Contact structures

and symplectic open books

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Contact forms and structures

A *contact form* on a $2n+1$-manifold V is a 1-form α whose product with $(d\alpha)^n$ nowhere vanishes.

A tangent hyperplane field ξ on V is a *contact structure* if it is the kernel field of a contact form α. Thus $d\alpha$ induces a symplectic form on ξ at each point whose conformal class is independent of α.

A contact manifold is a manifold equipped with a contact structure.

The geometry of a contact form has two components:

- the contact structure $\xi = \ker \alpha$;
- the kernel of $d\alpha$, which is a line field transversal to ξ.

The unique vector field ∂_α in $\ker d\alpha$ such that $\alpha(\partial_\alpha) = 1$ is called the *Reeb vector field* of α.

Examples

1) The staandard contact structure on $\mathbf{R}^{2n+1}$ is given by

$$dz \sum_{i=1}^{n} x_i \, dy_i, \quad (x, y, z) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R},$$

and is the universal local model.

2) Let V be an oriented real hypersurface in a complex manifold W. The complex hyperplanes of TW tangent to V form a (co) oriented hyperplane field ξ on V. If the Levi form

$$Q = d\alpha(., \sqrt{-1}.)\,|_{\,\xi} \quad (\text{where } \xi = \ker \alpha)$$

is non-degenerate, ξ is a contact structure.. A typical case is when V is *strictly pseudoconvex*, *i.e.*, when Q is positive definite.

For instance, the complex hyperplanes tangent to the boundary of the unit ball in $\mathbf{C}^{n+1}$ form the standard contact structure on $\mathbf{S}^{2n+1}$.

Automorphisms and stability

Theorem (Libermann 1959). *For any contact structure ξ on a manifold V, projection $TV \to TV/\xi$ induces a one-to-one correspondence between vector fields preserving ξ and sections of TV/ξ.*

Theorem (Gray 1959). *On a closed manifold, every component of the space of contact structures is an isotopy class.*

Two contact structures are *isotopic* is there exists a diffeomorphism isotopic to the identity that takes one to the other. isotope à l'identité qui envoie l'une sur l'autre par sa différentielle.

Corollary. *On a closed manifold, contact structures form at most countably many isotopy classes.*

Classification problem

Problem (Chern 1965). Which manifolds admit contact structures? More precisely, describe the space of contact structures inside the space of all tangent hyperplane fields equipped with a conformal symplectic structure.

Theorem (Gromov 1969). *On any open manifold, each component of the space of tangent hyperplane fields equipped with a conformal symplectic structure contains exactly one component of the space of contact structures.*

Theorem (Lutz-Martinet 1971). *On a closed orientable three-dimensional manifold, each component of the space of oriented tangent plane fields contains a contact structure.*

Summary presentation

F: a compact Stein manifold, with strictly pseudoconvex boundary $K = \partial F$;
$\phi\colon F \to F$: a symplectic diffeomorphism equal to the identity near K;
$\Sigma(F,\phi)$: the mapping torus of ϕ, that is

$$\Sigma(F,\phi) = \big(F \times [0,1]\big)/\sim, \quad (p,1) \sim (\phi(p),0).$$

$\partial\,\Sigma(F,\phi) = K \times \mathbf{S}^1$ — because $\phi = \text{id}$ near K – so

$$\overline{\Sigma}(F,\phi) = \Sigma(F,\phi) \cup_\partial (K \times \mathbf{D}^2)$$

is a closed manifold.

Theorem. *$\overline{\Sigma}(F,\phi)$ is naturally a contact manifold and every closed contact manifold can be obtained in this way.*

Stein manifolds

(F, J): a complex manifold.

A function $f: F \to \mathbf{R}$ is *strictly plurisubharmonic* if the 2-form

$$\omega_f = -dd^{\mathbf{c}} f, \quad \text{where} \quad d^{\mathbf{c}} f = (df \circ J),$$

satisfies $\omega_f(v, Jv) > 0$ for all $v \neq 0$.

(F, J) is a *Stein manifold* if it admits a positive, proper, and spsh function f for which ∂F is a regular level set.

Completion. Assume F is compact and extend ω_f to

$$\widetilde{F} = F \cup_\partial \big(\partial F \times [0, \infty)\big)$$

by $\widetilde{\omega}_f(p, t) = e^t \omega_f(p)$, $(p, t) \in \partial F \times [0, \infty)$.

Theorem (Gromov-Eliashberg). *The isotopy class of $\widetilde{\omega}_f$ on $\widetilde{F}$ is independent of f.*

Weinstein manifolds

(F, ω): a symplectic manifold.

A function $f: F \to \mathbf{R}$ is *ω-convex* if it admits a gradientlike vector field ν that expands ω exponentially, *i.e.*, satisfies $\nu \cdot \omega = \omega$.

(F, ω) is a *Weinstein manifold* if it admits a positive, proper, and ω-convex Morse function f for which ∂F is a regular level set.

Example. Every Stein manifold is a Weinstein manifold: the gradient ∇f of any spsh function f with respect to the metric $\omega_f(., J.)$ satisfies

$$\nabla f \cdot \omega_f = d(\nabla f \vee \omega_f) = -dd^{\mathbf{c}} f = \omega_f .$$

Theorem (Eliashberg). *Every Weinstein manifold is symplectically diffeomorphic to a Stein manifold.*

Contactization

F: a compact Stein manifold;
$K = \partial F$;
$f\colon F \to [0,\infty)$: a strictly plurisubharmonic function which is constant on K;
$\omega = \omega_f = -dd^{\mathbf{c}} f$;
$\beta = -d^{\mathbf{c}} f$ — so $d\beta = \omega$;
γ: the (contact) form induced by β on K.

Lemma. *The form*

$$\alpha = \begin{cases} \beta + d\theta & \text{on } F \times \mathbf{S}^1, \\ \gamma + r^2\, d\theta & \text{on } K \times \mathbf{D}^2, \end{cases}$$

is a contact form on $\overline{\Sigma}(F, id)$.

Contactization of a diffeomorphism

ϕ: a symplectic diffeomorphism of F equal to the identity near the boundary $K = \partial F$;
$f\colon F \to [0,\infty)$: a strictly plurisubharmonic function;
$\beta = -d^{\mathbf{c}} f$: a primitive of $\omega = -dd^{\mathbf{c}} f$.

$\phi^*\omega = \omega$ so $\phi^*\beta - \beta$ is closed.
Assume that $\phi^*\beta - \beta = -dh$ is exact.

The 1-form $\alpha = \beta + dt$ is a contact form on $\widetilde{V} = F \times \mathbf{R}$ and is invariant under the transformation

$$(x,t) \mapsto \big(\phi(x), t + h(x)\big).$$

The quotient of $\widetilde{V}$ by this transformation is the mapping torus $\Sigma(F,\phi)$.
Since ϕ is the identity near K, we can glue $K \times \mathbf{D}^2$ as before.

Isotopy Lemma

ϕ: a symplectic diffeomorphism of F equal to the identity near $K = \partial F$;

Lemma. *We can deform ϕ through diffeomorphisms of the same type to a diffeomorphism ϕ_1 for which $\phi_1^*\beta - \beta$ is an exact form.*

Proof. Let μ be the form $\phi^*\beta - \beta$ and η the vector field given by $\eta \vee \omega = \mu$. Since μ is closed, η preserves ω:

$$\eta \cdot \omega = d(\eta \vee \omega) = d\mu = 0.$$

Let ψ_t denote the flow of η. All the maps $\phi \circ \psi_t$ are symplectic diffeomorphisms equal to the identity near the boundary and $\phi_1 = \phi \circ \psi_1$ works.

Open books

V: a closed manifold.
An *open book* in V is a pair (K, θ) consisting of:

- a codimension 2 submanifold $K \subset V$ with trivial normal bundle;
- a fibration $\theta \colon V \setminus K \to \mathbf{S}^1$ which, in a neighborhood $K \times \mathbf{D}^2$ of K, is the normal angular coordinate.

Let (K, θ) be an open book and ν a vector field transversal to the fibers of θ and equal to ∂_θ near K.
If ϕ denotes the first return map of ν on an arbitrary fiber F of θ, then V can be identified with $\overline{\Sigma}(F, \phi)$.

Contact structures and open books

Definition. A contact structure ξ on V is *supported* by an open book (K, θ) if it is the kernel of a form α satisfying the following:

- α induces a contact form on K;
- $d\alpha$ induces a symplectic form on each fiber F of θ.
- the contact orientation of (K, α) coincides with the boundary orientation of the symplectic manifold $(F, d\alpha)$.

Theorem. *Any contact structure on a closed* $2n+1$*-manifold is supported by an open book* (K, θ) *whose fibers are Weinstein manifolds and whose monodromy can be represented by a symplectic diffeomorphism.*

Example: the Milnor fibration

$p: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$: a complex polynomial with an isolated critical point at 0;
$H = p^{-1}(0)$ is a complex hypersurface with an isolated singularity at 0;

Theorem (Milnor$+\varepsilon$). *There exist a closed (smooth) ball $B \subset \mathbf{C}^n$ around 0 and a foliation of $B \setminus \{0\}$ by spc spheres S_r, where $r \in (0, 1]$ and $S_1 = \partial B$, such that, for r sufficiently small, the following properties hold:*

- *the sphere S_r intersects H transversally along $K = H \cap S_r$;*
- *the map*

$$\theta = \arg P : S_r \setminus K \to \mathbf{S}^1$$

is a fibration and (K, θ) is an open book;

- *the open book (K, θ) supports the contact structure on S_r consisting of the complex hyperplanes tangent to S_r.*

First reduction

α: a contact form with kernel ξ;
J: a $d\alpha$-compatible almost complex structure on ξ;
g: the metric $d\alpha(.,J.)\big|_\xi \oplus^\perp \alpha^2\big|_{\ker d\alpha}$;
$L = V \times \mathbf{C}$: the trivial Hermitian bundle with unitary connection given by the 1-form $-i\alpha$.

Lemma. *Suppose there exist an open book (K,θ) and a neighborhood $N = K \times \mathbf{D}^2$ of $K = K \times \{0\}$ such that:*
- *θ is the normal angular coordinate in N;*
- *α induces a contact form on each submanifold $K_w = K \times \{w\}$ in N;*
- *$d\alpha$ induces a symplectic form on each fiber of θ in $V \setminus \operatorname{Int} N$;*
- *the contact orientation of (K_w,α), $w \in \mathbf{S}^1$, is the boundary orientation of $(F,d\alpha)$.*

Then the contact structure $\xi = \ker\alpha$ is supported by (K,θ).

Construction of functions

Theorem (Ibort-Martinez-Presas). *There exist constants $C, \delta > 0$ and smooth functions $s_k : V \to \mathbf{C}$, $k \geq 0$, with the following properties:*
1) *at each point $p \in V$,*

$$|s_k| \leq C,$$
$$|ds_k - iks_k\alpha| \leq Ck^{1/2}$$
$$\text{and} \quad |\overline{\partial}_\xi s_k| \leq C;$$

2) *at each point p where $|s_k(p)| \leq \delta$,*

$$|\partial_\xi s_k(p)| \geq \delta k^{1/2}.$$

$\partial_\xi s_k, \overline{\partial}_\xi s_k$ are the J-linear and J-antilinear parts of $ds_k\,|_\xi$.

Corollary. *For* $|w| \leq \delta$ *the set* $K_w = s_k^{-1}(w)$ *is a contact submanifold. Moreover, the map* $\arg s_k \colon V \setminus K_0 \to \mathbf{S}^1$ *is a fibration and the fibers are transverse to the Reeb vector field* ∂_α *of* α *on the set where* $|s_k| \geq \delta$.

Proof. $\xi \cap TK_w = \ker(ds_k \,|_\xi)$ and $ds_k \,|_\xi$ is surjective and almost J-linear. Therefore the intersection is transverse and $d\alpha$ is non-degenerate on $\xi \cap TK_w$.

On the other hand, $|ds_k(\partial_\alpha) - iks_k| \leq Ck^{1/2}$. So, for $|s_k| \geq \delta$ and k large, $ds_k(\partial_\alpha)$ is close to iks_k, *i.e.*, is non-zero and almost orthogonal to s_k. Therefore, the sets
$s_k^{-1}(R_\theta)$, where $R_\theta = \{re^{i\theta},\ r > 0\}$,
are submanifolds transverse to ∂_α.

The Weinstein structure

(W, ω): a closed integral symplectic manifold;
L: a hermitian line bundle with a unitary connection whose curvature form is $-i\omega$;
W_k: a symplectic submanifold Poincaré dual to $k[\omega]$ and obtained by Donaldson's construction.

Proposition. *For k large, $(W \setminus W_k, \omega)$ is a Weinstein manifold.*

Proof. $W_k = s_k^{-1}(0)$ where $s_k : W \to L^k$ can be assumed to satisfy $|\overline{\partial}_k s_k| \leq \frac{1}{\sqrt{2}} |\partial_k s_k|$.
In the trivialization of L^k over $W \setminus W_k$ given by the unit section $s_k/|s_k|$, the connection is given by a 1-form $-i\lambda$ where $d\lambda = k\omega$.
The vector field $k\omega$-dual to λ is gradientlike for $\log |s_k|$.

Construction of contact structures

Theorem (Bourgeois). *If a closed manifold V admits a contact structure, then $V \times \mathbf{T}^2$ also does.*

Proof. Let (K, θ) be a supporting open book for a given contact structure on V and let α denote the associated contact form. Choose a neighborhhood $\mathbf{D}^2 \times K$ of K in which θ is the normal angular coordinate and let r denote the normal radial coordinate.
Then set

$$\widetilde{\alpha} = f(r)(\cos\theta\, dx_1 - \sin\theta\, dx_2) + \alpha$$

where $f(r) = r$ for $r \leq r_0$, $f(r) = 1$ for $r \geq 2r_0$, and $f'(r) \geq 0$.
For r_0 small enough, $\widetilde{\alpha}$ is a contact form.

Dehn-Seidel twists

The (*right-handed*) *Dehn-Seidel twist* τ in the unit cotangent space (U, ω_0) of $\mathbf{S}^n$ is the composition of the following symplectomorphisms:

- the time π map of the geodesic flow of $\mathbf{S}^n$;
- the differential of the antipodal map.

Thus, τ is the identity on ∂U.

(F, ω): a symplectic $2n$-manifold;
$\underset{\sim}{\psi}\colon \mathbf{S}^n \to F$: a Lagrangian embedding;
$\psi\colon (U, \varepsilon\omega_0) \to (F, \omega)$: a symplectic extension of ψ.

τ induces a Dehn-Seidel twist τ_ψ in F with support in $\widetilde{\psi}(U)$.

Positive Lagrangian plumbing

(F, ω): a compact Weinstein $2n$-manifold;
$\widehat{\psi}: \mathbf{D}^n \to F$: a proper Lagrangian embedding.

There is a canonical way to attach a n-handle to (F, ω) along $\widehat{\psi}(\partial \mathbf{D}^n)$ so as to obtain a Weinstein manifold (F', ω').
Moreover, $\widehat{\psi}$ extends to a Lagrangian embedding $\psi: \mathbf{S}^n \to F'$.

Proposition. *Let $\phi: F \to F$ be a symplectic diffeomorphism with support in the interior and let $\phi' = \tau_\psi \circ \phi: F' \to F'$. There exists a diffeomorphism $\overline{\Sigma}(F, \phi) \to \overline{\Sigma}(F', \phi')$ which extends the inclusion $F \to F'$ and is a contact diffeomorphism.*

Stabilization

(K, θ), (K', θ'): two symplectic open books in the same closed manifold V.

(K', θ') is a *stabilization* of (K, θ) if it can be obtained from (K, θ) by a sequence of positive Lagrangian plumbings

Theorem.

a) *On a closed* 3*-manifold, two open books supporting the same contact structure have isotopic stabilizations.*

b) *On a closed manifold of higher dimension, two open books supporting the same contact structure and obtained by the Donaldson-Presas construction have isotopic stabilizations.*

Holomorphic fillability

A contact structure ξ on a closed manifold V is *holomorphically fillable* if there exists a compact Stein manifold W such that:

- V is the boundary of W;
- ξ is the field of complex hyperplanes tangent to $V = \partial W$.

Theorem. *A contact structure on a closed manifold V is holomorphically fillable iff it is supported by an open book whose monodromy is a product of right-handed Dehn-Seidel twists.*