The Imaging of Anisotropic Media in Inverse Electromagnetic Scattering

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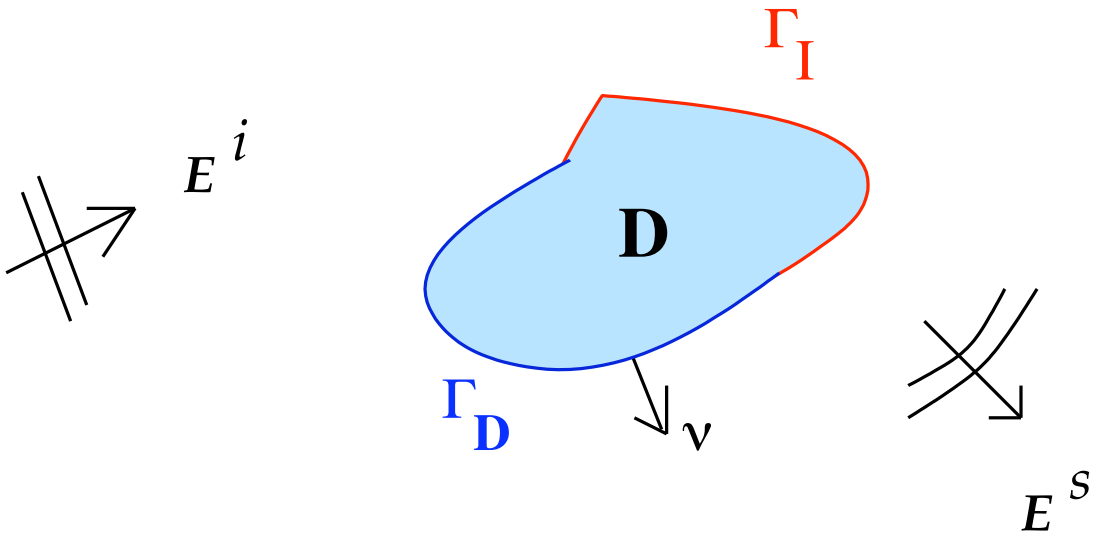
Introduction

Many materials in real life exhibit anisotropic properties to electromagnetic interrogation. Mathematically, the presence of anisotropy raises interesting and challenging questions.

Can qualitative methods in inverse scattering provide information about the physical properties of the scatterer in addition to the support? We will discuss here two problems:

- Inverse electromagnetic scattering for a mixed Dirichlet - impedance boundary value problem with matrix impedance function.
- Inverse electromagnetic scattering problem for an anisotropic inhomogeneous medium

Partially Coated Obstacles



We first consider the scattering by a perfect conductor partially coated with a thin layer of anisotropic conducting material. The first order approximate model for the forward scattering problem leads to an exterior mixed Dirichlet - impedance boundary value problem with matrix surface impedance.

Direct Scattering Problem

The direct scattering problem reads:

$$
\begin{aligned}
\nabla \times \nabla \times E - k^2 E &= 0 && \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \\
\nu \times E &= 0 && \text{on} \quad \Gamma_D \\
\nu \times (\nabla \times E) - ik\Lambda \cdot (\nu \times E) \times \nu &= 0 && \text{on} \quad \Gamma_I \\
E(x) &= E^s(x) + E^i(x) \\
\lim_{|x| \to \infty} (\nabla \times E^s \times x - ik|x| E^s) &= 0
\end{aligned}
$$

where $E^i(x) = ik(d \times p) \times d\, e^{ikx \cdot d}, \quad k^2 = \epsilon_0 \mu_0 \omega^2$ and

Λ is the real valued tensor impedance which maps a vector tangential to Γ_I at a point x to a vector tangential to Γ_I at the same point x.

Tensor Impedance

- Assume that ∂D is Lipschitz curvilinear polyhedron.

  For $x \in \Gamma_l^j$ (a smooth face), let $\nu(x)$ be the outward unit normal, and $\hat{t}_1(x)$ and $\hat{t}_2(x)$ two unit perpendicular tangential vectors.

- If $\xi(x) = \alpha\hat{t}_1(x) + \beta\hat{t}_2(x)$ then

$$\Lambda(x) \cdot \xi(x) = (\alpha\lambda_{11}(x) + \beta\lambda_{21}(x))\hat{t}_1(x) + (\alpha\lambda_{12}(x) + \beta\lambda_{22}(x))\hat{t}_2(x).$$

- $\Lambda \in L_\infty(\Gamma_l) \cup C(\Gamma_l^j)$, real valued, symmetric i.e. $\lambda_{12} = \lambda_{21}$ and uniformly positive definite i.e.

$$\xi^\top(x)\Lambda(x) \cdot \xi(x) = \alpha^2\lambda_{11}(x) + 2\alpha\beta\lambda_{12}(x) + \beta^2\lambda_{22}(x) \geq \gamma|\xi(x)|^2$$

  where $\gamma > 0$ is a constant independent of x.

The Inverse Problem

The scattered field E^s, which is in $H_{loc}(curl, \mathbb{R}^3 \setminus \overline{D})$ with $\nu \times E^s$ in $L^2_t(\Gamma_I)$, has the asymptotic behavior

$$E^s(x) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right)\right\}$$

where $E_\infty(\cdot, d, p)$ is the far field pattern defined on the unit sphere Ω.

The inverse scattering problem is

> Determine **both** D and Λ from a knowledge of $E_\infty(\hat{x}, d, p)$ for $d, -\hat{x} \in \Omega_0 \subset \Omega$ and two linearly independent polarizations p tangential to Ω

Uniqueness

Uniqueness of D

For fixed k, $E_\infty(\hat{x}, d, p)$ for $d, -\hat{x} \in \Omega_0 \subset \Omega$ and two linearly independent polarizations p perpendicular to d uniquely determine D

Uniqueness of Λ

For fixed k, $E_\infty(\hat{x}, d, p)$ for $\hat{x}, -d \in \Omega_0 \subset \Omega$ and two linearly independent polarizations p perpendicular to d, uniquely determine Γ_I and the tensor Λ.

Uniqueness

Assume that ∂D is known.

The far field pattern corresponding to one incident wave does not uniquely determine the matrix Λ (uniquely determines Γ_I)

Theorem

Given the far field pattern corresponding to one incident wave E^i, there exist a matrix $\tilde{\Lambda}$ depending on the geometry of ∂D and the corresponding total field $E^i + E^s$, and a unknown scalar function a defined on Γ_I such that $\Lambda = a\tilde{\Lambda}$ on Γ_I.

Conjecture: The far field pattern corresponding to two (appropriately chosen) incident waves uniquely determine Λ

The incident waves must be such that the corresponding total electric fields have linearly independent tangential components almost everywhere on Γ_I.

Far Field Equation

Reconstruction is based on the far field equation

$$\boxed{(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)}$$

where $E_{e,\infty}(\hat{x}, z, q)$ denotes the far field pattern of the electric dipole

$$E_e(x, z, q) := \frac{i}{k} \nabla_x \times \nabla_x \times q \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|},$$

and the far field operator $F : L^2_t(\Omega) \to L^2_t(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, ds(d)$$

which is the far field pattern produced by the Herglotz wave function

$$E_{g_z^\epsilon}(x) := ik \int_\Omega e^{ikx\cdot d} g_z^\epsilon(d) ds(d).$$

Solution of Far-Field Equation

Theorem

- *For* $z \in D$ *and an arbitrary* $\epsilon > 0$ *there exists an approximate solution* g_z^ϵ *of the far field equation*

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \epsilon \quad \textit{is such that}$$

$$\lim_{\epsilon \to 0} \|E_{g_z^\epsilon}\|_{L^2(D)} \quad \textit{exists}$$

- *For* $z \notin D$ *every approximate solution* g_z^ϵ

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \epsilon \quad \textit{is such that}$$

$$\lim_{\epsilon \to 0} \|E_{g_z^\epsilon}\|_{L^2(D)} \to \infty \qquad \textit{and} \qquad \lim_{\epsilon \to 0} \|g_z^\epsilon\|_{L^2(\Omega)} \to \infty.$$

Solution of Far-Field Equation

Theorem (cont.)

Furthermore, for $z \in D$, $E_{g_z^\epsilon}$ converges to the unique solution E_z of

$$\nabla \times \nabla \times E_z - k^2 E_z = 0 \quad in \quad D$$

$$\nu \times [E_z + E_e(\cdot, z, q)] = 0 \quad on \quad \Gamma_D$$

$$\nu \times \nabla \times [E_z + E_e(\cdot, z, q)] - ik\Lambda \cdot \nu \times [E_z + E_e(\cdot, z, q)] \times \nu = 0 \quad on \quad \Gamma_I$$

The Linear Sampling Method

The support D can be reconstructed using this behavior of the approximate (regularized) solution of the far field equation. In particular,

- Construct a grid $\mathcal{G}$ and for $z_i \in \mathcal{G}$, solve

$$(\alpha I + F^* F) g_{z_i,q} = F^* E_{e,\infty}$$

- For $z_i \in \mathcal{G}$ and three linearly independent $q_1, q_2, q_3 \in \mathbb{R}^3$ evaluate

$$G(z_i) = \frac{1}{3}\left(\|g_{z_i,q_1}\|_{\ell^2}^{-1} + \|g_{z_i,q_2}\|_{\ell^2}^{-1} + \|g_{z_i,q_3}\|_{\ell^2}^{-1}\right).$$

- Choose an appropriate $C > 0$ and visualize the boundary by plotting

$$\mathcal{G}(z) = C \max_{z_i \in \mathcal{G}} G(z_i).$$

Determination of Λ

Recall that the Herglotz wave function E_{g_z} with kernel g_z the (regularized) solution of the far field equation converges to E_z.

Two important properties of $W_z := E_z + E_e(\cdot, z, q) \approx E_{g_z} + E_e(\cdot, z, q)$.

- Let $z_1, z_2 \in D$. Then

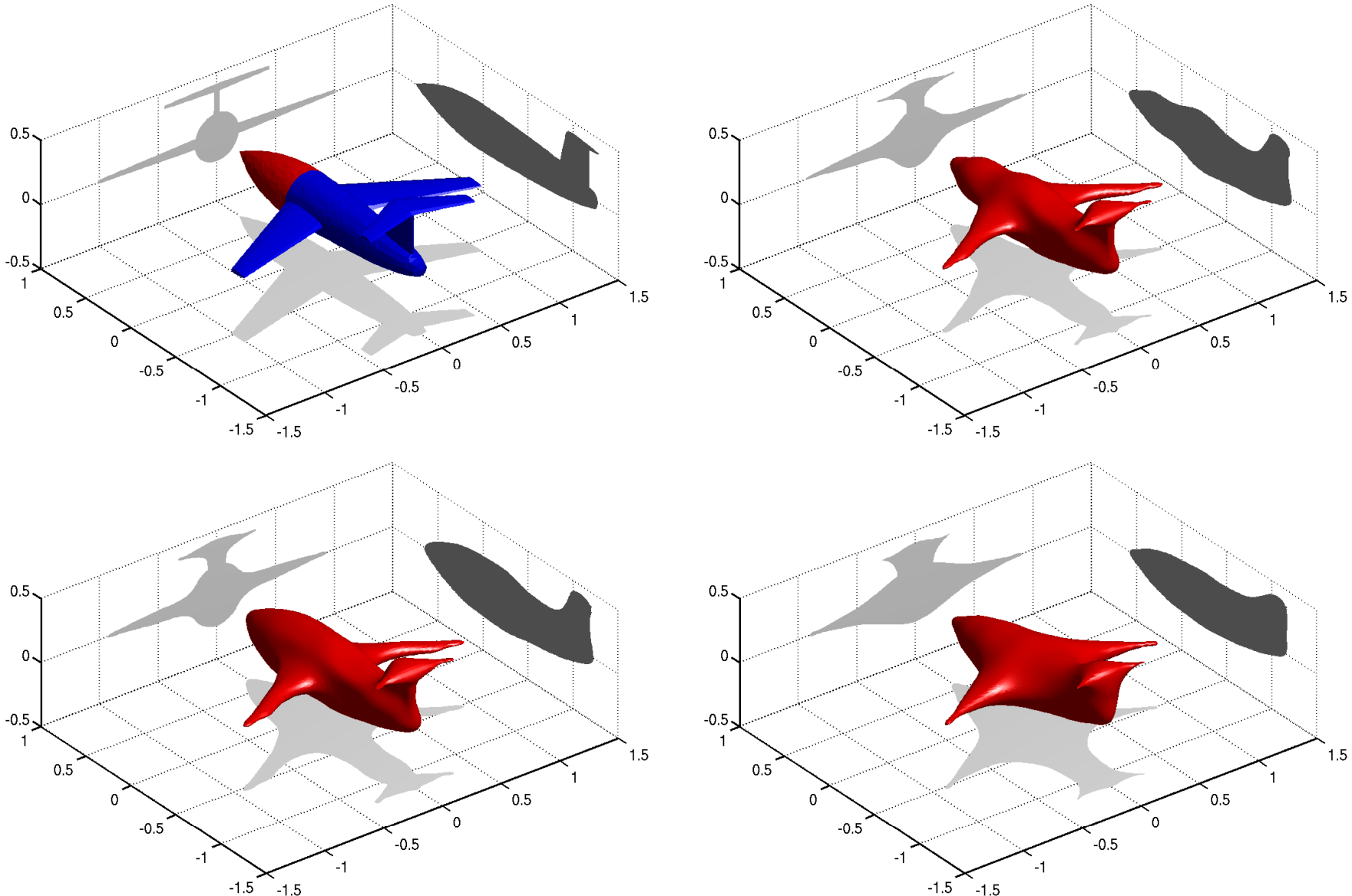
$$2\int_{\Gamma_I} (\nu \times W_{z_1})\, \Lambda \cdot (\nu \times \overline{W}_{z_2})\, ds = -\|q\|^2 A(z_1, z_2, k, q) + k(q \cdot E_{z_1}(z_2) + q \cdot \overline{E}_{z_2}(z_1))$$

  where $A(z_1, z_2, k, q)$ is a computable.

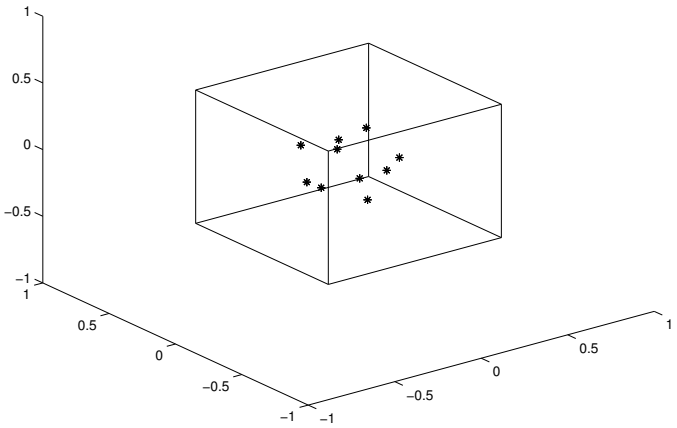
- The set of $\nu \times W_z|_{\Gamma_I}$ for z in a ball $B_r \subset D$ is complete in $L^2_t(\Gamma_I)$.
- Note that $\Gamma_D := \{z \in \partial D : \nu \times W_z = 0\}$.

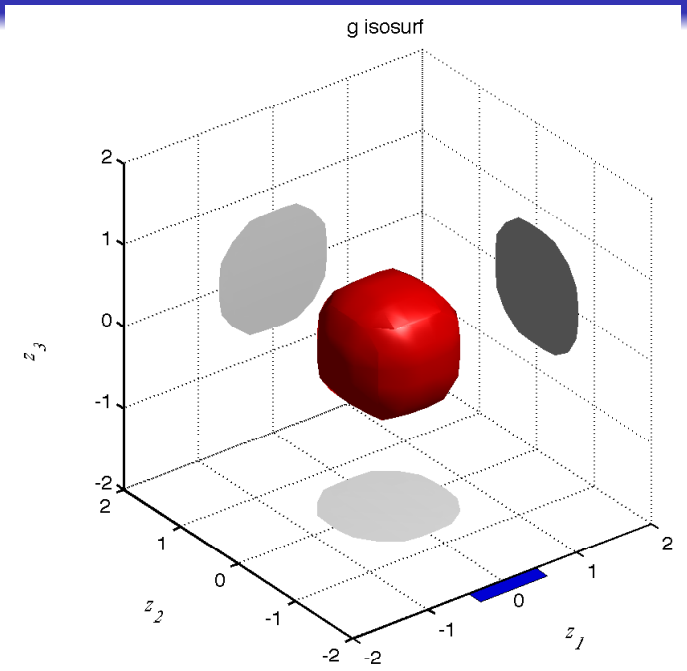
Examples of Reconstruction

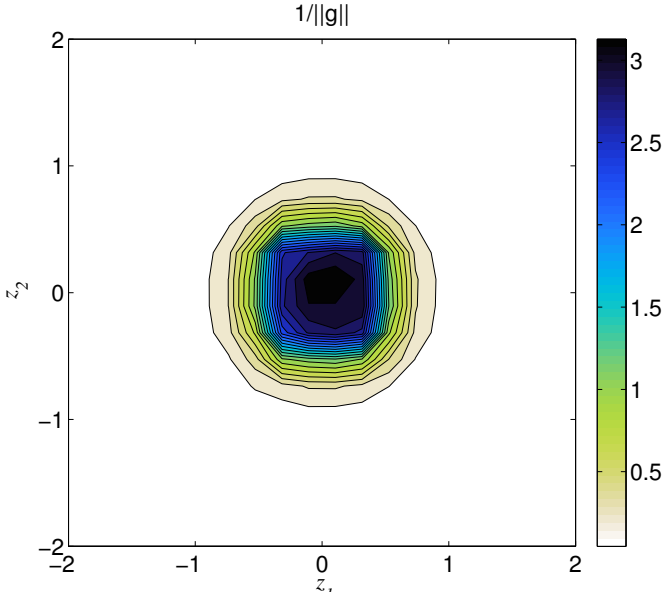
$\lambda = 1$, wavelength$= 0.7$

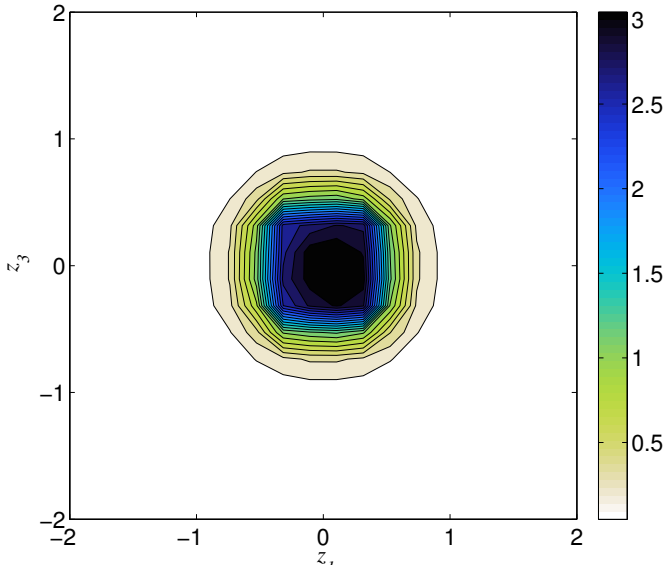


Examples of Reconstruction









Examples of Reconstruction

Face	Exact Λ	Matrix Reconstruction
$x = -.5$	$\begin{pmatrix} .1 & .05 \\ .05 & .1 \end{pmatrix}$	$\begin{pmatrix} 0.1026 & 0.01121 \\ 0.01121 & 0.1030 \end{pmatrix}$
$y = -.5$	$\begin{pmatrix} 2 & .03 \\ .03 & 1.5 \end{pmatrix}$	$\begin{pmatrix} 1.682 & -0.05898 \\ -0.05898 & 1.587 \end{pmatrix}$
$z = -.5$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1.148 & -0.05123 \\ -0.05123 & 1.231 \end{pmatrix}$
$z = .5$	$\begin{pmatrix} .3 & 0 \\ 0 & .3 \end{pmatrix}$	$\begin{pmatrix} 0.3028 & 0.005399 \\ 0.005399 & 0.2994 \end{pmatrix}$
$y = .5$	$\begin{pmatrix} .4 & 0 \\ 0 & .4 \end{pmatrix}$	$\begin{pmatrix} 0.3986 & -0.006338 \\ -0.006338 & 0.3942 \end{pmatrix}$
$x = .5$	$\begin{pmatrix} .7 & .4 \\ .4 & .7 \end{pmatrix}$	$\begin{pmatrix} 0.5933 & 0.02344 \\ 0.02344 & 0.5974 \end{pmatrix}$

Literature

These results are presented in

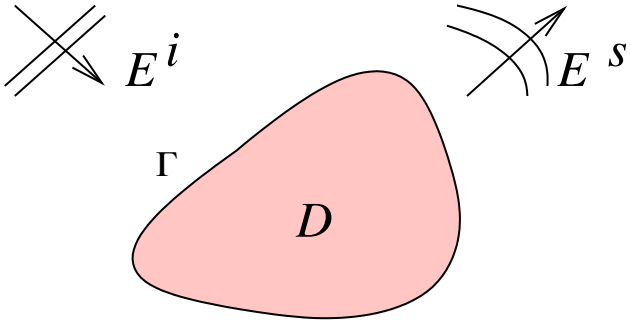
- F. Cakoni and P. Monk, The determination of anisotropic surface impedance in electromagnetic scattering, *J. Methods and Applications of Analysis*, 17, no 4, (2010).

which generalizes the approach in

- F. Cakoni and D. Colton, The determination of the surface impedance of a partially coated obstacle from far field data, *SIAM J. Appl. Math.* 64, 709-723 (2004).

The Scattering Problem

Consider now the scattering by an inhomogeneous anisotropic dielectric media. The goal is to determine the support of inhomogeneity and information about the anisotropy.



The relative electric permittivity $N(x) = \dfrac{\epsilon(x)}{\epsilon_0}$, $x \in \overline{D}$, is a positive definite symmetric matrix-valued function with bounded entries. The magnetic permeability is constant μ_0 and the wave number $k > 0$ is such that $k^2 = \epsilon_0 \mu_0 \omega^2$ where $\omega > 0$ is the frequency.

Maxwell's Equations

After eliminating the magnetic field, the scattering problem can be written in terms of electric fields as:

$$\begin{aligned}
\nabla \times \nabla \times E^s - k^2 E^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D} \\
\nabla \times \nabla \times E - k^2 N(x) E &= 0 && \text{in } D \\
\nu \times (E^s + E^i) &= \nu \times E && \text{on } \partial D \\
\nu \times \nabla \times (E^s + E^i) &= \nu \times \nabla \times E && \text{on } \partial D
\end{aligned}$$

where the scattered electric field satisfies the Silver-Muller radiation condition

$$\lim_{|x| \to \infty} (\nabla \times E^s \times x - ik|x| E^s) = 0.$$

and

$$E^i(x, d, p, k) = \frac{i}{k} \nabla \times \nabla \times p\, e^{ikx \cdot d}.$$

Inverse Problem

The inverse scattering problem is

> Determine **both** D and (some information about) N from a knowledge of $E_\infty(\hat{x}, d, p)$ for $d, -\hat{x} \in \Omega_0 \subset \Omega$ and two linearly independent polarizations p perpendicular to d, and a range of wave numbers $k \in [k_0\ k_1]$

Theorem (Uniqueness)

If $\|N\|_2 > 1$ or $\|N^{-1}\|_2 > 1$. Then D is uniquely determined from the above data.

There is no uniqueness proof for the determination of the matrix N for the above data.

Far Field Operator

Again, the scattered field E^s has the asymptotic behavior

$$E^s(x,d,p,k) = \frac{e^{ikr}}{r} E_\infty(\hat{x},d,p,k) + O\left(\frac{1}{r^2}\right)$$

as $r \to \infty$, $\hat{x} = x/|x|$, $r = |x|$.

Then far field operator $F : L^2(\Omega) \to L^2(\Omega)$

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x},d,g(d),k)\, ds(d)$$

is the far field pattern of the scattered field corresponding to the electric Helglotz wave function

$$E_g(x) := \int_\Omega e^{ikx\cdot d} g(d)\, ds(d)$$

as incident field.

Transmission Eigenvalues

Theorem

The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is injective and has dense range if and only if there does not exists a nontrivial solution to the homogeneous interior transmission problem

$$\begin{aligned} \nabla \times \nabla \times E - k^2 N(x) E &= 0 && \text{in} \quad D \\ \nabla \times \nabla \times E_0 - k^2 E_0 &= 0 && \text{in} \quad D \\ \nu \times E &= \nu \times E_0 && \text{on} \quad \partial D \\ \nu \times (\nabla \times E) &= \nu \times (\nabla \times E_0) && \text{on} \quad \partial D \end{aligned}$$

such that $E_0 := E_g$ is an electric Herglotz wave function with kernel g.

Transmission Eigenvalues

Values of k for which the homogenoeus interior transmission problem has a non trivial solution are called transmission eigenvalues.

Transmission eigenvalues are related to non scattering waves – if E_0 can be extended as a solution to the Maxwell's equation outside D then this extension is an incident wave that does not scatterer. Since Herglotz wave functions

$$E_g(x) := ik \int_\Omega e^{ikx\cdot d} g(d) ds(d), \qquad \text{for } \; g \in L^2_t(\Omega)$$

are dense in the space

$$\left\{ W \in L^2(\Omega) : \; \nabla \times \nabla \times W - k^2 W = 0 \right\}$$

then at a transmission eigenvalue there is an incident field that produces arbitrarily small scattered field.

Far Field Equation

We again consider the far field equation

$$\boxed{(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)}$$

where $E_{e,\infty}(\hat{x}, z, q)$ denotes the far field pattern of the electric dipole

$$E_e(x, z, q) := \frac{i}{k} \nabla_x \times \nabla_x \times q \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|},$$

and the far field operator $F : L^2_t(\Omega) \to L^2_t(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, ds(d).$$

Solution of Far Field Equation

Theorem

1. *Let* $z \in D$.
   - *If* k *is not a transmission eigenvalue then for every* $\epsilon > 0$ *there exists* $g_{z,\epsilon,k} \in L^2(\Omega)$ *satisfying*

$$\|Fg_{z,\epsilon,k} - E_{e,\infty}\|_{L^2(\Omega)} < \epsilon \quad \text{and} \quad \lim_{\epsilon \to 0} \|E_{g_{z,\epsilon,k}}\|_{L^2(D)} = \|E_0\|_{L^2(D)}.$$

   - *If* k *is a transmission eigenvalue* , *every* $g_{z,\epsilon,k} \in L^2(\Omega)$ *satisfying*

$$\|Fg_{z,\epsilon,k} - E_{e,\infty}\|_{L^2(\Omega)} < \epsilon \quad \text{is such that} \quad \lim_{\epsilon \to 0} \|E_{g_{z,\delta,k}}\|_{L^2(D)} = \infty.$$

2. *Let* $z \notin D$.
   - *For all* $k > 0$, *every* $g_{z,\epsilon,k} \in L^2(\Omega)$ *satisfying*

$$\|Fg_{z,\epsilon,k} - E_{e,\infty}\|_{L^2(\Omega)} < \epsilon \quad \text{is such that} \quad \lim_{\epsilon \to 0} \|E_{g_{z,\epsilon,k}}\|_{L^2(D)} = \infty.$$

Solution of Far Field Equation

- In practice one has available the noisy far field operator F^δ,

$$(F^\delta g)(\hat{x}) := \int_\Omega E^\delta_\infty(\hat{x}, d, g(d), k)\, ds(d)$$

  i.e. the far field operator with kernel the noisy measurements with δ being the noise level.

- For various of $z \in \mathbb{R}^3$ and $k > 0$, $g_{z,k}$ is computed using Tikhonov regularization technique, i.e by solving

$$(\alpha(\delta) I + F^{\delta *} F^\delta) g^\delta_{z,k} = F^{\delta *} E_{e,\infty}(\cdot, z, q)$$

  $\alpha(\delta) \to 0$ as $\delta \to 0$.

- The behavior of g in the above theorem is inherited by $g^\delta_{z,k}$ as $\delta \to 0$.

- This behavior for different z and k can be used to solve the inverse problem.

Reconstruction of the Support D

- For fixed frequency $k > 0$ not a transmission eigenvalue we have

  $\|g_{z,k}(\cdot)\|_{L^2(\Omega)}$ is bounded for $z \in D$ and very large for $z \notin D$.

- Using band limit multifrequency data for $k \in [k_0, k_1]$.

  Note: If $[k_0, k_1]$ contains at least a transmission eigenvalue than $\|g_{z,k}(\cdot)\|_{L^2(\Omega)}$ as function of k is not integrable in $[k_0, k_1]$.

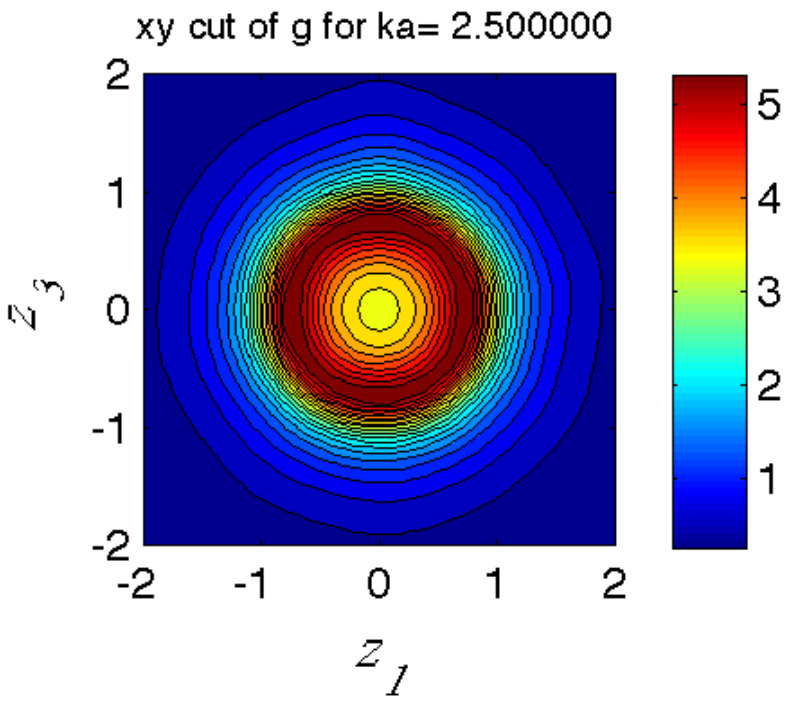
  In this case we can use

$$\mathcal{G}(z) := \int_{k_0}^{k_1} \frac{1}{\|g_{z,k}(\cdot)\|_{L^2(\Omega)}}$$

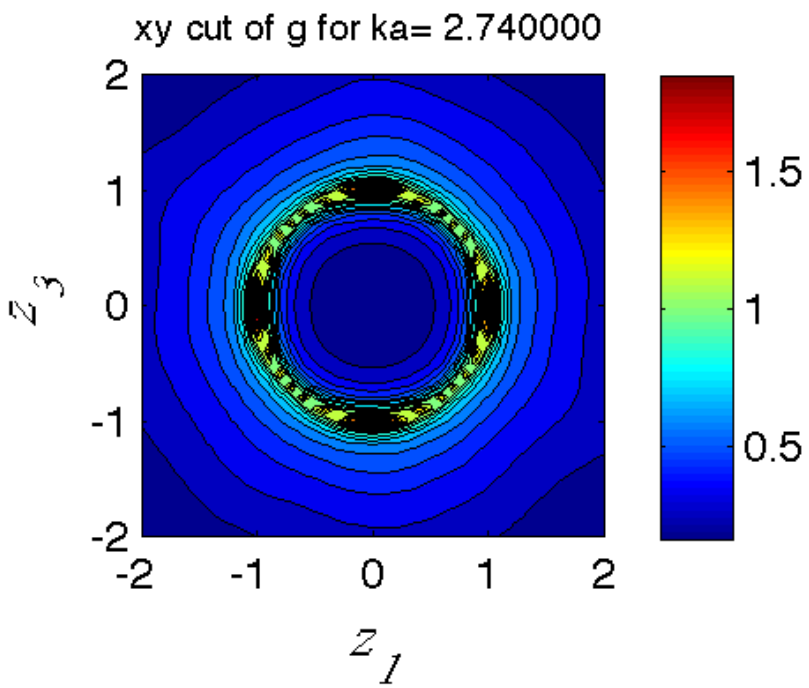
  as an indicator function for D.

Examples of Reconstruction

$N = 5l$



g isosurf for ka= 2.500000



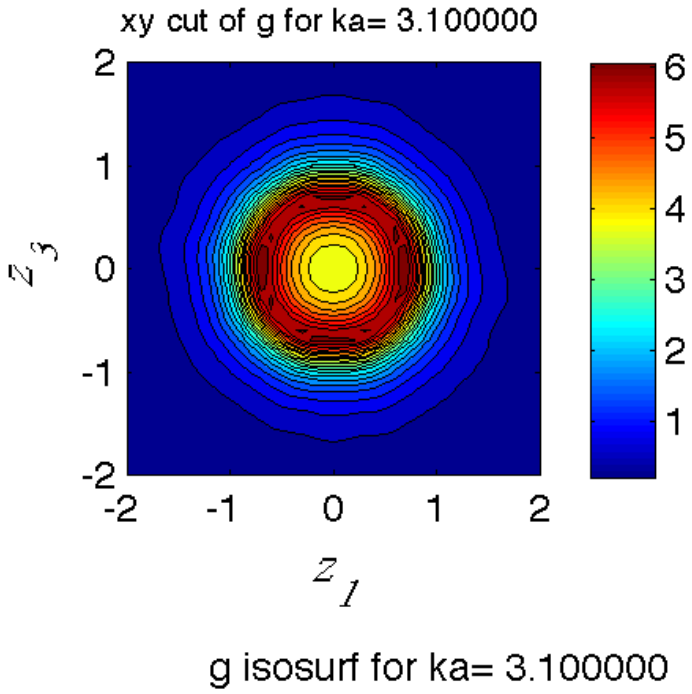
g isosurf for ka= 2.740000

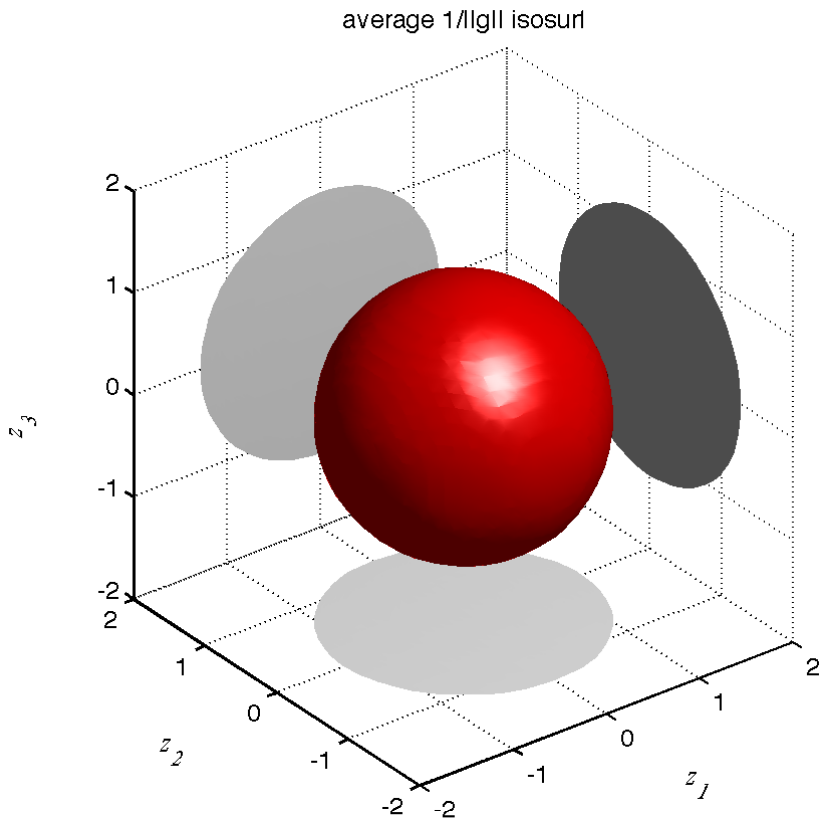
k is not a TE

k is a TE

Examples of Reconstruction

$N = 5I$



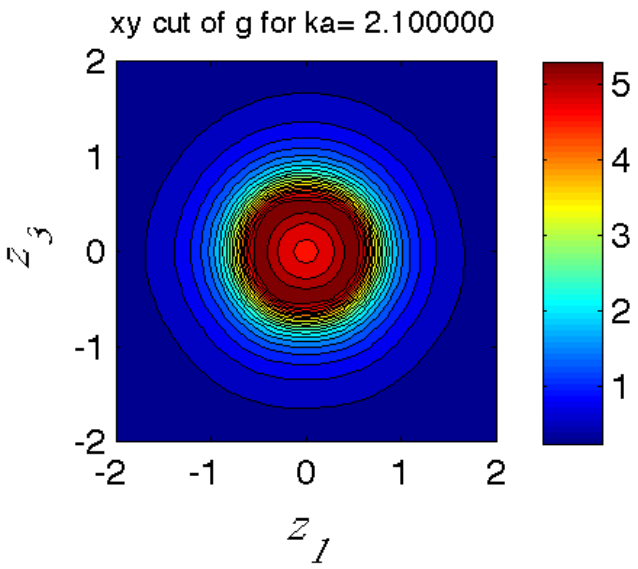


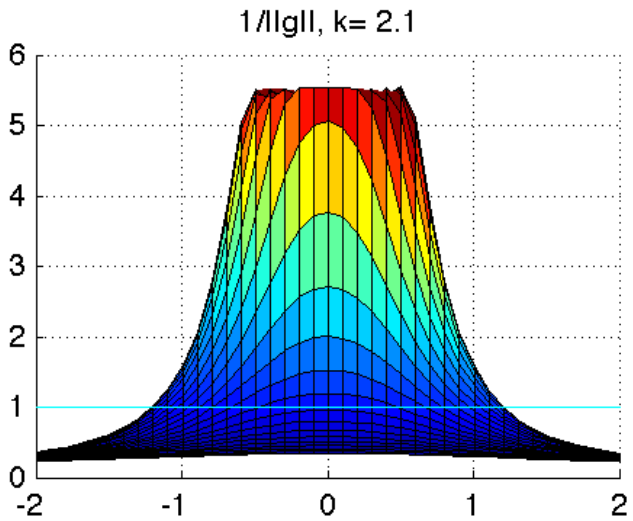
k is not a TE

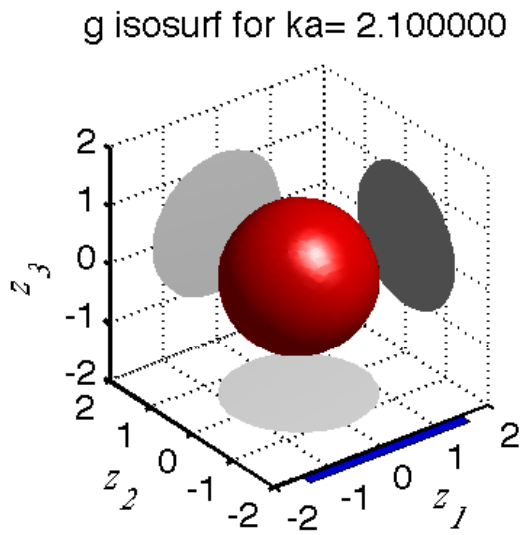
Reconstruction for $k \in [2, 3.5]$

Examples of Reconstruction

$N = 16l$, k is not TE

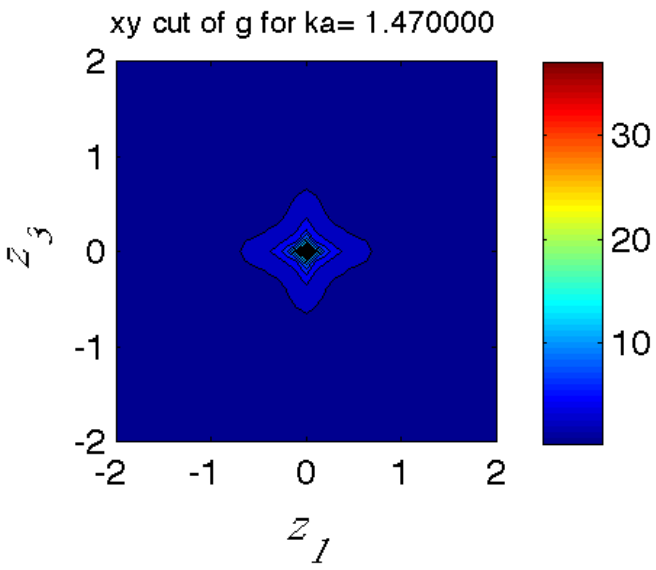


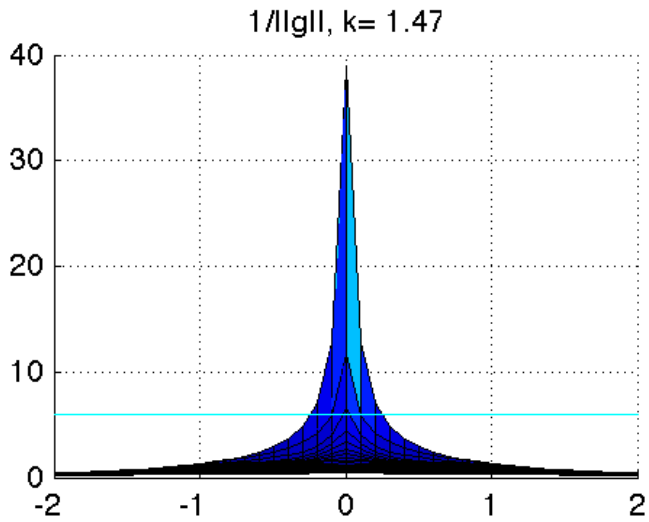


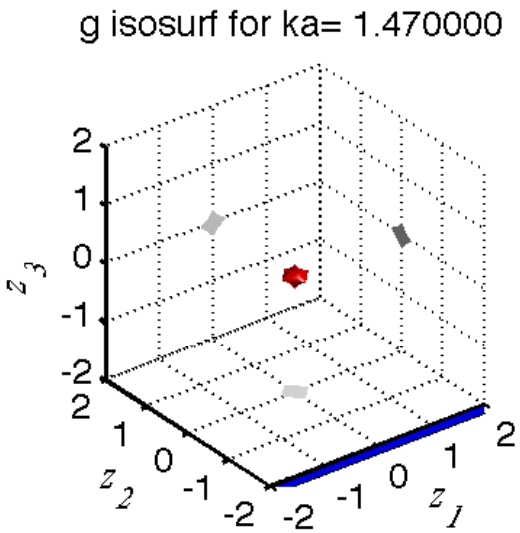


Examples of Reconstruction

$N = 16l$, k is a TE

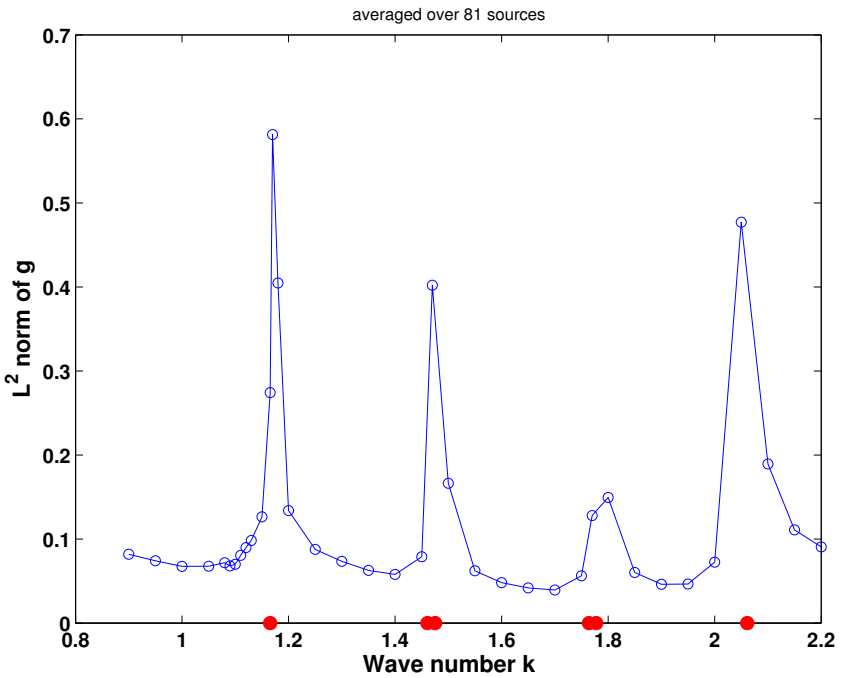




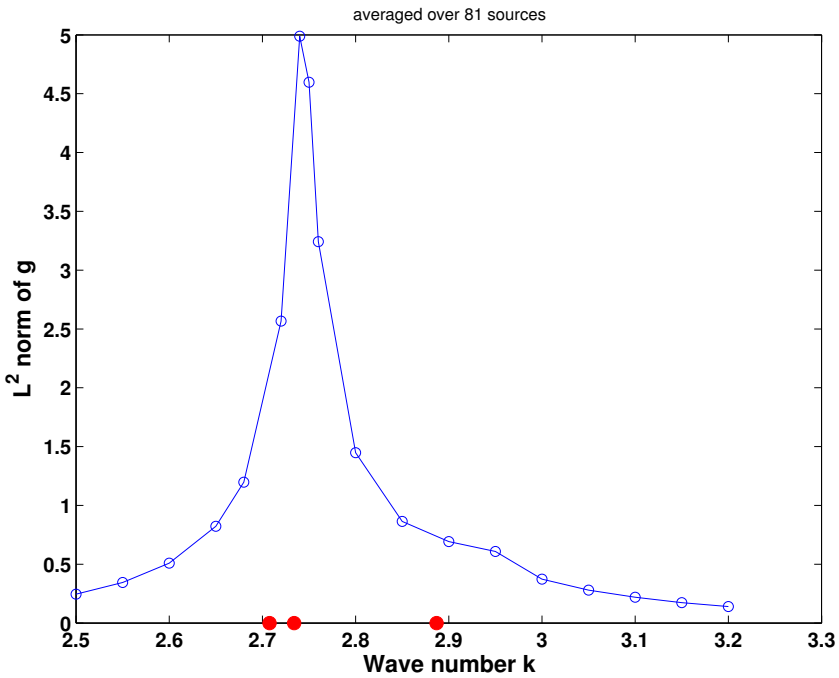


Computation of Real Transmission Eigenvalues

Results for an isotropic sphere of unit radius. Solving the far-field equation for several source points z inside the sphere (using 42 incoming directions and 42 measurements) gives obvious peaks at the first transmission eigenvalue.



$N = 16I$



$N = 5I$

Red dots indicate exact transmission eigenvalues. Using the far field pattern, peaks in $\|g\|$ indicate transmission eigenvalues.

Non-destructive Testing of Anisotropic Materials

What do transmission eigenvalues say about $N(x) = \epsilon(x)/\epsilon_0$?

To this end let $\lambda_1(x) \leq \lambda_2(x) \leq \lambda_3(x)$ be the eigenvalues of N and define

$$\lambda_{min} = \inf_{x \in D} \lambda_1(x) \qquad \text{and} \qquad \lambda_{max} = \sup_{x \in D} \lambda_3(x).$$

Let $k_{1,D,N(x)}$ be the first transmission eigenvalue for our scattering problem and let $k_{1,D,\lambda_{min}}$ and $k_{1,D,\lambda_{max}}$ be the first transmission eigenvalues for the case of $N = \lambda_{min} I$ and $N = \lambda_{max} I$, respectively.

Non-destructive Testing of Anisotropic Materials

Theorem

Let α and β be positive constants.

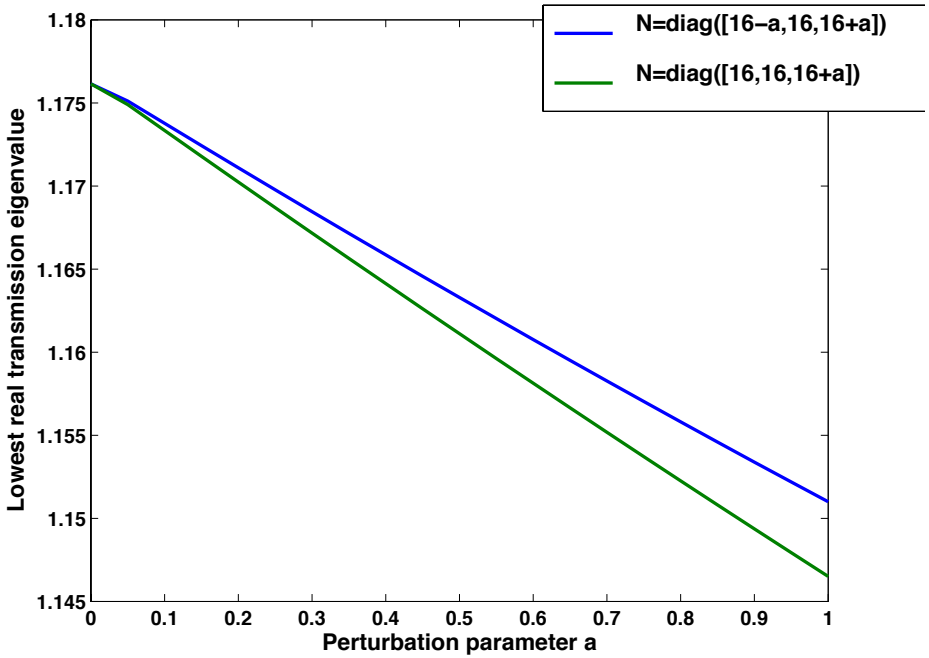
1. *If* $\|N(x)\|_2 \geq \alpha > 1$ *then* $k_{1,D,\lambda_{max}} \leq k_{1,D,N(x)} \leq k_{1,D,\lambda_{min}}$
2. *If* $\|N(x)\|_2 \leq \beta < 1$ *then* $k_{1,D,\lambda_{min}} \leq k_{1,D,N(x)} \leq k_{1,D,\lambda_{max}}$

Given $k_{1,D,N(x)}$ we now compute a constant n such that $k_{1,D,N(x)} = k_{1,D,n}$. Then the above theorem implies that

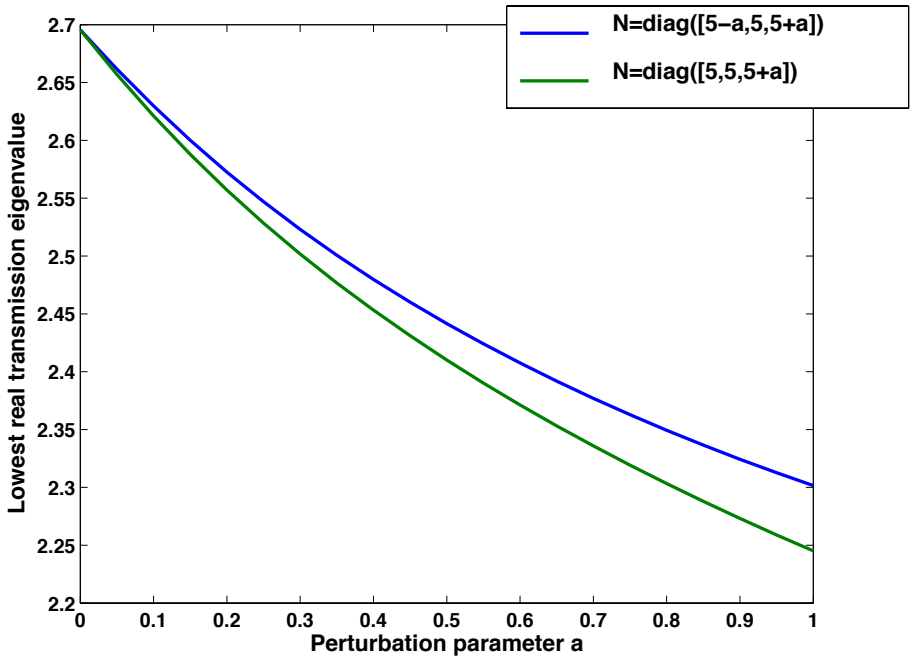
$$\lambda_{min} \leq n \leq \lambda_{max}$$

and that $k_{1,D,n}$ depends monotonically on n. It can in fact be shown that $k_{1,D,n}$ depends strictly monotonically on n.

Numerical Examples



Perturbation of $N = 16I$

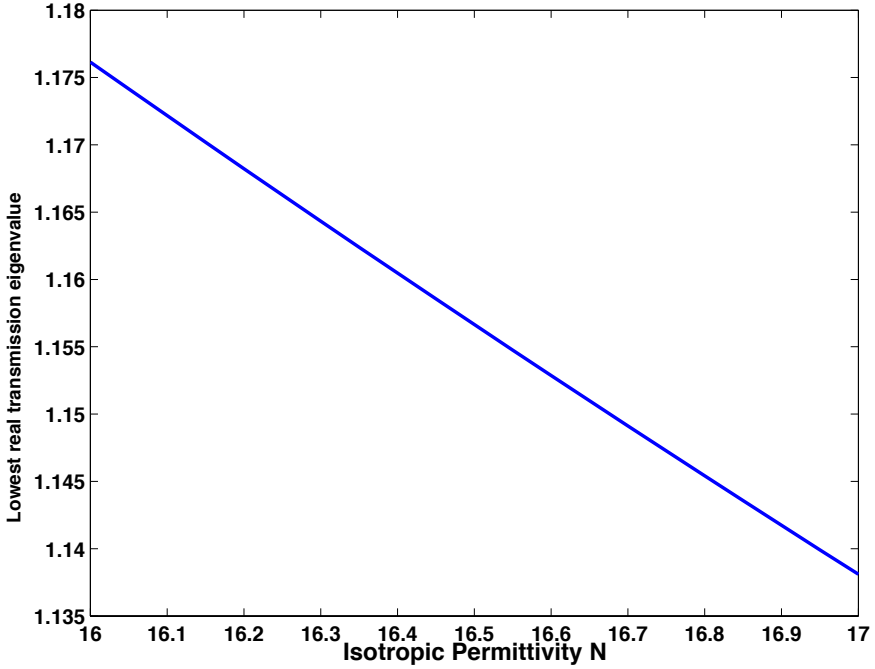


Perturbation of $N = 5I$

Numerical Examples (cont)

Having the first transmission eigenvalue for anisotropic N then compute the isotropic n discussed previously

$$\lambda_{min} \leq n \leq \lambda_{max}$$

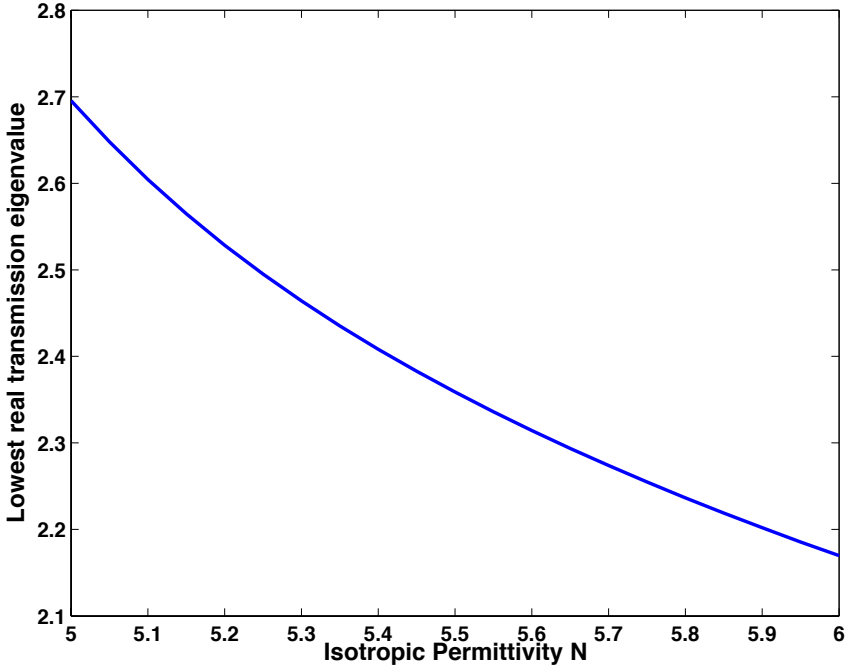


Lowest transmission eigenvalue against N (isotropic)

N	$k_{1,D,N(x)}$	n
$\mathrm{diag}([15.5, 16, 16.5])$	1.163	16.33
$\mathrm{diag}([15, 16, 17])$	1.151	16.65
$\mathrm{diag}([16, 16, 16.5])$	1.161	16.38
$\mathrm{diag}([16, 16, 17])$	1.146	16.77

Numerical Examples (cont)

The same procedure can be carried out at lower N



Lowest transmission eigenvalue against N (isotropic)

N	$\lambda_{1,D,N(x)}$	n
$\mathrm{diag}([4.5, 5, 5.5])$	2.442	5.339
$\mathrm{diag}([4, 5, 6])$	2.302	5.631
$\mathrm{diag}([5, 5, 5.5])$	2.410	5.397
$\mathrm{diag}([5, 5, 6])$	2.245	5.778

Absorbing Media

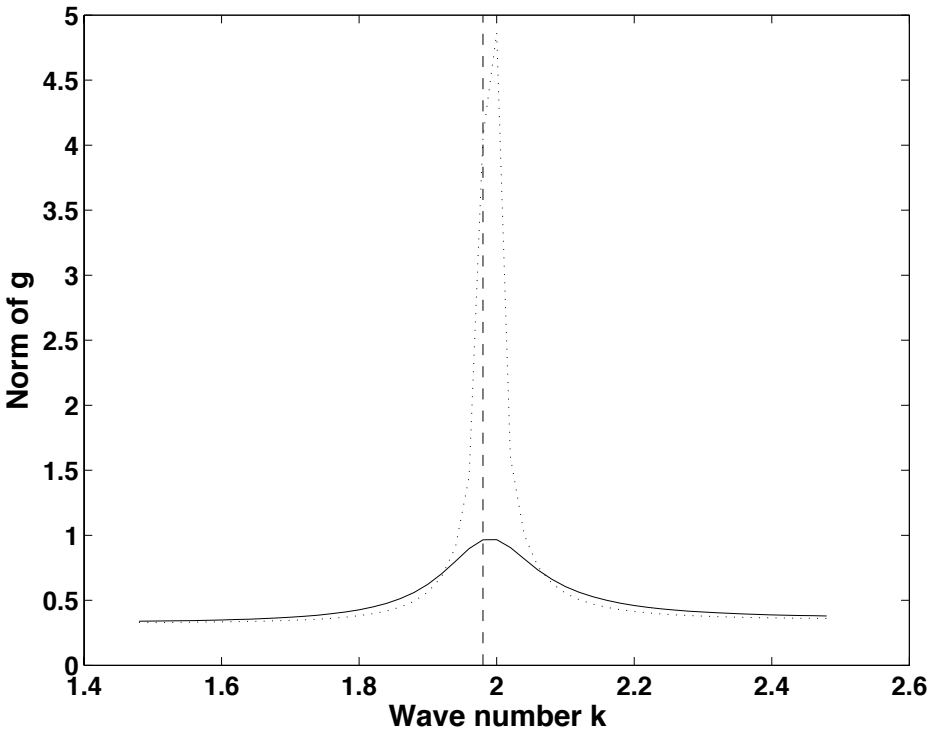
There is no material that is a perfect dielectric. So what happens to the above theory if there is a small amount of absorption in D?

Let $N_1(x) = \frac{\epsilon(x)}{\epsilon_0} + i\frac{\sigma(x)}{\epsilon_0\omega}$ describe the absorbing media. Using Kato's perturbation theory for linear operators, it is possible to prove the following theorem.

Theorem

Let $k^* > 0$ be a real transmission eigenvalue corresponding to $N(x) = \frac{\epsilon(x)}{\epsilon_0}$. Then if $\|\sigma(x)\| < \epsilon$ for ϵ sufficiently small there exists at least one (complex) transmission eigenvalue near k^* corresponding to $N_1(x)$.

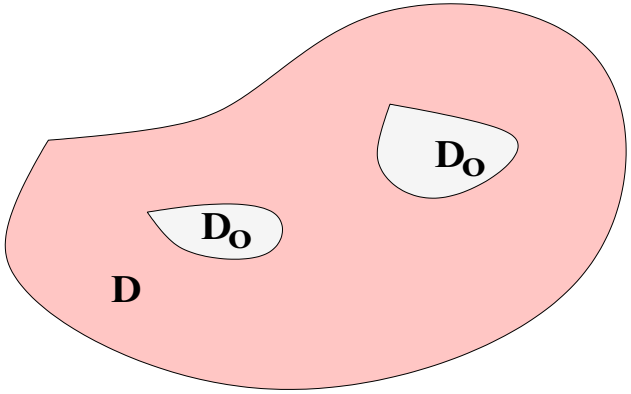
Numerical Example: Absorbing Media



D is a sphere of diameter 1 and $N = (16 + i)I$

Anisotropic Media with Cavities

Can the assumption $N > I$ or $0 < N < I$ in D be relaxed?



$$N = I \text{ in } D_0 \quad N - I \geq \delta > 0 \text{ in } D \setminus D_0$$

The case when there are regions D_0 in D where $N = I$ (i.e. cavities) is more delicate but the same type of analysis as above can be carried through.

Anisotropic Media with Cavities

Let $k_{1,D,D_0,N(x)}$ be the first transmission eigenvalue for D, D_0, $N(x)$.

- Monotonicity with respect to the index of refraction

$$k_{1,D,D_0,N(x)} \leq k_{1,D,D_0,\tilde{N}(x)}, \qquad \tilde{N}(x) \leq N(x).$$

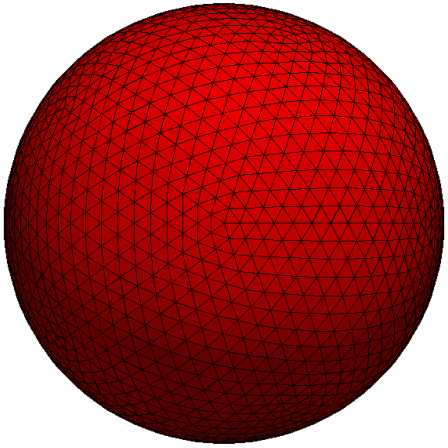
- Monotonicity with respect to voids

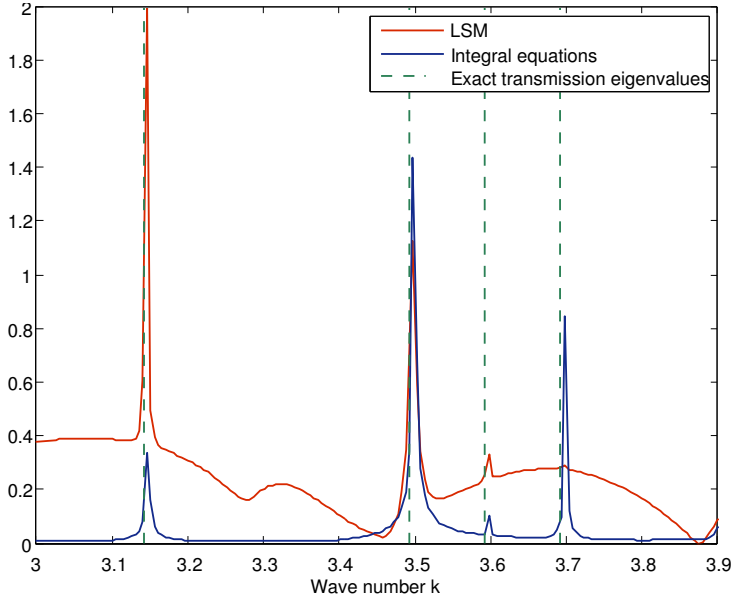
$$k_{1,D,D_0,N(x)} \leq k_{1,D,\tilde{D}_0,N(x)}, \qquad D_0 \subset \tilde{D}_0.$$

Other type of inclusions such as a perfect conductor or crack could also be considered inside the inhomogeneity.

Numerical Example: Media with Cavities

Due to A. Cossonniere, Ph.D. Thesis



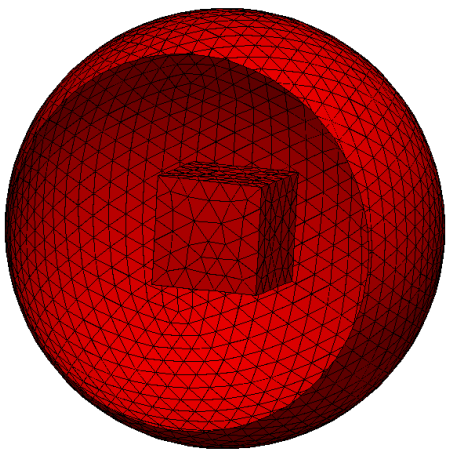


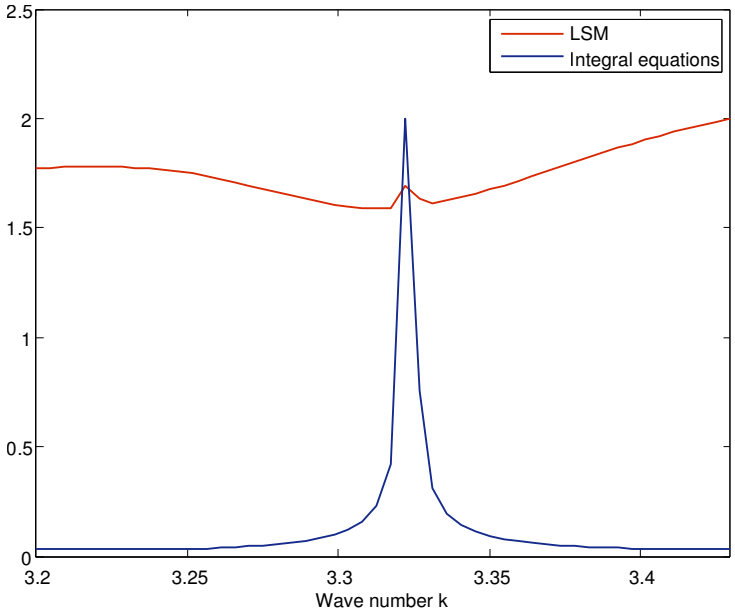
The sphere with radius 1 and $N = 4l$

$k_1 = 3.16$

Numerical Example: Media with Cavities

Due to A. Cossonniere, Ph.D. Thesis





The sphere with radius 1, $N = 4l$ containing a cubic cavity
$k_1 = 3.33$, i.e. k_1 is shifted to the right

Case with Contrast in ϵ and μ

If $\mu \neq \mu_0$ in D then the corresponding transmission eigenvalue problem written in terms of magnetic fields is

$$\begin{aligned} \nabla \times A \nabla \times H - k^2 n H &= 0 && \text{in} \quad D \\ \nabla \times \nabla \times H_0 - k^2 H_0 &= 0 && \text{in} \quad D \\ \nu \times H &= \nu \times H_0 && \text{on} \quad \partial D \\ \nu \times (A \nabla \times H_0) &= \nu \times (\nabla \times H_0) && \text{on} \quad \partial D. \end{aligned}$$

where $A^{-1} = \epsilon(x)/\epsilon_0$ and $n = \mu/\mu_0$

Here the electric permittivity $\epsilon(x)$ in the medium is a positive definite matrix with bounded entries wheres the magnetic permeability $\mu \neq \mu_0$ in the medium is a constant.

Case with Contrast in ϵ and μ

- The support D can be reconstructed by using the Linear Sampling Method.
- The following results can be obtained on transmission eigenvalues:
  - If the contrasts $A - I$ and $n - 1$ have the same fixed sign, then there exists an infinite discrete set of real transmission eigenvalues accumulating at $+\infty$.
  - If the contrasts $A - I$ and $n - 1$ have the opposite fixed sign, then there exits at least one real transmission eigenvalue providing that n is small enough.
- The first transmission eigenvalue in this case provides estimates for $\epsilon(x)/\mu$.

Literature

- F. Cakoni, D. Gintides and H. Haddar, The existence of an infinite discrete set of transmission eigenvalues, *SIAM J. Math. Anal.*, 42, no 1, 237-255 (2010).
- F. Cakoni, D. Colton, P. Monk and J. Sun, The inverse electromagnetic scattering problem for anisotropic media, *Inverse Problems*, 26, 074004 (2010).
- F. Cakoni and A. Kirsch, On the interior transmission eigenvalue problem, *Int. Jour. Comp. Sci. Math.*, 1-2, 142-167, (2010).
- F. Cakoni, D. Colton and H. Haddar, The interior transmission problem for regions with cavities, *SIAM J. Math. Anal.* 42, no 1, 145-162 (2010).
- F. Cakoni, D. Colton and H. Haddar, The interior transmission eigenvalue problem for absorbing media, *Inverse Problems*, (to appear).

Literature

Most of the results on the Qualitative Methods for Maxwell's equations can be found in the following monograph:

- F. Cakoni, P. Monk and P. Monk, The linear Sampling Method in Inverse Electromagnetic Scattering, *CBMS Series, SIAM Publications* 80, (2011).

A comprehensive study on transmission eigenvalues for Maxwell's equation can be found in the following thesis:

- A. Cossoniere, Transmission Eigenvalues for Maxwell's equation and their Application in Non-destructive Testing, Ph.D Thesis (2011).