}))$.
– $M = \Sigma_\varphi$ is a mapping torus. $\tau(\Sigma_\varphi) = \exp(\sum_{k \in \mathbb{Z}^+} t^k \,\#\mathrm{Fix}(\phi^k)/k)$ when $H_1(\Sigma_\varphi) = \mathbb{Z}$.
– For 3-manifolds there are surgery formulae to compute $\tau(M)$. e.g. $M = S^3_0(K)$ is 0-surgery of a knot $K \subset S^3$. Then $\tau(M) = \mathrm{Alex}(K)/(1-t)^2$.

By modifying the literature, it is not hard to define a Reidemeister torsion for the Floer complex $\tau_F = \tau_F(\_\_\_\_; J, X)$. ____ specifies the loop space used, it may be (M, γ_0, ϕ), the space of ϕ-twisted loop in M in the homotopy class of γ_0, or (L, L'), the space of paths from L to L', etc. Perturbed p-hol curves satisfying $\partial_s u + J_t(u)\partial_t u + \check{\theta}_{X_t})(u) = 0$ is considered. $\theta_X := \iota(X)\omega$.

However, τ_F **is not an invariant.**



Nevertheless, τ_F can be "corrected" by taking into account closed orbits in the flow on the loop space. They correspond to (perturbed) p-hol tori in M.

Analogous to the dynamical zeta function (twisted version), the Floer-theoretic *zeta function* is defined as

$$\zeta_F := \exp\left(\sum_{u\in\hat{\mathcal{M}}^0_O} \frac{\mathrm{sign}(u)}{m(u)}[u]\right). \quad \Rightarrow \text{multiplicity}$$

Coefficients in the exponent are "orbifold Euler numbers". The exponent is in Nov^+ by compactness theorems; and exp is defined in the formal sense. $\mathrm{Nov}^+(G,\mathbb{Q};N)\subset \mathrm{Nov}(G,\mathbb{Q};N)$ denote the subset consisting of formal sums $\sum_g a_g\cdot g$ for which $a_g\neq 0$ only when $N(g)>0$. Used Taube's ideas for the transversality problem with multiple covers.

Formally, ζ_F is very similar to the Gromov series (generating function of genus 1 Gromov invariants) defined by Ionel-Parker.

Theorem. (hard) $I_F = I_F(\underline{\quad\quad}; J, X) := \zeta_F \tau_F$ *is invariant under hamiltonian isotopies.* *(independent of J and addition to X by some X_H).*

Finite dimension version: (Topology 99'; G&T 99')

$$\zeta(\theta)\tau(\theta) = \iota\tau(M).$$

joint with M. Hutchings, motivated by Seiberg-Witten theory on 3-manifolds.

cpt

Example (4): Theorem. *Suppose M is monotone and ϕ is connected to the identity via the symplectic-isotopy ϕ_t, $t \in [0,1]$; $\phi_0 = \mathrm{Id}$; $\phi_1 = \phi$. Let $\gamma_0(t) = \phi_t(p_0)$, $p_0 \in M$. Then*

$$I_F(M, \gamma_0, \phi) = \iota\tau(M),$$

where ι is a natural inclusion.

It is hard to find *compact symplectic* manifolds with interesting torsion. Easy to find noncompact ones: e.g. Stein manifolds.

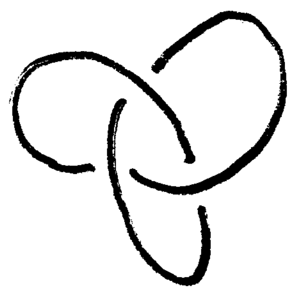
In this example and the next, I_F reduces to the classical version of torsion because the extra conditions (e.g. monotonicity) guarantee invariance of I_F under general *symplectic isotopies*, and the computation is done for small X as in Floer's proof.

Example (3): Theorem. *Suppose that M is complete, $L, L' \subset M$ are both compact, orientable lagragian submanifolds which are monotone, with minimal chern numbers $\Sigma(L), \Sigma(L') \geq 3$, and relatively spin.* (*for def of $I_F(L, L')$*) *Suppose $H_1(L) \to H_1(M)$ is injective* (*for inv under symp isotopy*), *and $\pi_2(M, L) = 1$* (*no 'quantum correction'*) *and $L' = \phi(L)$ for a symplectomorphism ϕ of M that is connected to the identity via a symplectic*[redacted] *isotopy ϕ_λ, $\lambda \in [0,1]$. Let $\gamma_0(t) = \phi_t(p)$, for a $p \in M$. Then $I_F^{\gamma_0}(L, L') = \tau(L)$.*

g.c.d of $c: \pi_2(M, L) \to \mathbb{Z}$

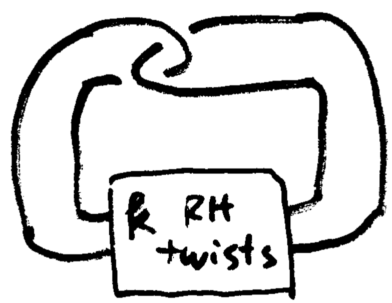
Prime example. $M = T^*U$, U is any compact spin manifold such as $U = S^3_0(K)$. Alex(K) can be any A-polynomial ($P(t), P(1) = 1$; symmetric). $\longrightarrow$ Lots of interesting I_F.

$L = U =$ zero section



Trefoil Knot

$A(K) = 1 - t + t^2$



k- twisted Knot

$A(K) = k - (2k+1)t + kt^2$

An example that is not *classical*:

Example (1) Let $\mathcal{L}(\gamma)$ be the space of loops in M in homotopy class γ, $[\gamma] \in H_1(M)$ being primitive. There is a free S^1 action on $\mathcal{L}(\gamma)$. When J, X are t-independent, this induces free S^1 action on $\mathcal{M}_P, \mathcal{M}_O$, making the usual definition of I_F vanish. It is more appropriate to consider the S^1-equivariant version of Floer theory here. (Model on Morse theory on $\mathcal{L}/S^1$). To better compare with the usual Gromov invariant, use the natural map $\mathrm{im} : H_1(\mathcal{L}(\gamma)) \to H_2(M)$ to define $I_F^{S^1}$ in the Novikov ring of $H_2(M)$ instead.

Let $\mathcal{H} = \ker c_1 \cap \underline{\mathrm{im}}(H_1(\hat{\mathcal{L}})) \subset H_2(M)$. In this case the Floer homology is trivial and $\ln I_F^{S^1}$ is well-defined. (easy for small X).

Theorem. *Let (M, ω) be a compact monotone symplectic manifold of dimension ≥ 4 (for transversality) and let $[\gamma] \in H_1(M; \mathbf{Z})$ be primitive as above. In addition, assume that $c_1 : \pi_2(M) \to \mathbf{Z}$ has trivial kernel. (E.g. $\pi_2(M) = 1$.) Let θ be a closed 1-form s.t. $\theta([\gamma]) = 0$ (for S^1-equiv). Then for any primitive class $A \in \mathcal{H}$, $\ln I_F^{S^1}(M, \gamma; J, X_\theta)(A) = RT_{A,1,0}(M)$.*

used for computation

basically, $\pi_2(M) = 0$ or $\mathbb{Z}$

t-independent

Gromov invt counting J-hol tori in class A of all possible conformal st. (used S^1-equiv theory)

Prime example. (*symplectic mapping tori*) $M = \Sigma_\varphi \times S^1$. Say (Σ, ω_Σ) is a closed surface; $\Sigma_\varphi = \Sigma \times [0,1]/(x,0) = (\varphi(x),1)$. Let ϑ_φ be the canonical 1-form on Σ_φ, and let τ parametrizes S^1. $\omega = * \cancel{\vartheta_\varphi}\, \omega_\Sigma + \vartheta_\varphi \wedge d\tau$. Then $I_F^{S^1}(M, \{p\} \times S^1) = \tau(\Sigma_\varphi)$, which is computable by Lefschetz numbers. (Using the computation of Gromov invariant by Ionel-Parker).

Example. $\Sigma_\varphi = S_0^3(K)$, where K is a fibered knot (such as the trefoil). Alex(K) can be any monic A-polynomial. $\longrightarrow$ Lots of interesting $I_F^{S^1}$. $= \text{Alex}(K)/(1-t)^2$

(II) Less-straightforward applications

Let $Y = \Sigma_\varphi$ be the mapping torus of a (not necessarily symplectic) diffeomorphism of a closed oriented manifold Σ. Let q be a nowhere vanishing closed 1-form in the class of the canonical 1-form ϑ_φ. Suppose there is a primitive class $b \in H_1(Y;\mathbf{Z})$ such that $[q](b) = m > 0$ and $\ln \tau[\xi](Y) : H_1(Y) \to \mathbf{Q}$ is nontrivial at b. (This is well-defined in this case and the Novikov homology is trivial). "manifold of Type F" $\Rightarrow$ q-essential class

(in particular $\tau(Y) \neq 1$)
$Y \neq T^{2n}$

Example. $Y = S^1$ or $S^3_0(K)$, K being the trefoil knot. Let b be a generator of $H_1(Y)$.

Theorem. *Under the above assumptions:* (Y: type F, b: q-essential) 24

(a) *Let* L, $L' \subset T^*Y$ *be respectively the zero section and the section corresponding to* q, *and let* J *be a* t*-independent compatible almost complex structure on* T^*Y. *Let* H *be a 1-periodic Hamiltonian on* T^*Y *separating* L, L'; *namely,* $H : T^*Y \times S^1_1 \to \mathbf{R}$, *and* $H_t := H(\cdot, t)$ *satisfies* $\mu := \int_{S^1_1} \left(\min H_t\big|_{L'} - \max H_t\big|_{L} \right) dt > 0$, *where* S^1_1 *is the unit circle. Suppose further that* ∇H_t *is compactly supported* $\forall t \in S^1_1$, *then there exists* $\lambda \in (0, m/\mu]$ *such that the Hamiltonian flow of* λH *has a 1-periodic orbit in the class* b.

λ: generalization of "period"

Theorem. *Under the same assumptions:*

(b) *Suppose furthermore that* Σ *be a monotone symplectic manifold with dimension at least 2, such that* $c_1(T\Sigma): H_2(\Sigma) \to \mathbf{Z}$ *has trivial kernel* (E.g. any surface) *and that* φ *is a symplectomorphism. Let* $M = Y \times S^1_1$ *be the symplectic mapping torus, and let* X *be a (t-independent) symplectic vector field on* M *with* $[\iota(X)\omega] = i_Y^*\beta \in H^1(M)$, *where* $\beta \in H^1(Y)$ *s.t.* $\beta(b) =: \mu' > 0$. *Then for* $\lambda = m/\mu'$ *there exists a 1-periodic orbit of* λX *in the homology class of* $\{p\} \times S^1 \subset M$.

In general.

(M,ω), cpt, monotone $\dim \geq 4$. $[\gamma_0] \in H_1(M;\mathbb{Z})$ primitive

$c_1(TM): \pi_2(M) \to \mathbb{Z}$ has $\ker = \phi$ $\quad \theta([\gamma_0]) = 0 \quad A \in \mathcal{H}$. $RT_{A,1,0}(M) \neq 0$. $\quad \omega(A) =: m > 0$

$\theta(\kappa(A)) =: \mu > 0 \quad (\kappa: \mathcal{H} \to H_1(M)/\mathbb{Z}[\gamma_0])$

Then $\exists$ P.O.(X_θ) of period m/μ.

Desirable result: existence for $\forall \lambda \geq m/\mu$

(True for [Biran-Polterovich-Salamon])

m is a kind of area—related to the geometry ("capacity") of M; μ, μ' "energy" measures how large X is. When X is t-independent, the above is equivalent to the existence of λ-periodic orbit of X.

Trivial examples:

(a) $Y = S^1$ and H t-independent. Each energy level in T^*S^1 carries a noncontractible periodic orbit.

(b) $q = \vartheta_\varphi$, and $\iota(X)\omega = i_Y^*\vartheta_\varphi$. Then the flow is just rotation along S^1, and there is a Y-family of 1-periodic orbits.

(I) and (II) are mutually complementary: (I) uses the nontriviality of the '0-th order terms' of I_F; in (II) the 0-th order terms are 1 but there are nontrivial higher order terms.

Gatien-Lalonde (Duke 2000) has the same statement as (a), but only for Y that are involutions of flat manifolds. Key idea: nonvanishing of a Gromov-type invariant of p-hol annuli between L and L'. This is perhaps the first "nonclassical" existence result of noncontractible P.O.

Idea of proof: relation between I_F and genus 1 Gromov invariants.

(a) The Gromov-type invariant of Gatien-Lalonde is computed by $I_F(L, L') = \tau(Y)$.

(b) $I_F^{S^1} = \tau(Y)$ is nontrivial (Theorem & Ionel-Parker), but $\zeta_F^{S^1}(J, m/\mu' X) = 1$ by energy computation.

In (b), by Gromov invariants alone we know there is a 1-periodic orbit of λX for some $\lambda \in (0, m/\mu']$; the Floer-theoretic torsion is more precise.

Compare: [Biran-Polterovich-Salamon] $M = T^*\mathbb{T}^n$ $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
H: cptly supported $e \in \mathbb{Z}^n$, $|e| \leq c := \inf_{[0,1]\times\mathbb{T}^n} H$
then $\exists$. P.O. (X_H) in class e.

The Dream: Give a Floer-theoretic interpretation of the genus 1 generating function F_1 of the A-model side of Mirror symmetry.

- Bershadsky-Cecotti-Ooguri-Vafa: higher genus mirror symmetry.

combination of genus 1 Gromov invariants	$\longleftrightarrow$	holomorphic analytic torsions.
A model side		B model side

- Kontsevitch's proposal: for genus 0,

Floer homologies	$\longleftrightarrow$	sheaf cohomologies.
symplectic side		complex side

Method of proof for the invariance theorem:
Direct *bifurcation analysis.*

So far all the invariance proofs in the Floer theory literature used the *continuation method*, which contructs a chain map between two different $\mathrm{CF}(J, X)$ and $\mathrm{CF}(J', X')$, which is a homotopy equivalence.

This obviously does not work in our situation (*non-homotopy* invariant).

This bifurcation analysis should be useful for:

- Defining other more refined invariants in Floer theory.
- Providing an alternative for more involved versions of Floer theories, e.g. contact homology.