The Imaging of Anisotropic Media in Inverse Electromagnetic Scattering

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Many materials in real life exhibit anisotropic properties to electromagnetic interrogation. Mathematically, the presence of anisotropy raises interesting and challenging questions.

Can qualitative methods in inverse scattering provide information about the physical properties of the scatterer in addition to the support? We will discuss here two problems:

- Inverse electromagnetic scattering for a mixed Dirichlet - impedance boundary value problem with matrix impedance function.

- Inverse electromagnetic scattering problem for an anisotropic inhomogeneous medium
We first consider the scattering by a perfect conductor partially coated with a thin layer of anisotropic conducting material. The first order approximate model for the forward scattering problem leads to an exterior mixed Dirichlet - impedance boundary value problem with matrix surface impedance.
The direct scattering problem reads:

\[
\nabla \times \nabla \times E - k^2 E = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}
\]

\[
\nu \times E = 0 \quad \text{on} \quad \Gamma_D
\]

\[
\nu \times (\nabla \times E) - ik\Lambda \cdot (\nu \times E) \times \nu = 0 \quad \text{on} \quad \Gamma_I
\]

\[
E(x) = E^s(x) + E^i(x)
\]

\[
\lim_{|x| \to \infty} (\nabla \times E^s \times x - ik|x|E^s) = 0
\]

where \( E^i(x) = ik(d \times p) \times d e^{ikx \cdot d} \), \( k^2 = \epsilon_0 \mu_0 \omega^2 \) and \( \Lambda \) is the real valued tensor impedance which maps a vector tangential to \( \Gamma_I \) at a point \( x \) to a vector tangential to \( \Gamma_I \) at the same point \( x \).
Assume that $\partial D$ is Lipschitz curvilinear polyhedron. For $x \in \Gamma_j^i$ (a smooth face), let $\nu(x)$ be the outward unit normal, and $\hat{t}_1(x)$ and $\hat{t}_2(x)$ two unit perpendicular tangential vectors.

If $\xi(x) = \alpha \hat{t}_1(x) + \beta \hat{t}_2(x)$ then

$$\Lambda(x) \cdot \xi(x) = (\alpha \lambda_{11}(x) + \beta \lambda_{21}(x))\hat{t}_1(x) + (\alpha \lambda_{12}(x) + \beta \lambda_{22}(x))\hat{t}_2(x).$$

$\Lambda \in L_\infty(\Gamma_j^i) \cup C(\Gamma_j^i)$, real valued, symmetric i.e. $\lambda_{12} = \lambda_{21}$ and uniformly positive definite i.e.

$$\xi^\top(x)\Lambda(x) \cdot \xi(x) = \alpha^2 \lambda_{11}(x) + 2\alpha\beta \lambda_{12}(x) + \beta^2 \lambda_{22}(x) \geq \gamma |\xi(x)|^2$$

where $\gamma > 0$ is a constant independent of $x$. 

Tensor Impedance
The Inverse Problem

The scattered field $E^s$, which is in $H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D})$ with $\nu \times E^s$ in $L^2_t(\Gamma_I)$, has the asymptotic behavior

$$E^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right) \right\}$$

where $E_\infty(\cdot, d, p)$ is the far field pattern defined on the unit sphere $\Omega$.

The inverse scattering problem is

Determine both $D$ and $\Lambda$ from a knowledge of $E_\infty(\hat{x}, d, p)$ for $d, -\hat{x} \in \Omega_0 \subset \Omega$ and two linearly independent polarizations $p$ tangential to $\Omega$.
Uniqueness of $D$

For fixed $k$, $E_\infty(\hat{x}, d, p)$ for $d, -\hat{x} \in \Omega_0 \subset \Omega$ and two linearly independent polarizations $p$ perpendicular to $d$ uniquely determine $D$.

Uniqueness of $\Lambda$

For fixed $k$, $E_\infty(\hat{x}, d, p)$ for $\hat{x}, -d \in \Omega_0 \subset \Omega$ and two linearly independent polarizations $p$ perpendicular to $d$, uniquely determine $\Gamma_I$ and the tensor $\Lambda$. 
Uniqueness

Assume that $\partial D$ is known.

The far field pattern corresponding to one incident wave does not uniquely determine the matrix $\Lambda$ (uniquely determines $\Gamma_i$)

**Theorem**

*Given the far field pattern corresponding to one incident wave $E^i$, there exist a matrix $\tilde{\Lambda}$ depending on the geometry of $\partial D$ and the corresponding total field $E^i + E^s$, and a unknown scalar function $a$ defined on $\Gamma_i$ such that $\Lambda = a\tilde{\Lambda}$ on $\Gamma_i$.*

**Conjecture:** The far field pattern corresponding to two (appropriately chosen) incident waves uniquely determine $\Lambda$

The incident waves must be such that the corresponding total electric fields have linearly independent tangential components almost everywhere on $\Gamma_i$. 
Reconstruction is based on the far field equation

\[(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)\]

where \(E_{e,\infty}(\hat{x}, z, q)\) denotes the far field pattern of the electric dipole

\[E_e(x, z, q) := \frac{i}{k} \nabla x \times \nabla x \times q \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|},\]

and the far field operator \(F : L^2_t(\Omega) \rightarrow L^2_t(\Omega)\) is defined by

\[(Fg)(\hat{x}) := \int_{\Omega} E_\infty(\hat{x}, d, g(d)) \, ds(d)\]

which is the far field pattern produced by the Herglotz wave function

\[E_{g_\epsilon}(x) := ik \int_{\Omega} e^{ikx \cdot d} g_\epsilon^{\epsilon}(d) \, ds(d).\]
Solution of Far-Field Equation

Theorem

- For \( z \in D \) and an arbitrary \( \epsilon > 0 \) there exists an approximate solution \( g_z^\epsilon \) of the far field equation

\[
\| F g_z^\epsilon - E_{e,\infty}(\cdot, z, q) \|_{L_t^2(\Omega)} < \epsilon
\]
is such that

\[
\lim_{\epsilon \to 0} \| E g_z^\epsilon \|_{L^2(D)} \quad \text{exists}
\]

- For \( z \notin D \) every approximate solution \( g_z^\epsilon \)

\[
\| F g_z^\epsilon - E_{e,\infty}(\cdot, z, q) \|_{L_t^2(\Omega)} < \epsilon
\]
is such that

\[
\lim_{\epsilon \to 0} \| E g_z^\epsilon \|_{L^2(D)} \to \infty \quad \text{and} \quad \lim_{\epsilon \to 0} \| g_z^\epsilon \|_{L^2(\Omega)} \to \infty.
\]
Theorem (cont.)

Furthermore, for \( z \in D \), \( E_{g^z} \) converges to the unique solution \( E_z \) of

\[
\nabla \times \nabla \times E_z - k^2 E_z = 0 \quad \text{in} \quad D
\]

\[
\nu \times [E_z + E_e(\cdot, z, q)] = 0 \quad \text{on} \quad \Gamma_D
\]

\[
\nu \times \nabla \times [E_z + E_e(\cdot, z, q)] - ik\Lambda \cdot \nu \times [E_z + E_e(\cdot, z, q)] \times \nu = 0 \quad \text{on} \quad \Gamma_I
\]
The support $D$ can be reconstructed using this behavior of the approximate (regularized) solution of the far field equation. In particular,

- Construct a grid $\mathcal{G}$ and for $z_i \in \mathcal{G}$, solve
  \[(\alpha I + F^* F)g_{z_i,q} = F^* E_{e,\infty}\]

- For $z_i \in \mathcal{G}$ and three linearly independent $q_1, q_2, q_3 \in \mathbb{R}^3$ evaluate
  \[G(z_i) = \frac{1}{3} \left( \|g_{z_i,q_1}\|_{\ell^2}^{-1} + \|g_{z_i,q_2}\|_{\ell^2}^{-1} + \|g_{z_i,q_3}\|_{\ell^2}^{-1} \right).\]

- Choose an appropriate $C > 0$ and visualize the boundary by plotting
  \[G(z) = C \max_{z_i \in \mathcal{G}} G(z_i).\]
Determination of $\Lambda$

Recall that the Herglotz wave function $E_{gz}$ with kernel $g_z$ the (regularized) solution of the far field equation converges to $E_z$.

Two important properties of $W_z := E_z + E_e(\cdot, z, q) \approx E_{gz} + E_e(\cdot, z, q)$.

- Let $z_1, z_2 \in D$. Then
  \[
  2 \int_{\Gamma_I} (\nu \times W_{z_1}) \cdot (\nu \times \overline{W}_{z_2}) \, ds = -\|q\|^2 A(z_1, z_2, k, q) \\
  + k(q \cdot E_{z_1}(z_2) + q \cdot \overline{E}_{z_2}(z_1))
  \]
  where $A(z_1, z_2, k, q)$ is a computable.

- The set of $\nu \times W_z|_{\Gamma_I}$ for $z$ in a ball $B_r \subset D$ is complete in $L_t^2(\Gamma_I)$.
- Note that $\Gamma_D := \{ z \in \partial D : \nu \times W_z = 0 \}$. 
Examples of Reconstruction

\( \lambda = 1 \), wavelength = 0.7
Examples of Reconstruction
Examples of Reconstruction

<table>
<thead>
<tr>
<th>Face</th>
<th>Exact $\Lambda$</th>
<th>Matrix Reconstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = -.5$</td>
<td>$\begin{pmatrix} .1 &amp; .05 \ .05 &amp; .1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0.1026 &amp; 0.01121 \ 0.01121 &amp; 0.1030 \end{pmatrix}$</td>
</tr>
<tr>
<td>$y = -.5$</td>
<td>$\begin{pmatrix} 2 &amp; .03 \ .03 &amp; 1.5 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1.682 &amp; -0.05898 \ -0.05898 &amp; 1.587 \end{pmatrix}$</td>
</tr>
<tr>
<td>$z = -.5$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1.148 &amp; -0.05123 \ -0.05123 &amp; 1.231 \end{pmatrix}$</td>
</tr>
<tr>
<td>$z = .5$</td>
<td>$\begin{pmatrix} .3 &amp; 0 \ 0 &amp; .3 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0.3028 &amp; 0.005399 \ 0.005399 &amp; 0.2994 \end{pmatrix}$</td>
</tr>
<tr>
<td>$y = .5$</td>
<td>$\begin{pmatrix} .4 &amp; 0 \ 0 &amp; .4 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0.3986 &amp; -0.006338 \ -0.006338 &amp; 0.3942 \end{pmatrix}$</td>
</tr>
<tr>
<td>$x = .5$</td>
<td>$\begin{pmatrix} .7 &amp; .4 \ .4 &amp; .7 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0.5933 &amp; 0.02344 \ 0.02344 &amp; 0.5974 \end{pmatrix}$</td>
</tr>
</tbody>
</table>
These results are presented in


which generalizes the approach in

Consider now the scattering by an inhomogeneous anisotropic dielectric media. The goal is to determine the support of inhomogeneity and information about the anisotropy.

The relative electric permittivity \( N(x) = \frac{\epsilon(x)}{\epsilon_0} \), \( x \in \overline{D} \), is a positive definite symmetric matrix-valued function with bounded entries. The magnetic permeability is constant \( \mu_0 \) and the wave number \( k > 0 \) is such that \( k^2 = \epsilon_0 \mu_0 \omega^2 \) where \( \omega > 0 \) is the frequency.
After eliminating the magnetic field, the scattering problem can be written in terms of electric fields as:

\[
\nabla \times \nabla \times E^s - k^2 E^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}
\]
\[
\nabla \times \nabla \times E - k^2 N(x) E = 0 \quad \text{in } D
\]
\[
\nu \times (E^s + E^i) = \nu \times E \quad \text{on } \partial D
\]
\[
\nu \times \nabla \times (E^s + E^i) = \nu \times \nabla \times E \quad \text{on } \partial D
\]

where the scattered electric field satisfies the Silver-Muller radiation condition

\[
\lim_{|x| \to \infty} \left( \nabla \times E^s \times x - i k |x| E^s \right) = 0.
\]

and

\[
E^i(x, d, p, k) = \frac{i}{k} \nabla \times \nabla \times p \ e^{ikx \cdot d}.
\]
The inverse scattering problem is

Determine both $D$ and (some information about) $N$ from a knowledge of $E_\infty(\hat{x}, d, p)$ for $d, -\hat{x} \in \Omega_0 \subset \Omega$ and two linearly independent polarizations $p$ perpendicular to $d$, and a range of wave numbers $k \in [k_0, k_1]$.

Theorem (Uniqueness)

If $\|N\|_2 > 1$ or $\|N^{-1}\|_2 > 1$. Then $D$ is uniquely determined from the above data.

There is no uniqueness proof for the determination of the matrix $N$ for the above data.
Far Field Operator

Again, the scattered field $E^s$ has the asymptotic behavior

$$E^s(x, d, p, k) = \frac{e^{ikr}}{r} E_\infty(\hat{x}, d, p, k) + O\left(\frac{1}{r^2}\right)$$

as $r \to \infty$, $\hat{x} = x/|x|$, $r = |x|$.

Then the far field operator $F : L^2(\Omega) \to L^2(\Omega)$

$$(F g)(\hat{x}) := \int_{\Omega} E_\infty(\hat{x}, d, g(d), k) \, ds(d)$$

is the far field pattern of the scattered field corresponding to the electric Helgloetz wave function

$$E_g(x) := \int_{\Omega} e^{ikx \cdot d} g(d) \, ds(d)$$

as incident field.
Theorem

The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is injective and has dense range if and only if there does not exists a nontrivial solution to the homogeneous interior transmission problem

\[
\nabla \times \nabla \times E - k^2 N(x) E = 0 \quad \text{in} \quad D \\
\nabla \times \nabla \times E_0 - k^2 E_0 = 0 \quad \text{in} \quad D \\
\nu \times E = \nu \times E_0 \quad \text{on} \quad \partial D \\
\nu \times (\nabla \times E) = \nu \times (\nabla \times E_0) \quad \text{on} \quad \partial D
\]

such that $E_0 := E_g$ is an electric Herglotz wave function with kernel $g$. 
Transmission Eigenvalues

Values of $k$ for which the homogeneous interior transmission problem has a non trivial solution are called transmission eigenvalues.

Transmission eigenvalues are related to non scattering waves – if $E_0$ can be extended as a solution to the Maxwell’s equation outside $D$ then this extension is an incident wave that does not scatter. Since Herglotz wave functions

$$E_g(x) := ik \int_\Omega e^{ikx \cdot d} g(d) ds(d), \quad \text{for } g \in L^2_t(\Omega)$$

are dense in the space

$$\{ \mathcal{W} \in L^2(\Omega) : \nabla \times \nabla \times \mathcal{W} - k^2 \mathcal{W} = 0 \}$$

then at a transmission eigenvalue there is an incident field that produces arbitrarily small scattered field.
We again consider the far field equation

\[(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)\]

where \(E_{e,\infty}(\hat{x}, z, q)\) denotes the far field pattern of the electric dipole

\[E_e(x, z, q) := \frac{i}{k} \nabla_x \times \nabla_x \times q \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|},\]

and the far field operator \(F : L^2_t(\Omega) \to L^2_t(\Omega)\) is defined by

\[(Fg)(\hat{x}) := \int\limits_{\Omega} E_\infty(\hat{x}, d, g(d)) \, ds(d).\]
Theorem

1. Let $z \in D$.
   - If $k$ is not a transmission eigenvalue then for every $\epsilon > 0$ there exists $g_{z,\epsilon,k} \in L^2(\Omega)$ satisfying
     \[
     \|Fg_{z,\epsilon,k} - E_{e,\infty}\|_{L^2(\Omega)} < \epsilon \quad \text{and} \quad \lim_{\epsilon \to 0} \|E_{g_{z,\epsilon,k}}\|_{L^2(D)} = \|E_0\|_{L^2(D)}.
     \]
   - If $k$ is a transmission eigenvalue, every $g_{z,\epsilon,k} \in L^2(\Omega)$ satisfying
     \[
     \|Fg_{z,\epsilon,k} - E_{e,\infty}\|_{L^2(\Omega)} < \epsilon
     \]
     is such that \( \lim_{\epsilon \to 0} \|E_{g_{z,\epsilon,k}}\|_{L^2(D)} = \infty \).

2. Let $z \notin D$.
   - For all $k > 0$, every $g_{z,\epsilon,k} \in L^2(\Omega)$ satisfying
     \[
     \|Fg_{z,\epsilon,k} - E_{e,\infty}\|_{L^2(\Omega)} < \epsilon
     \]
     is such that \( \lim_{\epsilon \to 0} \|E_{g_{z,\epsilon,k}}\|_{L^2(D)} = \infty \).
In practice one has available the noisy far field operator $F^\delta$

$$(F^\delta g)(\hat{x}) := \int_{\Omega} E^\delta_{\infty}(\hat{x}, d, g(d), k) \, ds(d)$$
i.e. the far field operator with kernel the noisy measurements with $\delta$ being the noise level.

For various of $z \in \mathbb{R}^3$ and $k > 0$, $g_{z,k}$ is computed using Tikhonov regularization technique, i.e by solving

$$(\alpha(\delta) I + F^\delta F^\delta) g^\delta_{z,k} = F^\delta E_{e,\infty}(\cdot, z, q)$$

$\alpha(\delta) \to 0$ as $\delta \to 0$.

The behavior of $g$ in the above theorem is inherited by $g^\delta_{z,k}$ as $\delta \to 0$.

This behavior for different $z$ and $k$ can be used to solve the inverse problem.
Reconstruction of the Support $D$

- For fixed frequency $k > 0$ not a transmission eigenvalue we have
  \[ \| g_{z,k}(\cdot) \|_{L^2(\Omega)} \text{is bounded for } z \in D \text{ and very large for } z \notin D. \]

- Using band limit multifrequency data for $k \in [k_0, k_1]$.

**Note:** If $[k_0, k_1]$ contains at least a transmission eigenvalue than $\| g_{z,k}(\cdot) \|_{L^2(\Omega)}$ as function of $k$ is not integrable in $[k_0, k_1]$.

In this case we can use
\[
G(z) := \int_{k_0}^{k_1} \frac{1}{\| g_{z,k}(\cdot) \|_{L^2(\Omega)}}
\]

as an indicator function for $D$. 
Examples of Reconstruction

\[ N = 5I \]

\[ k \text{ is not a TE} \]

\[ k \text{ is a TE} \]
Examples of Reconstruction

\[ N = 5 \]

\[ k \text{ is not a TE} \quad \text{Reconstruction for } k \in [2, 3.5] \]
Examples of Reconstruction

$N = 16I$, $k$ is not TE

![Graph 1](xy cut of g for ka= 2.100000)

![Graph 2](1/|g|, k= 2.1)

![Graph 3](g isosurf for ka= 2.100000)
Examples of Reconstruction

$N = 16I$, $k$ is a TE

$g$ isosurf for $ka = 1.470000$
Results for an isotropic sphere of unit radius. Solving the far-field equation for several source points $z$ inside the sphere (using 42 incoming directions and 42 measurements) gives obvious peaks at the first transmission eigenvalue.

Red dots indicate exact transmission eigenvalues. Using the far field pattern, peaks in $\|g\|$ indicate transmission eigenvalues.
What do transmission eigenvalues say about $N(x) = \varepsilon(x)/\varepsilon_0$?

To this end let $\lambda_1(x) \leq \lambda_2(x) \leq \lambda_3(x)$ be the eigenvalues of $N$ and define

$$\lambda_{\text{min}} = \inf_{x \in D} \lambda_1(x) \quad \text{and} \quad \lambda_{\text{max}} = \sup_{x \in D} \lambda_3(x).$$

Let $k_{1,D,N(x)}$ be the first transmission eigenvalue for our scattering problem and let $k_{1,D,\lambda_{\text{min}}}$ and $k_{1,D,\lambda_{\text{max}}}$ be the first transmission eigenvalues for the case of $N = \lambda_{\text{min}}I$ and $N = \lambda_{\text{max}}I$, respectively.
Non-destructive Testing of Anisotropic Materials

**Theorem**

Let $\alpha$ and $\beta$ be positive constants.

1. If $\|N(x)\|_2 \geq \alpha > 1 \text{ then } k_{1,D},\lambda_{\text{max}} \leq k_{1,D},N(x) \leq k_{1,D},\lambda_{\text{min}}$

2. If $\|N(x)\|_2 \leq \beta < 1 \text{ then } k_{1,D},\lambda_{\text{min}} \leq k_{1,D},N(x) \leq k_{1,D},\lambda_{\text{max}}$

Given $k_{1,D},N(x)$ we now compute a constant $n$ such that $k_{1,D},N(x) = k_{1,D},n$. Then the above theorem implies that

$$\lambda_{\text{min}} \leq n \leq \lambda_{\text{max}}$$

and that $k_{1,D},n$ depends monotonically on $n$. It can in fact be shown that $k_{1,D},n$ depends strictly monotonically on $n$. 
Numerical Examples

Perturbation of $N = 16I$

Perturbation of $N = 5I$
Having the first transmission eigenvalue for anisotropic $N$ then compute the isotropic $n$ discussed previously

$$\lambda_{min} \leq n \leq \lambda_{max}$$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$k_{1,D,N(x)}$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{diag}([15.5, 16, 16.5])$</td>
<td>1.163</td>
<td>16.33</td>
</tr>
<tr>
<td>$\text{diag}([15, 16, 17])$</td>
<td>1.151</td>
<td>16.65</td>
</tr>
<tr>
<td>$\text{diag}([16, 16, 16.5])$</td>
<td>1.161</td>
<td>16.38</td>
</tr>
<tr>
<td>$\text{diag}([16, 16, 17])$</td>
<td>1.146</td>
<td>16.77</td>
</tr>
</tbody>
</table>
The same procedure can be carried out at lower $N$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\lambda_{1,D,N(x)}$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>diag([4.5, 5, 5.5])</td>
<td>2.442</td>
<td>5.339</td>
</tr>
<tr>
<td>diag([4, 5, 6])</td>
<td>2.302</td>
<td>5.631</td>
</tr>
<tr>
<td>diag([5, 5, 5.5])</td>
<td>2.410</td>
<td>5.397</td>
</tr>
<tr>
<td>diag([5, 5, 6])</td>
<td>2.245</td>
<td>5.778</td>
</tr>
</tbody>
</table>
There is no material that is a perfect dielectric. So what happens to the above theory if there is a small amount of absorption in $D$?

Let $N_1(x) = \frac{\epsilon(x)}{\epsilon_0} + i\frac{\sigma(x)}{\epsilon_0 \omega}$ describe the absorbing media. Using Kato’s perturbation theory for linear operators, it is possible to prove the following theorem.

**Theorem**

*Let $k^* > 0$ be a real transmission eigenvalue corresponding to $N(x) = \frac{\epsilon(x)}{\epsilon_0}$. Then if $\|\sigma(x)\| < \epsilon$ for $\epsilon$ sufficiently small there exists at least one (complex) transmission eigenvalue near $k^*$ corresponding to $N_1(x)$.***
Numerical Example: Absorbing Media

\[ D \text{ is a sphere of diameter } 1 \text{ and } N = (16 + i)l \]
Can the assumption $N > I$ or $0 < N < I$ in $D$ be relaxed?

$$N = I \text{ in } D_0 \quad N - I \geq \delta > 0 \text{ in } D \setminus D_0$$

The case when there are regions $D_0$ in $D$ where $N = I$ (i.e. cavities) is more delicate but the same type of analysis as above can be carried through.
Let $k_{1,D,D_0,N(x)}$ be the first transmission eigenvalue for $D, D_0, N(x)$.

- **Monotonicity with respect to the index of refraction**
  
  $$k_{1,D,D_0,N(x)} \leq k_{1,D,D_0,\tilde{N}(x)}, \quad \tilde{N}(x) \leq N(x).$$

- **Monotonicity with respect to voids**
  
  $$k_{1,D,D_0,N(x)} \leq k_{1,D,\tilde{D}_0,N(x)}, \quad D_0 \subset \tilde{D}_0.$$ 

Other type of inclusions such as a perfect conductor or crack could also be considered inside the inhomogeneity.
Numerical Example: Media with Cavities

Due to A. Cossonniere, Ph.D. Thesis

The sphere with radius 1 and $N = 4I$

$k_1 = 3.16$
Numerical Example: Media with Cavities

Due to A. Cossonniere, Ph.D. Thesis

The sphere with radius 1, $N = 4I$ containing a cubic cavity $k_1 = 3.33$, i.e. $k_1$ is shifted to the right.
Case with Contrast in $\epsilon$ and $\mu$.

If $\mu \neq \mu_0$ in $D$ then the corresponding transmission eigenvalue problem written in terms of magnetic fields is

$$\nabla \times A \nabla \times H - k^2 n H = 0 \quad \text{in} \quad D$$

$$\nabla \times \nabla \times H_0 - k^2 H_0 = 0 \quad \text{in} \quad D$$

$$\nu \times H = \nu \times H_0 \quad \text{on} \quad \partial D$$

$$\nu \times (A \nabla \times H_0) = \nu \times (\nabla \times H_0) \quad \text{on} \quad \partial D.$$

where $A^{-1} = \frac{\epsilon(x)}{\epsilon_0}$ and $n = \frac{\mu}{\mu_0}$

Here the electric permittivity $\epsilon(x)$ in the medium is a positive definite matrix with bounded entries whereas the magnetic permeability $\mu \neq \mu_0$ in the medium is a constant.
Case with Contrast in $\epsilon$ and $\mu$

- The support $D$ can be reconstructed by using the Linear Sampling Method.

- The following results can be obtained on transmission eigenvalues:
  - If the contrasts $A - I$ and $n - 1$ have the same fixed sign, then there exists an infinite discrete set of real transmission eigenvalues accumulating at $+\infty$.
  - If the contrasts $A - I$ and $n - 1$ have the opposite fixed sign, then there exists at least one real transmission eigenvalue providing that $n$ is small enough.

- The first transmission eigenvalue in this case provides estimates for $\epsilon(x)/\mu$. 
Literature


Most of the results on the Qualitative Methods for Maxwell’s equations can be found in the following monograph:


A comprehensive study on transmission eigenvalues for Maxwell’s equation can be found in the following thesis: