Curvature and spanning trees in metric spaces

Raanan Schul

UCLA

Mathematics of Knowledge and Search Engines: Tutorials

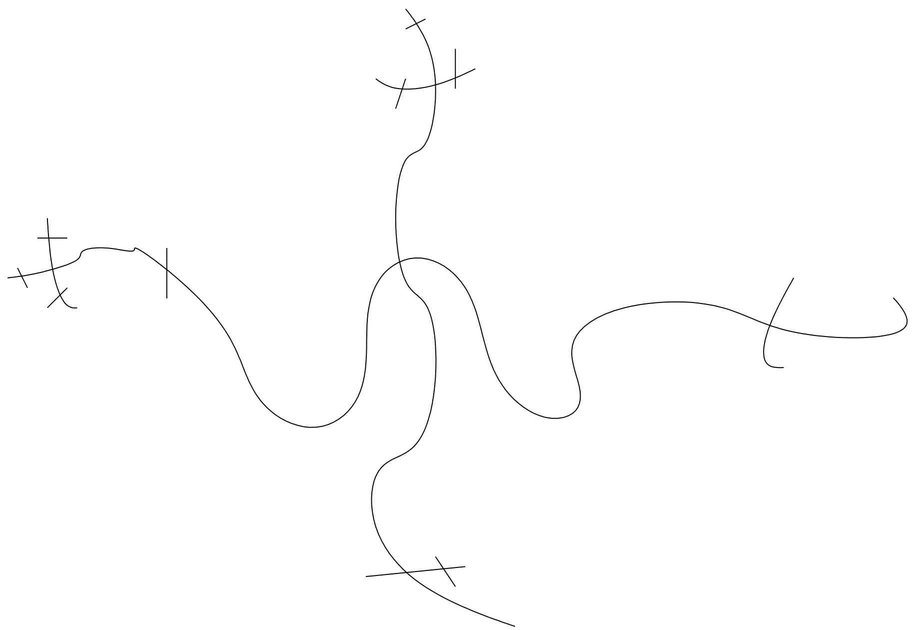
Sep. 19, 2007

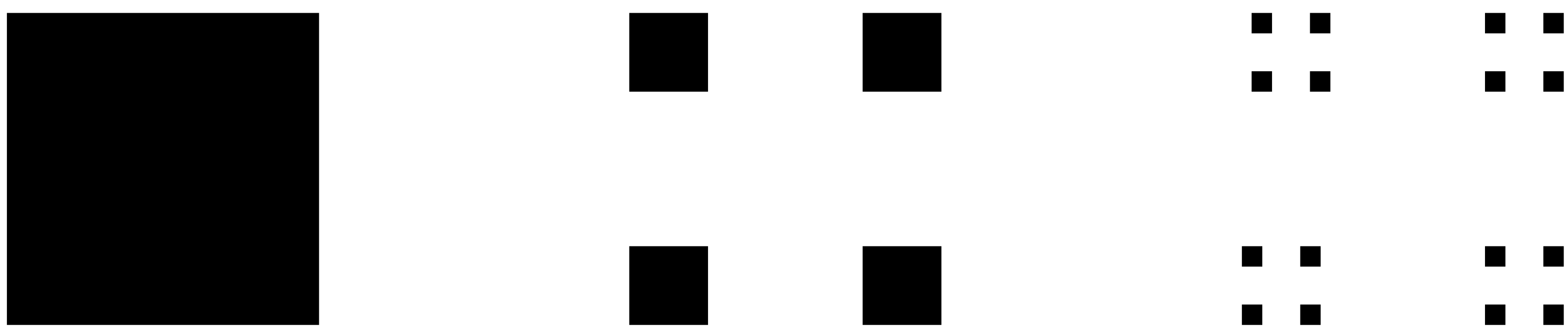
How I was brought into this...

- Want to discuss the geometry of sets of points in $\mathbb{R}^d$ or Hilbert space.
- Results about sets lying in $\mathbb{R}^d$ usually have constants that depend **exponentially** on d.
- This is called ‘the curse of dimensionality’
- We will discuss a theorem where this ‘curse’ disappears...
- And then move on to metric spaces...

Basic Questions:

- When is a set $K \subset \mathbb{R}^d$ contained inside a single connected set of finite length?
- Can we estimate the length of the shortest connected set containing K?
- What do these estimates depend on?
  - Number of points??
  - Ambient dimension (=d for $\mathbb{R}^d$) ??
- (we will see how does this connects to metric embeddings)





Quantitative Rectifiability

Peter Jones and Kate Okikiolu

- Intuitive Picture:
  - A connected set (in $\mathbb{R}^d$) of finite length is ‘flat’ on most scales and in most locations.
  - This can be used to characterize subsets of finite length connected sets.
  - One can give a quantitative version of this using multiresolutional analysis.

Quantitative Rectifiability

Definition: (Jones β number)

$$\beta_K(Q) = \frac{2}{\mathrm{diam}(Q)} \inf_{L \text{ line}} \sup_{x \in K \cap Q} \mathrm{dist}(x, L)$$

$$= \frac{\text{width of the thinest tube containing } K \cap Q}{\mathrm{diam}(Q)}.$$

(draw)

Quantitative Rectifiability

Theorem 1:[P. Jones]

For any connected $\Gamma \subset \mathbb{C}$

$$\sum_{Q \in \textit{dyadic grid}} \beta_\Gamma^2(3Q)\mathrm{diam}(Q) \lesssim \ell(\Gamma)$$

Quantitative Rectifiability

Theorem 1:[P. Jones]

For any connected $\Gamma \subset \mathbb{C}$

$$\sum_{Q \in \textit{dyadic grid}} \beta_\Gamma^2(3Q)\mathrm{diam}(Q) \lesssim \ell(\Gamma)$$

Theorem 2:[P. Jones] *For any set* $K \subset \mathbb{R}^d$*, there exists* $\Gamma_0 \supset K$*,* Γ_0 *connected, such that*

$$\ell(\Gamma_0) \lesssim \sum_{Q \in \textit{dyadic grid}} \beta_K^2(3Q)\mathrm{diam}(Q) + \mathrm{diam}(K)$$

(and in particular $K \subset \mathbb{C}$*).*

Quantitative Rectifiability

Theorem 1:[P. Jones] [K. Okikiolu]
For any connected $\Gamma \subset \mathbb{C}$ **or** $\Gamma \subset \mathbb{R}^d$

$$\sum_{Q \in \textit{dyadic grid}} \beta_\Gamma^2(3Q)\mathrm{diam}(Q) \lesssim \ell(\Gamma)$$

Theorem 2:[P. Jones] *For any set* $K \subset \mathbb{R}^d$*, there exists* $\Gamma_0 \supset K$*,* Γ_0 *connected, such that*

$$\ell(\Gamma_0) \lesssim \sum_{Q \in \textit{dyadic grid}} \beta_K^2(3Q)\mathrm{diam}(Q) + \mathrm{diam}(K)$$

Corollary:

For any connected set $\Gamma \subset \mathbb{R}^d$

$$\mathrm{diam}(\Gamma) + \sum_{Q \in \textit{dyadic grid}} \beta_\Gamma^2(3Q)\mathrm{diam}(Q) \sim \ell(\Gamma)$$

More generally:

For any set $K \subset \mathbb{R}^d$

$$\operatorname{diam}(K) + \sum_{Q \in \textit{dyadic grid}} \beta_K^2(3Q)\operatorname{diam}(Q) \sim \ell(\Gamma_{MST})$$

where Γ_{MST} *is the shortest curve containing* K*.*

Proof of corollary:

$$\begin{aligned}
\ell(\Gamma_{MST}) &\leq \ell(\Gamma_0) \\
&\lesssim \operatorname{diam}(K) + \sum_{Q \in \text{dyadic grid}} \beta_K^2(3Q)\operatorname{diam}(Q) \\
&\leq \operatorname{diam}(K) + \sum_{Q \in \text{dyadic grid}} \beta_{\Gamma_{MST}}^2(3Q)\operatorname{diam}(Q) \\
&\lesssim \ell(\Gamma_{MST})
\end{aligned}$$

Dictionary discovered by Peter Jones.

wavelets	Jones' β numbers
$\{a_{j,k}\}$ for function f	$\{\beta(Q)\}$ for set K
analysis and synthesis of the function f	analysis and synthesis of curve $\Gamma \supseteq K$
$\|f\|^2 = \sum \lvert a_{j,k}\rvert^2$	$l(\Gamma) \sim$ $\sum \beta(Q)^2 \cdot \mathrm{diam}(Q) + \mathrm{diam}(\Gamma)$
Wavelet square function $W_\psi(x)^2$	Jones' function $J(x)$

Slide by Gilad Lerman.

Issue to fix

Thm 1: $\forall$ connected $\Gamma \subset \mathbb{R}^d$

$$\sum_Q \beta_\Gamma^2(3Q)\operatorname{diam}(Q) \lesssim \ell(\Gamma)$$

Thm 2: $\forall K \subset \mathbb{R}^d$, $\exists$ connected $\Gamma_0 \supset K$, s.t.

$$\ell(\Gamma_0) \lesssim \operatorname{diam}(K) + \sum_Q \beta_K^2(3Q)\operatorname{diam}(Q)$$

Constants that make inequalities true are **exponential in d**.

Example: three points!

Result

Thm 1: $\forall$ connected $\Gamma \subset \mathbb{R}^d$

$$\sum_Q \beta_\Gamma^2(3Q)\mathrm{diam}(Q) \lesssim \ell(\Gamma)$$

Thm 2: $\forall K \subset \mathbb{R}^d$, $\exists$ connected $\Gamma_0 \supset K$, s.t.

$$\ell(\Gamma_0) \lesssim \mathrm{diam}(K) + \sum_Q \beta_K^2(3Q)\mathrm{diam}(Q)$$

"Theorem" :

One can reformulate theorems 1 and 2 in a way which will give constants **independent of dimension**

(Actually, reformulated theorems are true for Γ or K in **Hilbert space**).

Definitions

- let $K \subset \mathbb{R}^d$ be a subset with $\mathrm{diam}(K) = 1$.
- $X_n \subset K$ is 2^{-n} net for K means
  - $x, y \in X_n$ then $\mathrm{dist}(x, y) \geq 2^{-n}$
  - For any $y \in K$ exits an $x \in X_n$ with $\mathrm{dist}(x, y) < 2^{-n}$
- Take $X_n \subset K$ a 2^{-n} net for K, with $X_n \supset X_{n-1}$
- Define the multiresolution

$$\mathcal{G}^K = \{B(x, A2^{-n}) : x \in X_n; n \geq 0\}$$

- $\mathcal{G}^K$ replaces the dyadic grid

Result

Constants that make inequalities true are **independent of dimension** d (Theorems hold in Hilbert Spaces.)

Theorem 1':(R.S.) *For any connected* $\Gamma \subset H, \Gamma \supset K$

$$\sum_{Q \in \mathcal{G}^K} \beta_\Gamma^2(Q) \mathrm{diam}(Q) \lesssim \ell(\Gamma)$$

Theorem 2':(R.S.) *For any set* $K \subset H$*, there exists* $\Gamma_0 \supset K$*,* Γ_0 *connected, such that*

$$\ell(\Gamma_0) \lesssim \sum_{Q \in \mathcal{G}^K} \beta_K^2(Q) \mathrm{diam}(Q) + \mathrm{diam}(K)$$

Result

Corollary:
For any set $K \subset$ Hilbert Space

$$\mathrm{diam}(K) + \sum_{Q \in \mathcal{G}^K} \beta_K^2(Q)\mathrm{diam}(Q) \sim \ell(\Gamma_{MST})$$

where Γ_{MST} is the shortest curve containing K.

Metric spaces

- Menger curvature
  Let $x_1, x_2, x_3 \in \mathcal{M}$ be three distinct points. Take $x_1', x_2', x_3' \in \mathbb{C}$ such that $\mathrm{dist}(x_i, x_j) = |x_i' - x_j'|$ for $1 \leq i, j \leq 3$. If x_1', x_2', x_3' are collinear then define

$$c(x_1, x_2, x_3) := 0.$$

  Otherwise, let R be the radius of the circle going through x_1', x_2', x_3'. In this case define

$$c(x_1, x_2, x_3) := \frac{1}{R}.$$

Metric Spaces

- Define β_∞ (I. Hahlomaa)

$$\beta^2_{\mathcal{M},\infty,K}(Q)\mathrm{diam}(Q) = \mathrm{diam}(Q)^3 \sup_{\substack{x_1,x_2,x_3\in Q \\ \mathrm{dist}(x_i,x_j)\geq A^{-1}\mathrm{diam}(Q)}} c^2(x_1,x_2,x_3).$$

- draw triangle. hight and menger curvature...

Metric Spaces

Thm 1': $\forall$ connected $\Gamma \subset H$

$$\sum_Q \beta_\Gamma^2(Q)\mathrm{diam}(Q) \lesssim \ell(\Gamma)$$

Thm 2': $\forall K \subset H$, $\exists$ connected $\Gamma_0 \supset K$, s.t.

$$\ell(\Gamma_0) \lesssim \mathrm{diam}(K) + \sum_Q \beta_K^2(Q)\mathrm{diam}(Q)$$

Theorem 2": [I. Hahlomaa] *There is an $A > 1$ such that, for any set $K \subset \mathcal{M}$, there exists $E \subset [0,1]$ and $\gamma : E \to K$ such that*

$$\|\gamma\|_{Lip} \leq \mathrm{diam}(K) + \sum_n \sum_{Q=\mathrm{Ball}(x, A2^{-n}, x \in X_n)} \beta_{X_n}^2(Q)\mathrm{diam}(Q).$$

Metric spaces

Thm 1': $\forall$ connected $\Gamma \subset H$

$$\sum_Q \beta_\Gamma^2(Q)\mathrm{diam}(Q) \lesssim \ell(\Gamma)$$

Thm 2': $\forall K \subset H$, $\exists$ connected $\Gamma_0 \supset K$, s.t.

$$\ell(\Gamma_0) \lesssim \mathrm{diam}(K) + \sum_Q \beta_K^2(Q)\mathrm{diam}(Q)$$

- When is the converse (i.e. *Theorem 1"*) true?
- Not L^1!
- draw counter example.
  - length $= 1 + N\delta$
  - $\sum \sim N\delta \log(\delta^{-1})$
  - Off by a factor of $\log(\delta^{-1})$.
- Ferrari, Franchi and Pajot have a version of *Theorem 2* for the Heisenberg group. They use a different definition of β.

Go back in time...

- A set K is called j-Ahlfors-David-regular if for all $r \leq \mathrm{diam}(K)$ and $x \in K$ we have

$$\frac{1}{C} \cdot r^j \;\leq\; H^j(K \cap \mathrm{Ball}(x,r)) \;\leq C \cdot r^j.$$

In the Euclidean setting:

- Average deviation from j-plane. Let K be a j-Ahlfors-David regular set and let $\mu = H^j|_K$.

$$\beta_{q,\mu}(B) = \frac{1}{\mathrm{diam}(B)} \inf_{L=\text{j-plane}} \left\{ \int_B \mathrm{dist}(y,L)^q \, \frac{d\mu(y)}{\mu(B)} \right\}^{1/q}.$$

Go back in time...

Theorem 1''' + 2''':[David-Semmes]

Let $K \subset \mathbb{R}^d$ *be a 1-Ahlfors-Regular set and* $1 \leq q \leq \infty$. **TFAE:**

- K *is contained in a connected 1-Ahlfors-Regular set.*
- *Carleson conditon: for all* $z \in K$ *and* $0 < R < \operatorname{diam}(K)$

$$\int_0^R \int_{\mathrm{Ball}(z,R)} \beta_{q,H^1|_K}(\mathrm{Ball}(x,t))^2 dH^1|_K(x) \frac{dt}{t} \lesssim R.$$

- K *has BPLI*
- K *has BPBI*
- K *is* $good$ *for all* $\mathcal{K}$ *kernels.*

(j-Ahlfors-Regular, Big Pieces, Singular integrals, Uniform Rectifiability)

Theorem 1''' + 2''':[David-Semmes]

Let $K \subset \mathbb{R}^d$ *be a j-Ahlfors-David-regular set and* $1 \leq q < \frac{2j}{j-2}$ *for* $j \geq 2$ *and* $1 \leq j \leq \infty$ *for* $j = 1$. **TFAE**

- *Carleson condition: for all* $z \in K$ *and* $0 < R < \mathrm{diam}(K)$

$$\int_0^R \int_{\mathrm{Ball}(z,R)} \beta_q(\mathrm{Ball}(x,t))^2 dH^j|_K(x) \frac{dt}{t} \lesssim R^j.$$

- K *has BPLI*
- K *has BPBI (....note CS connection with embedabilty theorems)*
- K *is* $good$ *for all* $\mathcal{K}$ *kernels.*

(j-Ahlfors-Regular, Big Pieces, Singular integrals, Uniform Rectifiability)

BPLI-definition

A j-regular set $K \subset \mathcal{M}$ is said to have BPLI (Big Pieces of Lipschitz Images)

$\Longleftrightarrow$

for any $x \in K$, and $r < \mathrm{diam}(K)$ we have a L-Lipschitz map $f : E \to \mathrm{Ball}_{\mathcal{M}}(x, r)$ where $E \subset \mathrm{Ball}_{\mathbb{R}^j}(x, r)$ and such that

$$\mathcal{H}^j(f(E) \cap K) \geq \theta \mathcal{H}^j(\mathrm{Ball}_{\mathcal{M}}(x, r) \cap K)$$

BPBI-definition

A j-regular set $K \subset \mathcal{M}$ is said to have BPBI (Big Pieces of BiLipschitz Images)

$\iff$

for any $x \in K$, and $r < \mathrm{diam}(K)$ we have a L, L^{-1}-BiLipschitz map $f : E \to \mathrm{Ball}_{\mathcal{M}}(x, r)$ where $E \subset \mathrm{Ball}_{\mathbb{R}^j}(x, r)$ and such that

$$\mathcal{H}^j(f(E) \cap K) \geq \theta \mathcal{H}^j(\mathrm{Ball}_{\mathcal{M}}(x, r) \cap K)$$

good for all $\mathcal{K}$ kernels-definition

A j-regular set $K \subset \mathbb{R}^d$ is said to be
$good$ for all $\mathcal{K}$ kernels iff
for any kernel $k : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ satisfying
(i) k odd,
(ii) $|x|^{j+m}|\nabla^m k(x)| \in L^\infty(\mathbb{R}^d \setminus \{0\})$ for $m = 0, 1, 2, 3, ...$
we have that

$$T^* := \sup_{\epsilon>0} T_\epsilon f(x) = \sup_{\epsilon>0} \int_{\substack{y\in K \\ |x-y|>\epsilon}} k(x-y) f(y) d\mathcal{H}^j|_K(y)$$

is $L^2(K, \mathcal{H}^j|_K) \to L^2(K, \mathcal{H}^j|_K)$ bounded.

Question

Can we push the David-Semmes theorem into a metric space setting??? (singular integrals aside)

In the metric space setting:

*****$j = 1$*****:

Theorem 2””: [I. Hahlomaa, 2006] *Let K be a 1-Ahlfors-Regular set in a complete geodesic metric space $\mathcal{M}$. Assume that for all $z \in K$ and $R > 0$*

$$\int\int\int c^2(x_1, x_2, x_3) dH^1|_K(x_3) dH^1|_K(x_2) dH^1|_K(x_1) \leq C_0 R$$

where the integral on the left hand side is over all triples $x_1, x_2, x_3 \in K \cap \mathrm{Ball}(z, R)$ such that

$$A \cdot \mathrm{dist}(x_i, x_j) \geq \mathrm{diam}\{x_1, x_2, x_3\}.$$

Then there is a 1-Ahlfors-Regular connected set $\Gamma_0 \supset K$, whose constant depends only on C_0 (linearly) and on the 1-Ahlfors-Regularity constant of K. (gives: Carleson condition $\Longrightarrow$ BPLI)

In the metric space setting:

*****$j=1$*****:

Theorem 1""": [R.S.] *Let* Γ *be a connected 1-Ahlfors-Regular set in a metric space* $\mathcal{M}$ *with metric* $\mathrm{dist}(\cdot,\cdot)$. *Then for all* $z \in \Gamma$ *and* $R > 0$

$$\int\int\int c^2(x_1,x_2,x_3)dH^1|_\Gamma(x_3)dH^1|_\Gamma(x_2)dH^1|_\Gamma(x_1) \lesssim R$$

with constant depending only on the 1-Ahlfors-Regularity constant of Γ, *where the integral on the left hand side is over all triples* $x_1, x_2, x_3 \in \Gamma \cap \mathrm{Ball}(z,R)$ *such that*

$$A \cdot \mathrm{dist}(x_i, x_j) \geq \mathrm{diam}\{x_1,x_2,x_3\}.$$

(with a little work gives: BPLI $\Longrightarrow$ Carleson condition)

In the metric space setting:

*****$j \geq 1$*****:

Theorem: [R.S.] Let $K \subset \mathcal{M}$ be j-Ahlfors-David regular. If K has BPLI then K has BPBI.

The converse is clear

*****$j = 1$*****:

Corollary:

- Singular integrals aside, this gives, the David-Semmes theorem in a metric space setting for $j = 1$, i.e.
- Carleson condition $\iff$ BPLI $\iff$ BPBI for $j = 1$

In the metric space setting:

*****$j \geq 1$*****:
BPLI $\to$ BPBI follows from

Theorem: [R.S.] Let $0 < \alpha < 1$ and $k \geq 1$ be given. There are universal constants $M = M(\alpha, k)$, $c_1 = c_1(k)$ and c_2 such that the following statements hold. Let $\mathcal{M}$ be any metric space. Let $f : [0,1]^k \to \mathcal{M}$ be a 1-Lipschitz function, i.e. such that

$$\mathrm{dist}(f(x), f(y)) \leq |x - y| \,.$$

Then there are sets $F_1, ..., F_M \subset [0,1]^k$ so that for $1 \leq i \leq M$, $x, y \in F_i$ we have

$$\alpha |x - y| \leq \mathrm{dist}(f(x), f(y)) \leq |x - y| \,,$$

$$h^k(f([0,1]^k \setminus (F_1 \cup ... \cup F_M))) \leq c_1 \alpha \,.$$

(define h^k). (Jones ~ 84 for $M = \mathbb{R}^d$)

Question which remain

- What about the rest of the conditions for $j > 1$??
  In particular:
- What is the correct Carleson condition??
- Can one replace the Ahlfors-David condition with a doubling condition (lose IFF? lose $\log n$ factors?)

These questions are very related to CS embedability questions, especially in the applied setting.