Branched Covers and Seiberg-Witten Invariants

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Fix 2-dim'l homology class in s.c. alg. surface. Up to smooth isotopy — at most one emb. holo curve in this class

Symplectic case -

Th'm (F-S) T: emb sympl torus square 0 in X: s.c. sympl. 4-mfd (+ minor technical condition) $\Rightarrow$ $\forall$ $m \geq 2$ $\exists$ inf. many emb. sympl. tori $\Sigma_{m,i} \sim 2mT$ and pairwise nonsmoothly isotopic.

Impetus - Belief that for $c_1^2 > 0$ any emb sympl. surface is smoothly isotopic to emb. holo. curve.

Siebert-Tian Vidussi Etgu-Park

I. Smith Auroux-Donaldson-Katzarkov

Technique for proving thm

$$\text{Nbd } T \cong S^1 \times (S \times D^2)$$

$$\Sigma_{m,i} = S^1 \times \text{Braid of } 2m \text{ strands}$$

Distinguish via double covers br over $\Sigma_{m,i}$

SW(double covers) depend on

Alex poly of 2 comp link obtained lifting axis of braid to cover

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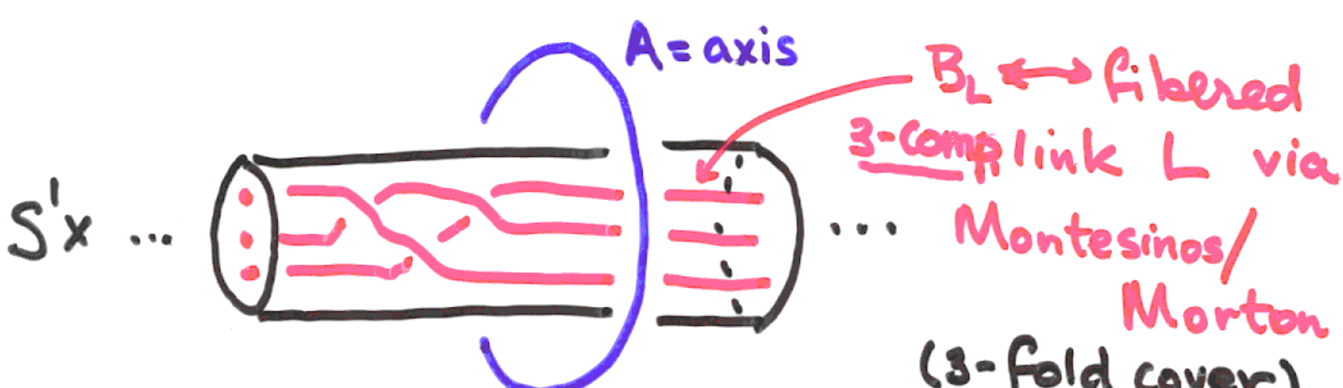
Thm (Montesinos Morton) Every fibered link in S^3 with k components can obtained as the preimage of the braid axis for a d sheeted simple br cover of S^3 branched along a suitable closed braid where $d = \max\{k, 3\}$

Thm (Kanenobu) Let $f(t)$ = symm. poly. of even degree with integral coeff's, $f(0)=\pm 1$, then for any $k \geqslant 2$ $\exists$ fibered link L k components with $f(t)$ as its Hosokawa poly $\nabla_L(t) = \Delta_L(t,\dots,t)/(t-1)^{k-2}$

Use these thms :

X: s.c. sympl. 4-mfd $\hookleftarrow T$ emb sympl torus, square 0

Nbd $N = T \times D^2 = S^1 \times (S^1 \times D^2)$



$T_L = S^1 \times B_L \subset N$ sympl & if B_L has m strands

$T_L \sim mT$

3-fold Simple Cover:

$$\begin{array}{c} (S^3, L) \\ \downarrow \pi \\ (S^3, A) \end{array}$$

Branch set B_L

$\pi|_A$ = trivial 3-fold cover

$\Rightarrow$ cover induced over $N = S^1 \times (S^1 \times D^2)$ extends trivially over X

$$\tilde{X} = \bigcup_{3\,\text{copies}} (X \setminus N) \cup \left(\tilde{N} = S^1 \times (S^3 \setminus L)\right)$$

'link surgery mfd'

Seiberg-Witten inv't of cover:

(Taubes, Park, F-S)

$$SW_{\tilde{X}} = \prod_1^3 SW_{X_i} \cdot (t_i - t_i^{-1}) \cdot \Delta_L^{\text{sym}}(t_1^2, t_2^2, t_3^2) \in \mathbb{Z}H_2(\tilde{X})$$

↳ i^{th} copy

$\downarrow \pi_*$

$t_i \leftrightarrow S^1 \times$ (merid. i^{th} comp. of L)

$\downarrow$

$t \leftrightarrow S^1 \times$ (merid of A) $= T$

$$\pi_* SW_{\tilde{X}} = SW_X^3 \cdot (t - t^{-1})^3 \, \Delta_L^{\text{sym}}(t^2, t^2, t^2)$$

Any isotopy of X taking T_L to $T_{L'}$ for another $L', B_{L'}$ and which also carries the covering data for π_L to $\pi_{L'}$

$$\begin{tikzcd}
\tilde{X}_L \arrow[r, "\hat{f}\ \cong"] \arrow[d, "\pi_L"'] & \tilde{X}_{L'} \arrow[d, "\pi_L"] \\
X \arrow[r, "f\ \cong"] & X \\
T_L \arrow[r] & T_L
\end{tikzcd}$$

& f_* id on

$$\hat{f}_*(SW_{\tilde{X}_L}) = SW_{X_L} \Rightarrow$$

$$\Delta_L^{sym}(t^2 t^2 t^2) = \Delta_{L'}^{sym}(t^2 t^2 t^2)$$

$$\begin{array}{ccc} \| & & \| \\ \nabla_L^{sym}(t^2)\,(t-t^{-1}) & & \nabla_{L'}^{sym}(t^2)(t-t^{-1}) \end{array}$$

Using Kanenobu's Thm build these at will with arb. even degree

$\frac{1}{2} \deg \nabla_L(t) =$ genus of fiber g

determines # strands of B_L m $2g+4$

$\Rightarrow$ For fixed $m \geq 6$ we get sympl tori $\sim mT$ with ∞ly many 3-fold simple br covers

Each braided torus T_L admits at most fin many simple 3-fold br covers of X with T_L as br set

$\Rightarrow$ Thm (F S) X sympl 4-mfd T emb sympl torus of square 0. Each $m \geq 6$ $\exists$ infinitely many pairwise nonsmoothly isotopic emb sympl tori homologous to mT.

Technique 2 - Fiber sums

Etgu-Park

Same hypotheses - Emb sympl torus square 0

$T \subset X$: sympl.

Same construction - Braid B

Braided torus $T_B = S^1 \times B \subset \underbrace{S^1 \times S^1}_{T} \times D^2$

T_B isotopic to $T_{B'} \Rightarrow \exists$ diffeo $f: X \to X$

$f(T_B) = T_{B'}$, $f(m_B) = m_{B'}$, $f_* = \text{id}$ on $H_* X$

Use relative SW-inv't of (X, T_B).

$SW_{\substack{X \# E(1) \\ T_B = F}}$ to distinguish the T_B

$$X \setminus N(T_B) = X \setminus \overset{\text{orig. } S^1 \times S^1 \times D^2}{N(T)} \cup S^1 \times (S^1 \times D^2 \setminus N(B))$$

$\Downarrow$

$$SW_{\substack{X \# E(1) \\ T_B = F}} = SW_X (x - x^{-1}) \Delta^{sym}_{L_B}(x^2, t^2)$$

$L_B = B \cup (A = \text{axis})$ $\quad x \longleftrightarrow T - S \times m_A$

$t \longleftrightarrow S \times m_B$

duals to R_B

$$H_2(X \#_{T_B = F} E(1)) - H_2(X) \oplus (R_B \oplus D_B) \oplus \langle S \rangle \oplus E_8$$

Rim group of $T_B \cong \mathbb{Z} \oplus \mathbb{Z}$

dual to $T_B = F$

$\hat{H}_2(E(1))$

If $\exists$ isotopy T_B to $T_{B'}$ $\exists$ diffeo

$$f \quad X \#_{T_B = F} E(1) \longrightarrow X \#_{T_{B'}, F} E(1$$

$\bar{f}_* \;{}_{H_2(X)}$ id $\qquad \bar{f}_*(R_B) = R_{B'} \qquad \Rightarrow$

$\bar{f}_*(SW_X) - SW_X \qquad \bar{f}_*(x) = x \qquad \Rightarrow$

$$\Delta^{sym}_{L_B}(x^2, f_*(t)^2) = \Delta^{sym}_{L_{B'}}(x^2, t^2)$$

Easy to use this to show

$$\Delta_{L_B}(x,t) = \Delta_{L_{B'}}(x,t)$$

Th'm $\forall\, m \geq 2$ $\exists$ ∞'ly many (distinct up to smooth isotopy) emb sympl tori $\sim mT$.

For proof need -

Given $m \geq 2$ $\exists$ ∞'ly many braids $B(m,i)$ of m-strands $\ni$ Alex poly's of links $L(m,i) = B(m,i) \cup \text{Axis}$ are distinct poly's of 2 vbles.

<u>Ex</u>. Etgu-Park give such examples.

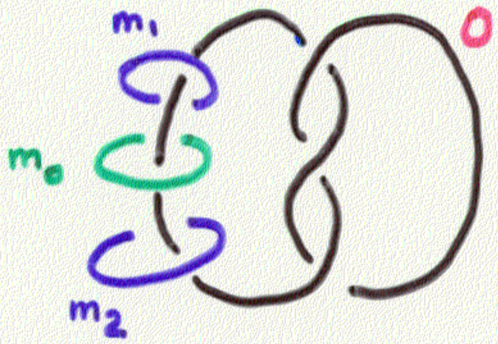
Lagrangian Tori

Stefano - detect via knot surgery - or even π_1.

Our point of view - log transforms and branched covers

Example :

$K =$ trefoil



$M_K = \Sigma_1 \times_\varphi S^1$

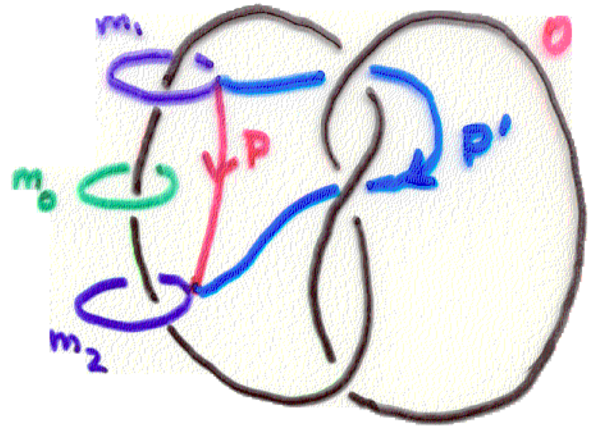
$E(1)_K = E(1) \underset{F = S^1 \times m_0}{\#} S^1 \times M_K$

$X =$ double br cover of $E(1)_K$ br over $S^1 \times (m_1 \cup m_2)$

$X = E(1)' \underset{F' = S^1 \times m_0'}{\#} S^1 \times \tilde{M}_K \underset{S^1 \times m_0'' = F''}{\#} E(1)''$

$\tilde{M}_K = M_{K \# K}$

Pair of Lagr tor in X



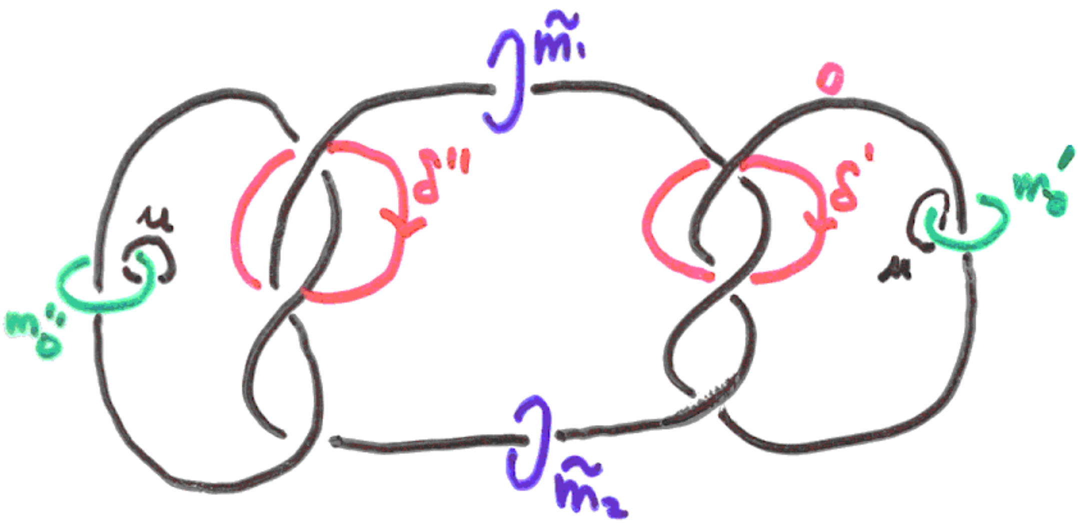
P P' lift to circles γ γ on fibers of

$$\begin{array}{c} \Sigma_2 \longrightarrow M_{K\#K} \\ \downarrow \\ S^1 \end{array}$$

Get Lagr tori T $T_\gamma = S^1 \times \gamma$ $T = T_\gamma$ $S^1 \times \gamma'$

in $S^1 \times (M_{K\#K} \setminus (m_0 \cup m_0''))\subset X$

$H_2(M_{K\#K} \setminus (m_0' \cup m_0''))$ $\mathbb{Z}\oplus\mathbb{Z}$ gen by $m_0 \sim m_0''$ & μ = merid to $m_0' \cap m_0''$



γ $\gamma \sim \delta' + \delta''$

$\delta' \sim 0$ $\delta'' \sim 0$

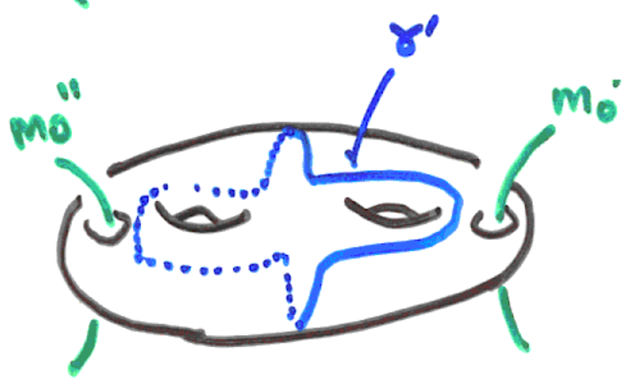
in $H_2(M_{K\#K} \setminus (m_0' \cup m_0''))$

$\Rightarrow$ tori $\Sigma' = S^1 \times \delta$ $\quad \Sigma'' = S^1 \times \delta''$

nullhomologous (& Lagr) in $S^1 \times (M_{K\#K} \setminus (m_0' \cup m_0''))$

$T' \; T \sim \Sigma' + \Sigma''$ $\Rightarrow$ $T \; T'$ homologous in X

fiber of $M_{K\#K}$



$T = S \times \gamma$ homologous to $S \times \mu$ rim torus of fibers of $E(1$ $\; E(1)''$ where gluing takes place

$\Rightarrow$ $\underline{T \text{ essential in } X}$

Claim $T \; T'$ not isotopic in X

If T' isotopic to T $(m_{T'} \to m_T)$

Any surgery on T' $\longleftrightarrow$ Some surgery on T

$$\underset{\overset{\shortparallel}{X}}{\text{Surgery killing } m_{T'}} \longleftrightarrow \underset{\overset{\shortparallel}{X}}{\text{Surgery killing } m_T}$$

Basis for $H_2(T^3 = \partial(T \times D^2))$

$[S^1 \times pt \times pt]$ $\quad$ $[pt \times \gamma_L \times pt]$ $\quad$ m_T

pushoff in fiber of $M_{K\#K}$

$X(p,q,r)$: result of surgery killing

$$p[S^1 \times pt \times pt] + q[pt \times \gamma_L \times pt] + r\, m_T$$

$X'(p,q,r)$: same for T' (using $\gamma_{L'}$)

$\{SW_{X(p,q,r)}\}$ is isotopy inv't of T.

$$SW_{X(0,0,1)} = \Delta^{sym}_{K\#K}(t_F^2, t_F^2) = (t_F^2 - 1 + t_F^{-2})^2$$

$t_F \leftrightarrow F(=$ fiber of $E(1)) = S^1 \times m_0' = S^1 \times m_0''$

Morgan - Mrowka - Szabo formula :

$$\sum SW_{X(p,q,r)}(k'''+2iT) = p\sum SW_{X(1,0,0)}(k'+2iT)$$
$$+$$
$$q\sum SW_{X(0,1,0)}(k''+2iT)$$
$$+$$
$$r\sum SW_{X(0,0,1)=X}(k+2iT)$$

where T = core of surgery in approp. mfd

k''' = homology class of $X(p,q,r)$ which

$$\begin{array}{ccc} H_2(X(p,q,r)) & \longrightarrow & H_2(X(p,q,r), N(T)) \\ & & \| \\ k''' & \longrightarrow & H_2(X \setminus N(T), \partial) \\ k & \longrightarrow & \| \\ H_2(X) & \longrightarrow & H_2(X, N(T)) \end{array}$$

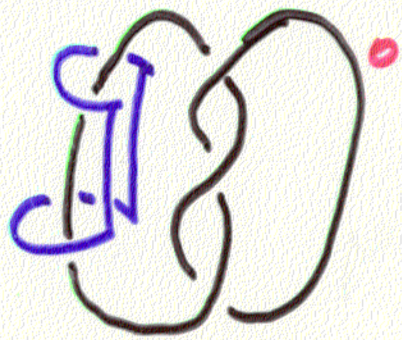
Same for k', k''

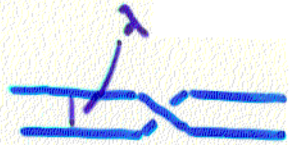
$$\begin{array}{ccccc} SW_{X(p,q,r)} & \in & \mathbb{Z}H_2(X(p,q,r)) & & t_T \\ \downarrow & & \downarrow & & \downarrow \\ \overline{SW}_{X(p,q,r)} & \in & \mathbb{Z}H_2(X(p,q,r), N(T)) & & 1 \end{array}$$

This gets rid of the sums in the formula:

$$\overline{SW}_{X(p,q,r)} = p\,\overline{SW}_{X(1,0,0)} + q\,\overline{SW}_{X(0,1,0)} + r\,\overline{SW}_{X}$$

$X(0,1,0)$ double br cover of $E(1)_K$ with branch set:

$S^1\times$ 

Idea: Double br cover of $S^1\times$ 

$= +1$ surgery on $S^1\times\tilde{\lambda}$ in

$\left(S^1\times\right.$  $\left.\right)^\sim$

0-surgery on $S^1\times\tilde{\lambda}$ = double br. cover of $S^1\times$ 

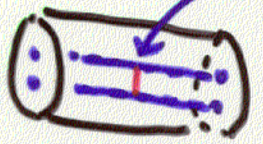
$$\Rightarrow \quad X(0,1,0) = E(1 \;\#_F\; [S^1\times(M_K\#M_K)] \;\#_F\; E(1)$$

$$SW_{X(0,1,0)} = 0$$

$X(1,0,0)$ Also a br. cover —

Base Y = result of (honest) surgery on

$$S^1\times pt \subset S^1\times(M_K\setminus m_0)\subset E(1)_K$$

$S^1\times\Big(D^3 =$  $\Big)$

replaced by $S^2\times D^2$.

In each torus in br set $S^1\times I$ removed

replaced by $D^2\amalg D^2$

New br set $= S^2\amalg S^2 = S^2\times (\,:\,)$

$$\Rightarrow\ X(1,0,0) = [Y\setminus(S^2\times D^2)]\cup S^2\times S^1\times I\ \cup[Y\setminus(S^2\times D^2)]$$

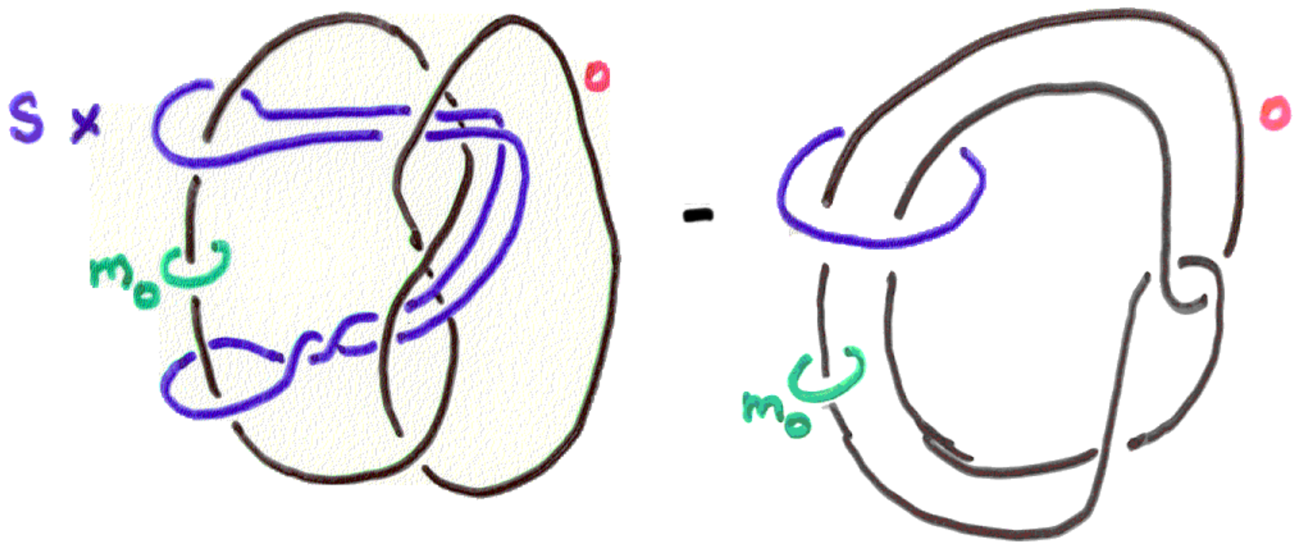
$$SW_{X(1,0,0)}=0$$

Same for T' $\qquad SW_{X'(1,0,0)}=0$

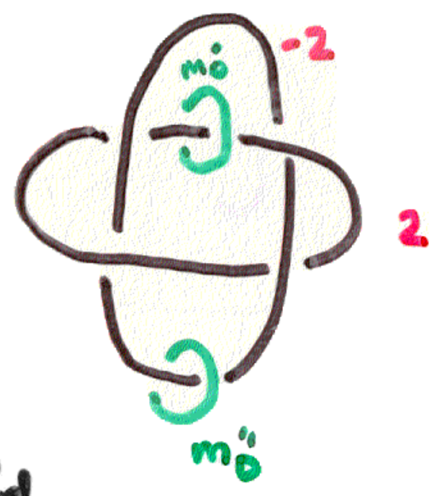
$\Rightarrow$ For T get invts $\{r(t_F^2-1+t_F^{-2})^2\}$

For T' still need to calculate $SW_{X'(0,1,0)}$

$X'(0,1,0)$ = br cover of $E(1)_K$ br over



Double br cover is



Torsion of link

$m_0 \cup m_0''$ in this mfd

1

$SW_{X'(0,\ ,0)} = 1$

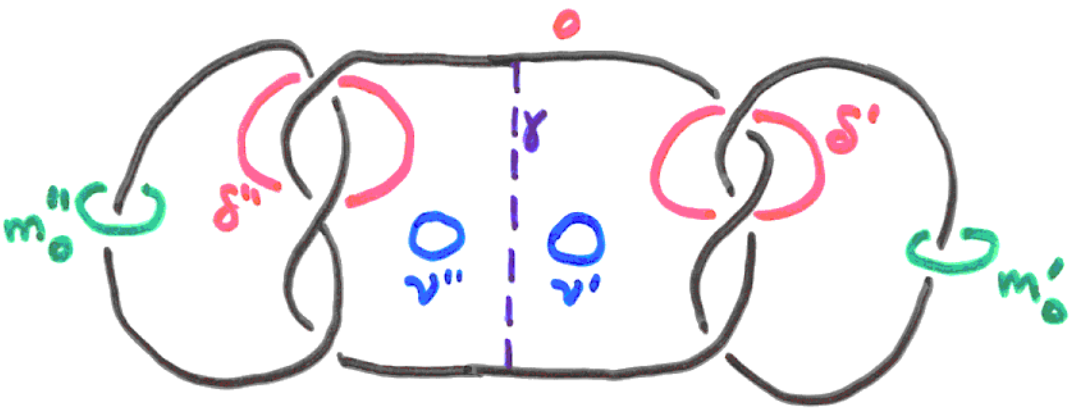
Invt's for T $\{q + r(t_F^2 + t_F^{-2})^2\}$

$\Rightarrow$ T' not isotopic to T

Rk Auroux Donaldson Katzarkov show surgery mfds $X'(0\ k, \ )$ are sympl. $\forall\ k$

Corr SW is $k + SW_X$ (& leading coeff $= 1$)

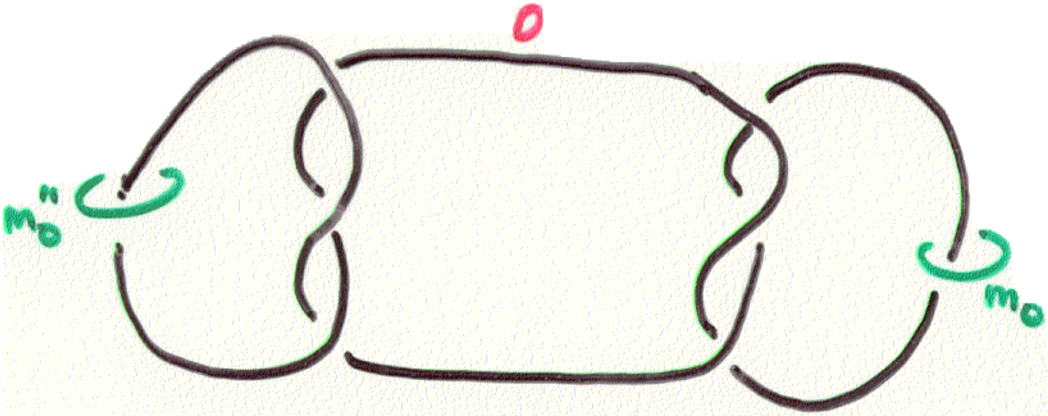
Key to construction Equivariant pair of nullhomologous tori not isotopic to equivt pair of 'trivial tori

$S^1 \times$  $\subset X$

(Equivariant) Lagrangian Circle Sum

The 0 (Lagrangian) framing on δ' or δ''
= usual (-1) framing

Do on both δ' & δ'' get



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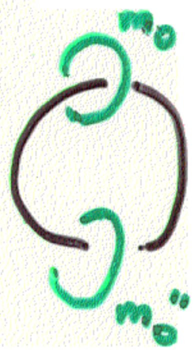


I.e. $E(1) \#_F T^2 \times S^2 \#_F E(1) = E(2)$

SW = 1

Same surgery on $S^1 \times (\nu \cup \nu'')$ yields

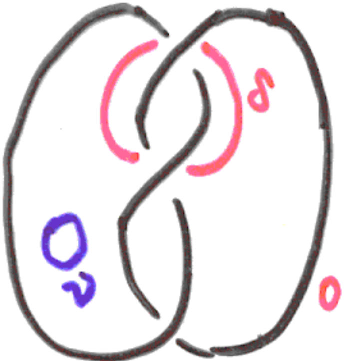
$X \# S^2 \times T^2 \# S^2 \times T^2$ SW = 0

Goal Prove that T not isotopic to T' directly from this

F nal comment

n $E(\ )_K$

$S^1 \times \delta$ **nullhomologous** Lagr torus not isotopic to $S^1 \times \nu$

$S^1 \times$ 

Same ana ysis as above

(now need $SW^{\pm}$)