Minimizing Finite Sums with the Stochastic Average Gradient Algorithm

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Joint work with Nicolas Le Roux and Francis Bach

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Context: Machine Learning for "Big Data"

- **Large-scale machine learning**: large N, large P
  - N: number of observations (inputs)
  - P: dimension of each observation
- **Regularized empirical risk minimization**: find x^* solution of

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- Applications to any data-oriented field:
  - Vision, bioinformatics, speech, natural language, web.
- Main practical challenges:
  - Choosing regularizer r and data-fitting term f.
  - Designing/learning good features a_i.
  - Efficiently solving the problem when N or P are very large.

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- Simplest example is ℓ_2-regularized least-squares,

$$f_i(x) := (a_i^T x - b_i)^2 + \frac{\lambda}{2}\|x\|^2.$$

- Other examples include any ℓ_2-regularized convex loss:
  - logistic regression, Huber regression, smooth SVMs, CRFs, etc.

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- Stochastic gradient method [Robbins & Monro, 1951]:
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  - Iteration cost is independent of N.
  - Sublinear convergence rate: $O(1/t)$.

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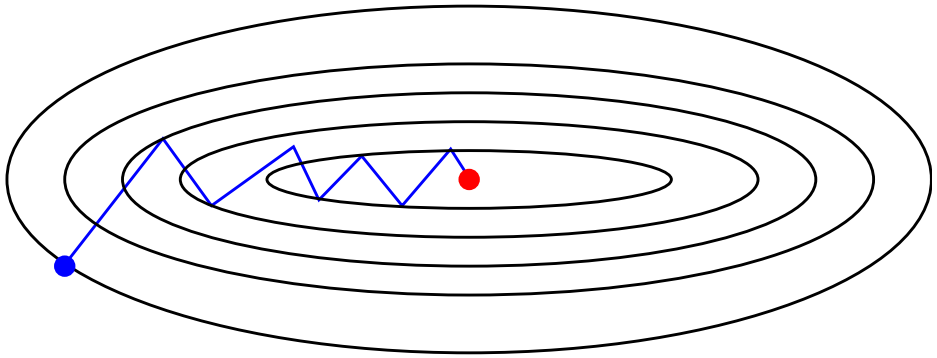

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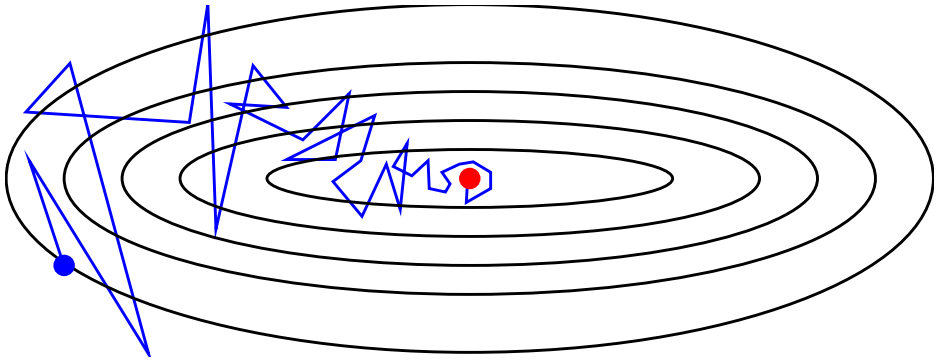

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- FG method has $O(N)$ cost with linear rate.
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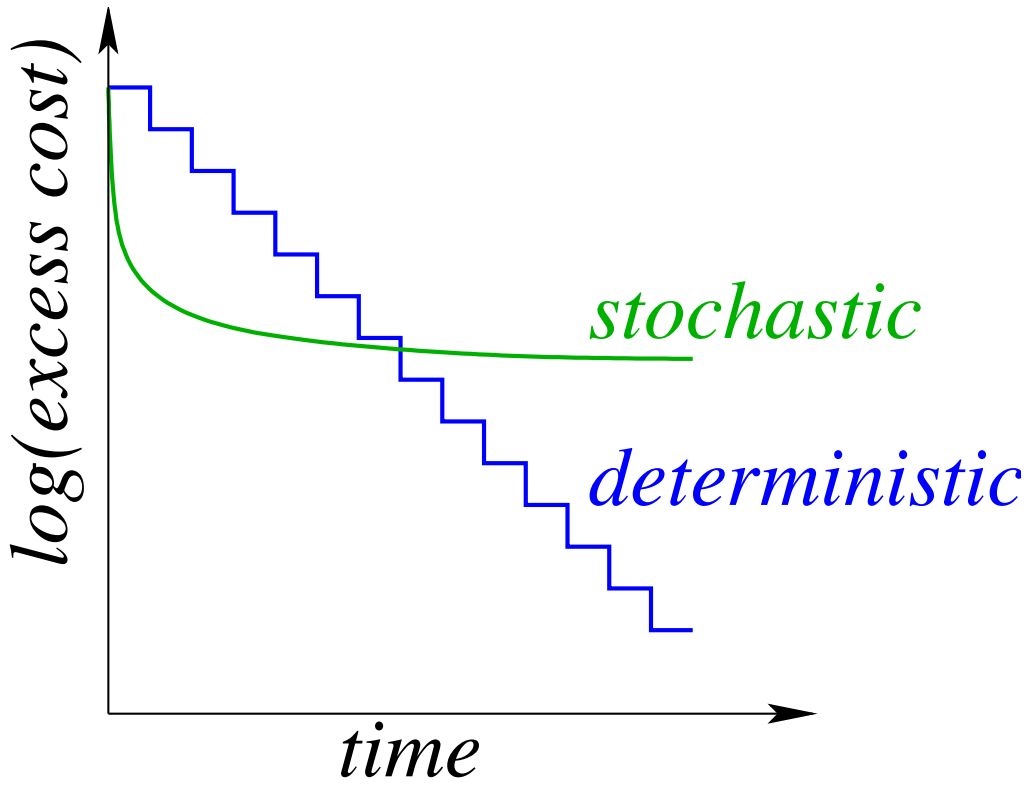

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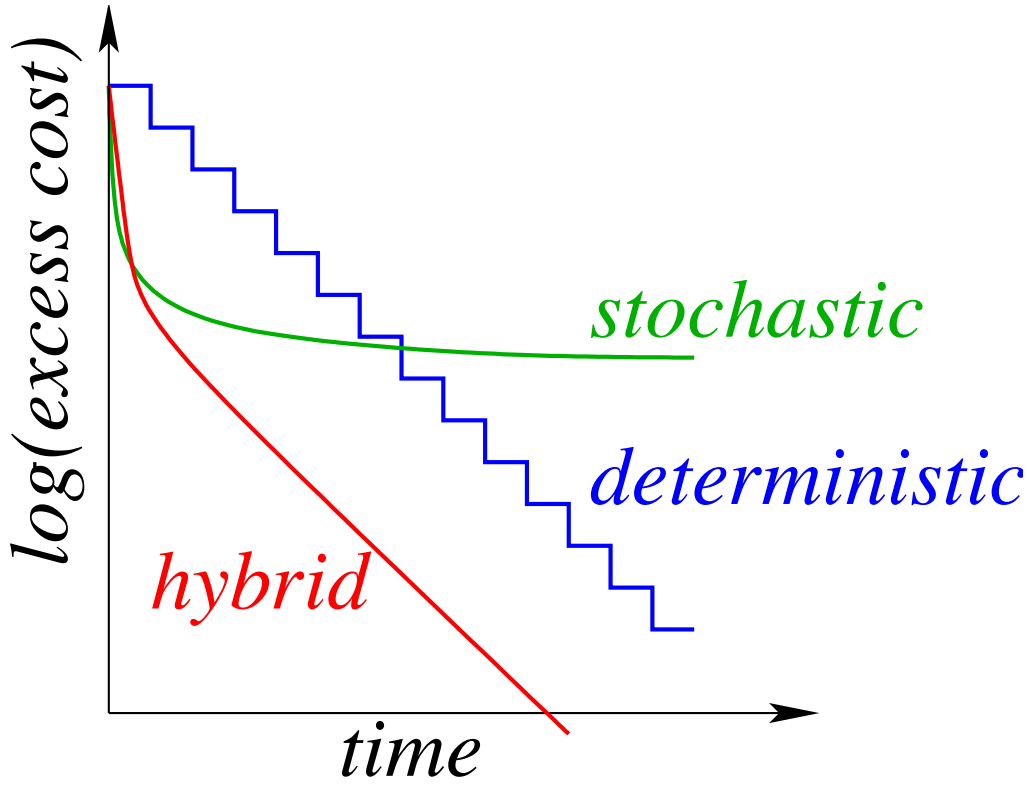

- Goal is linear rate **and** $O(1)$ cost.

Prior Work on Speeding up SG Methods

A variety of methods have been proposed to speed up SG methods:

- **Step-size strategies, momentum, gradient/iterate averaging**
  - Polyak & Juditsky (1992), Tseng (1998), Kushner & Yin (2003) Nesterov (2009), Xiao (2010), Hazan & Kale (2011), Rakhlin et al. (2012)
- **Stochastic version of accelerated and Newton-like methods**
  - Bordes et al. (2009), Sunehag et al. (2009), Ghadimi and Lan (2010), Martens (2010), Xiao (2010), Duchi et al. (2011)

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- None of these methods improve on the $O(1/t)$ rate

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Existing linear convergence results:

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  - Linear convergence but only up to a fixed tolerance
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- **Special Problems Classes**
  - Collins et al. (2008), Strohmer & Vershynin (2009), Schmidt and Le Roux (2012), Shalev-Shwartz and Zhang (2012)
  - Linear rate but limited choice for the f_i's

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    - **Memory**: $y_i^t = f_i'(x^k)$ from the last t where i was selected.
  - Assumes that gradients of other examples don't change.
  - This assumption becomes accurate as $||x^{t+1} - x^t|| \to 0$.
  - Stochastic variant of increment average gradient (IAG). [Blatt et al. 2007]

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  - Well-conditioned problems: constant non-trivial reduction per pass:

$$\left(1 - \frac{1}{8N}\right)^N \leq \exp\left(-\frac{1}{8}\right) = 0.8825.$$

  - Badly-conditioned problems, almost same as deterministic method. (gradient method has rate $\left(1 - \frac{\mu}{L}\right)^{2t}$ if $\alpha_t = \frac{1}{L}$, but N times slower)

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  - SAG: $O(\max\{N, \frac{L}{\mu}\}\log(1/\epsilon))$.

Constants in Convergence Rate

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where if we initialize with $y_i^0 = 0$ *we have*

$$C = [g(x^0) - g(x^*)] + \frac{4L}{N}\|x^0 - x^*\|^2 + \frac{\sigma^2}{16L},$$

and if we initialize with $y_i^0 = f_i'(x^0) - g'(x^0)$ *we have*

$$C = \frac{3}{2}[g(x^0) - g(x^*)] + \frac{4L}{N}\|x^0 - x^*\|^2.$$

- If we initialize with N iterations of SG, $[g(x^0) - g(x^*)]$ and $\|x^0 - x^*\|^2$ are in $O(1/N)$ so $C = O(1/N)$.

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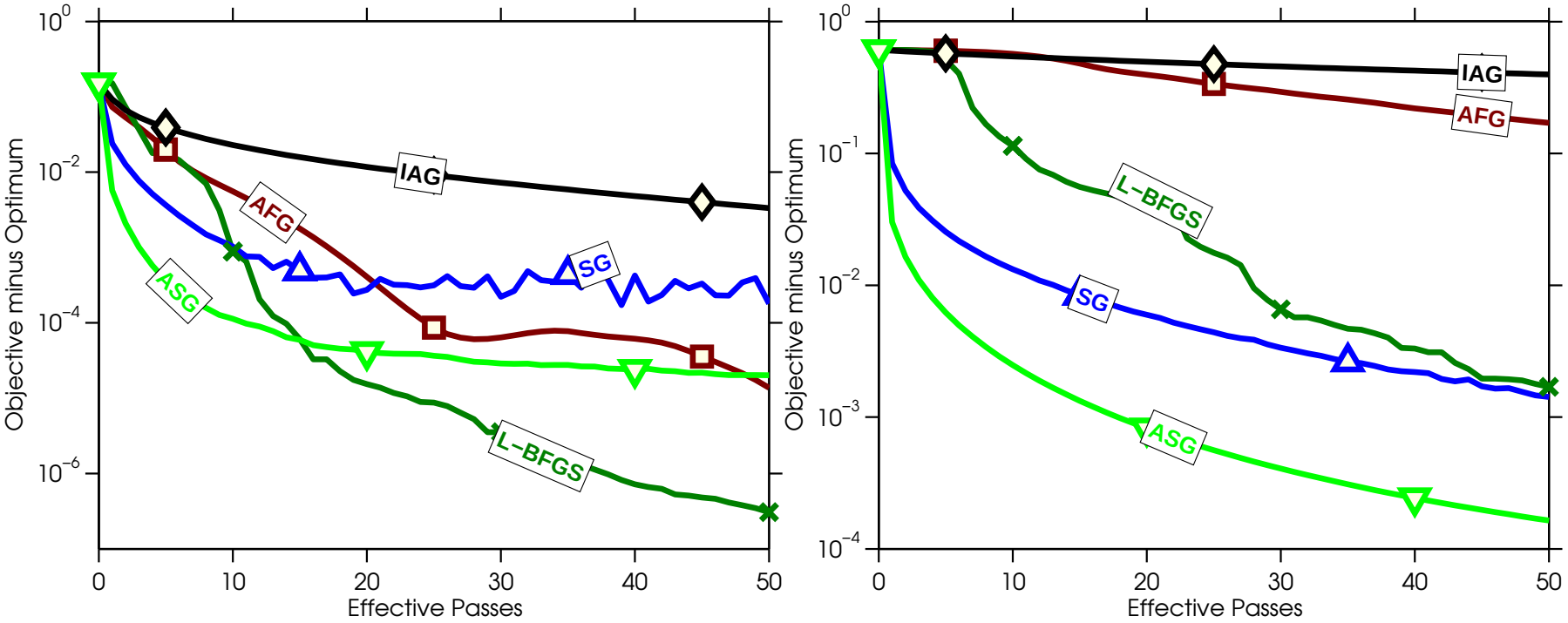
- Same algorithm could be used in non-convex case.
- Contrast with SDCA:
  - Requires explicit strongly-convex regularizer.
  - Not adaptive to μ, does not allow $\mu = 0$.

Comparing FG and SG Methods

- quantum ($n = 50000$, $p = 78$) and rcv1 ($n = 697641$, $p = 47236$)



- Comparison of competitive deterministic and stochastic methods.

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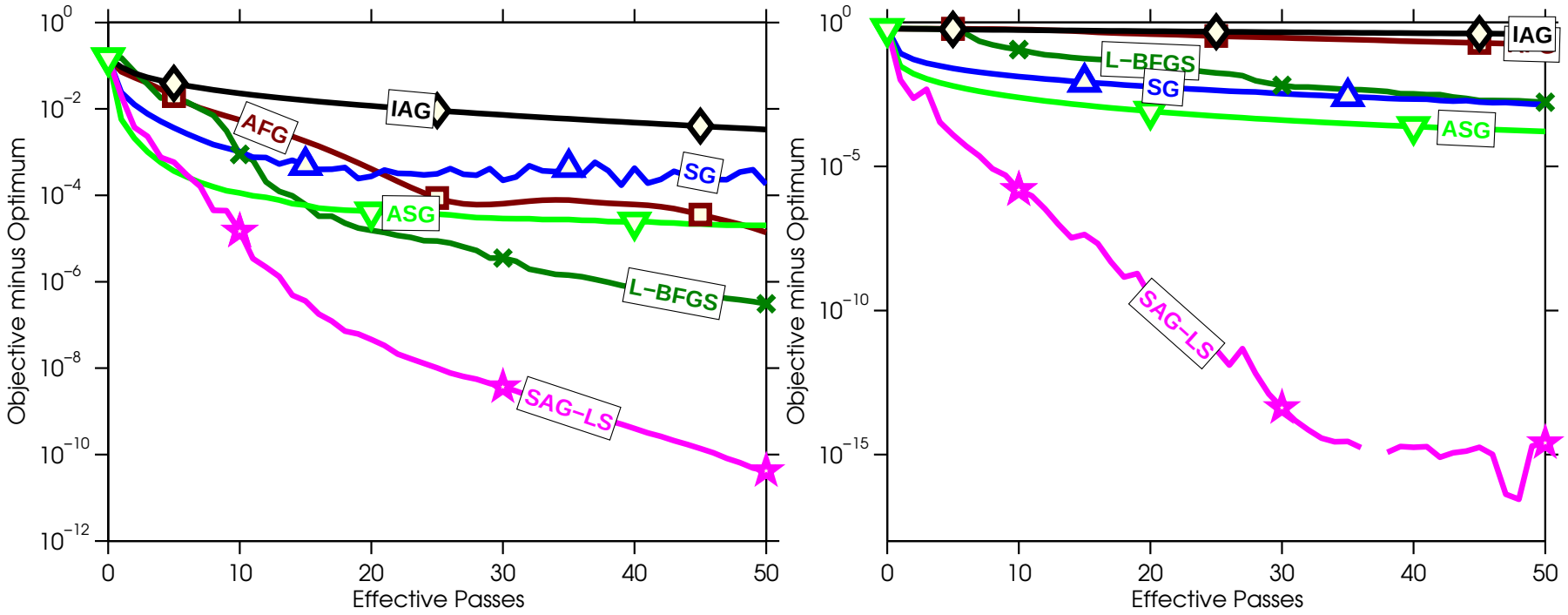

- SAG starts fast and stays fast.

SAG Compared to Coordinate-Based Methods

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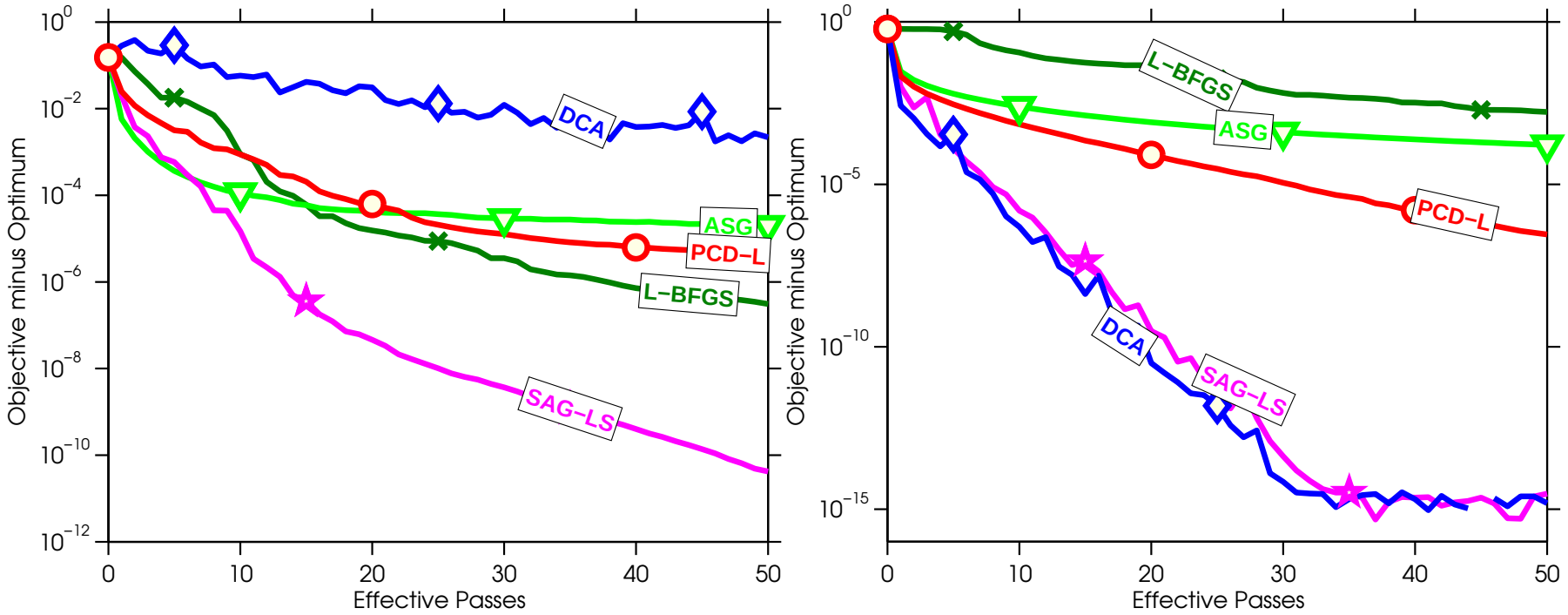

- PCD/DCA are similar on some problems, much worse on others.

SAG Implementation Issues

- while(1)
  - Sample i from $\{1, 2, \ldots, N\}$.
  - Compute $f_i'(x)$.
  - $d = d - y_i + f_i'(x)$.
  - $y_i = f_i'(x)$.
  - $x = x - \frac{\alpha}{N} d$.

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  - Can we handle sparse data?
  - How should we set the step size?
  - When should we stop?
  - Can we use mini-batches?
  - Should we shuffle the data?

Implementation Issues: Normalization

- Should we normalize by N in the early iterations?
- The parameter update:
  - $x = x - \frac{\alpha}{N} d$.

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- Should we normalize by N in the early iterations?
- The parameter update:
  - $x = x - \frac{\alpha}{M} d$.
- We normalize by number of examples seen (M).
- Better performance on early iterations.

Implementation Issues: Memory Requirements

- Can we reduce the memory?
- The memory update for $f_i(a_i^T x)$:
  - Compute $f_i'(a_i^T x)$.
  - $d = d - (y_i - f_i'(a_i^T x))$.
  - $y_i = f_i'(a_i^T x)$.

Implementation Issues: Memory Requirements

- Can we reduce the memory?
- The memory update for $f_i(a_i^T x)$:
  - Compute $f_i'(\delta)$, with $\delta = a_i^T x$.
  - $d = d - a_i(y_i - f'(\delta))$.
  - $y_i = f_i'(\delta)$.
- Only store the scalars $f_i'(\delta)$.
- Reduces the memory from $O(NP)$ to $O(N)$.

Implementation Issues: Sparsity

- Can we handle sparse data?
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- For sparse data, d_j is typically constant.
- Apply previous k updates when it changes.
- Reduces the iteration cost from $O(P)$ to $O(\|f_i'(x)\|_0)$.
- Standard tricks allow ℓ_2-regularization and ℓ_1-regularization.

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  - Increase L until we satisfy:

$$f_i(x^+ - \frac{1}{L}f_i'(x)) \leq f_i'(x) - \frac{1}{2L}\|f_i'(x)\|^2.$$

    (assuming $\|f_i'(x)\|^2 \geq \epsilon$)
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- For $f_i'(a_i^T x)$, this line-search is $O(1)$ in N and P:

$$f_i'(a_i^T x - \frac{f'(\delta)}{L}\|a_i\|^2).$$

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- And SAG has $y_i \to f_i'(x)$.
- We can check the size of $\|\frac{1}{N}d\| = \|\frac{1}{N}\sum_{i=1}^{N} y_i\| \to \|f'(x)\|$

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  - Yes, define each f_i to include more than one example.
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  - In practice, use the line-search on the batch to determine $L_{\mathcal{B}}$.

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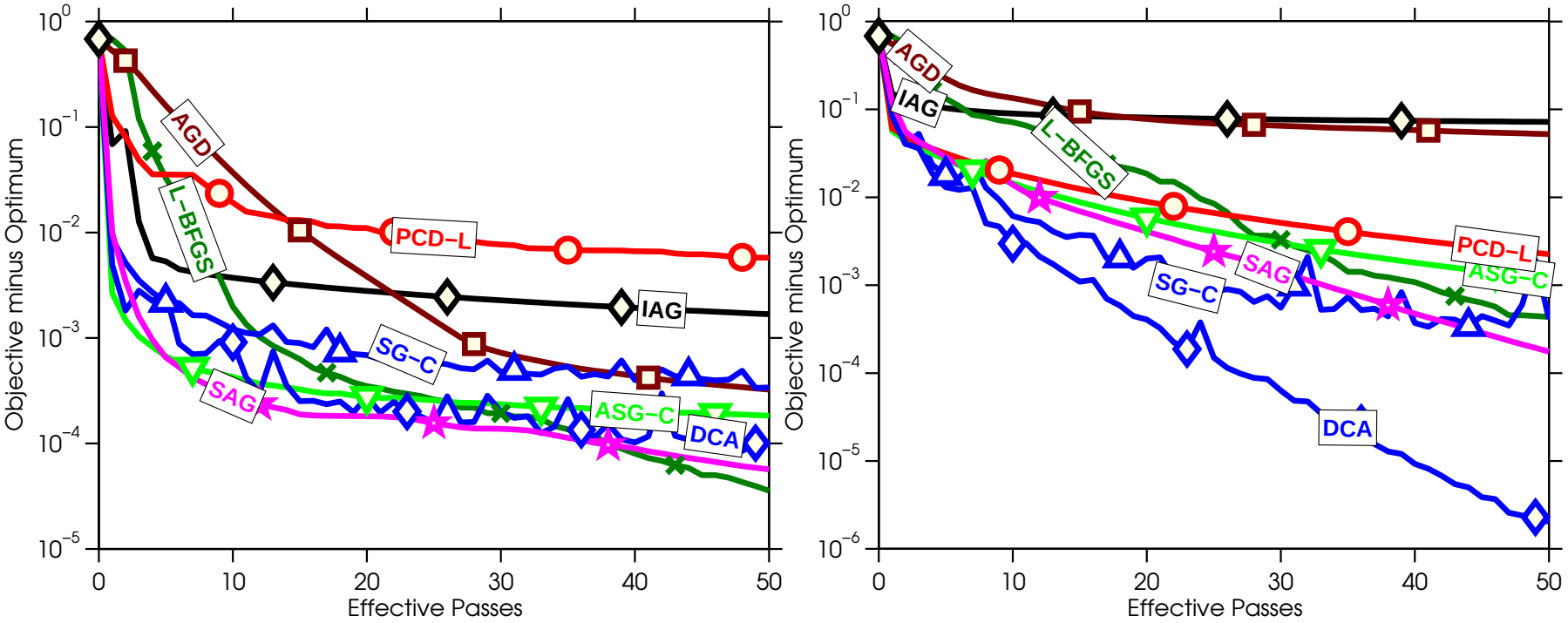
  - Better performance using partially biased sampling.
  - Combine with the line-search for adaptive sampling.

    (see paper/code for details)

SAG with Non-Uniform Sampling

- protein ($n = 145751$, $p = 74$) and sido ($n = 12678$, $p = 4932$)



- Datasets where SAG had the worst relative performance.

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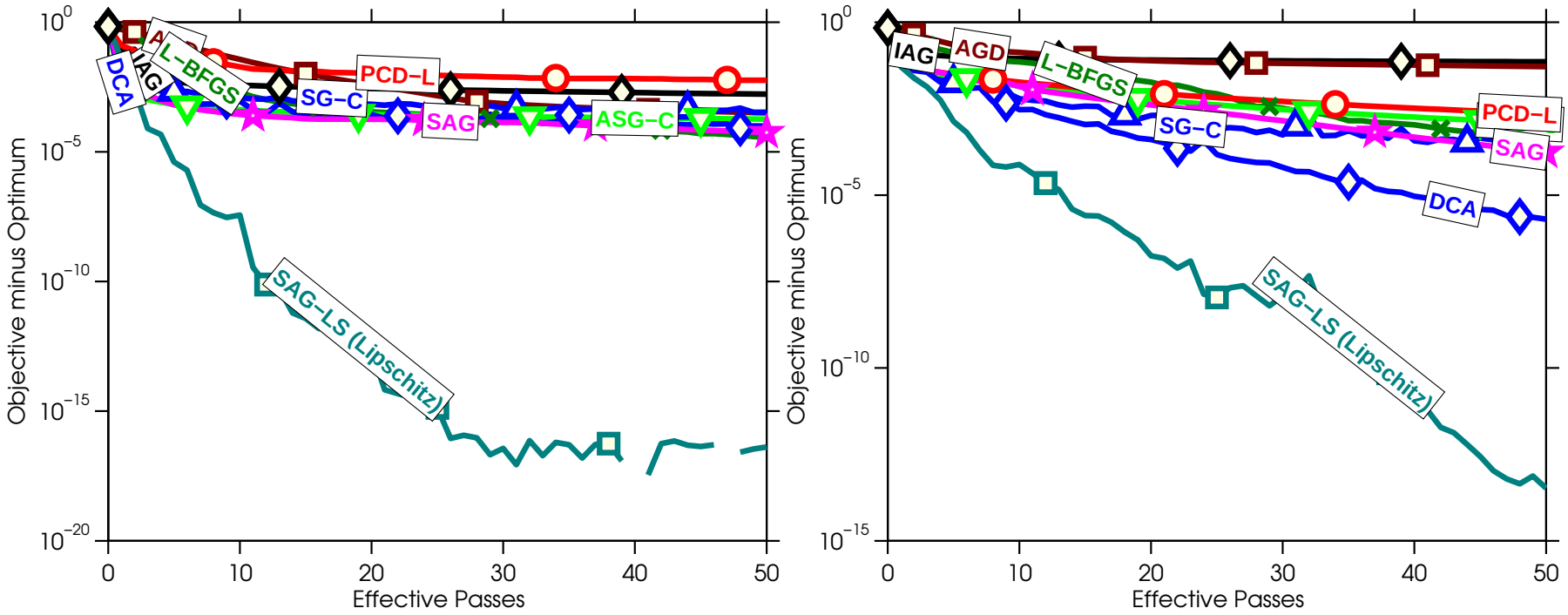

- Lipschitz sampling helps a lot.

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- For the adventurous:
  - Accelerated (seems to work, but requires a small step size).
  - Newton-like (diagonal-scaling didn't seem to help).
  - Going asynchronous (algorithm seems to be robust to this).
  - Simpler proof technique.