Deep Neural Networks: A Universal Classification Strategy?

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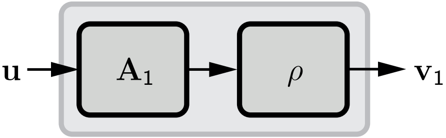
Based on joint work with Raja Giryes and Guillermo Sapiro

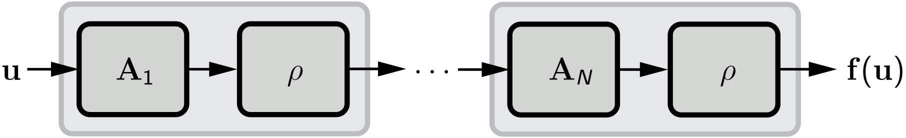
(Deep) neural network



Single layer:

$$\mathbf{v}_1 = \rho(\mathbf{A}_1\mathbf{u})$$

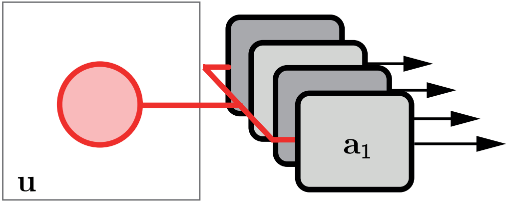
(Deep) neural network



Whole net response:

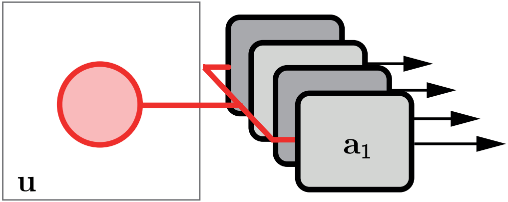
$$\mathbf{f}(\mathbf{u}) = \rho(\mathbf{A}_N\, \rho(\mathbf{A}_{N-1}\, \rho(\cdots \rho(\mathbf{A}_1 \mathbf{u}) \cdots)))$$

Linear part



- Convolutional layer: shift-invariant filter bank $\mathbf{v}_i = \mathbf{a}_i * \mathbf{u}_i$

Linear part



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  $\mathbf{A}$ is block-Töplitz

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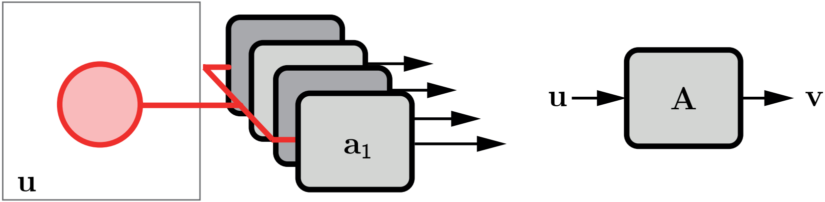
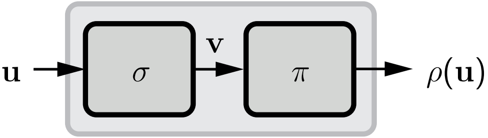


- Convolutional layer: shift-invariant filter bank $\mathbf{v}_i = \mathbf{a}_i * \mathbf{u}_i$
  $\mathbf{A}$ is block-Töplitz
- Fully-connected layer: $\mathbf{v} = \mathbf{A}\mathbf{u}$

Non-linear part



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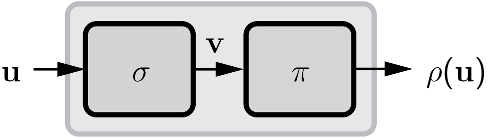
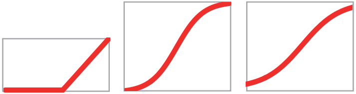
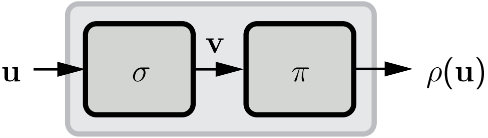




ReLU $\max\{u, 0\}$ | **tanh** $\tanh u$ | **Sigmoid** $\dfrac{1}{1+e^{-u}}$

- Element-wise activation function $\sigma(u)$

Non-linear part



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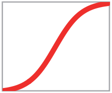
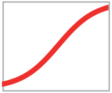


tanh
$\tanh u$



Sigmoid
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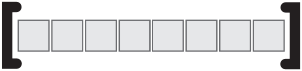
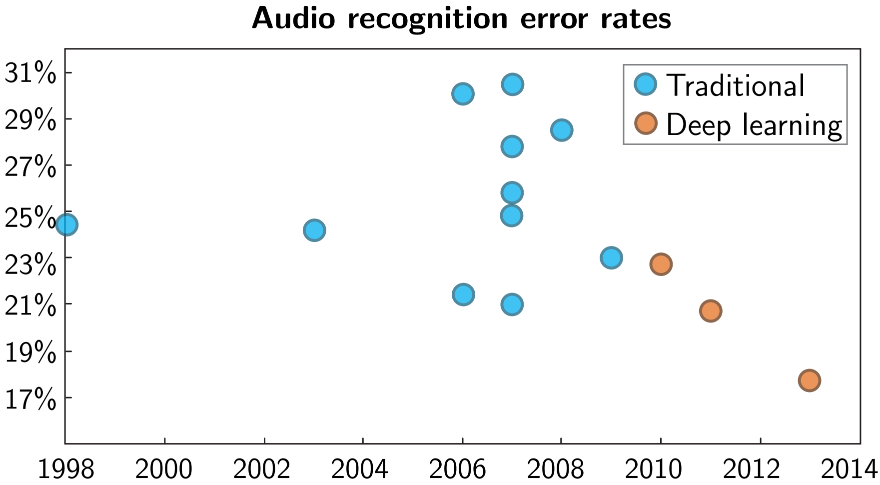


Max
$$\max_i v_i$$

Mean
$$\frac{1}{m}\sum_{i=1} v_i$$

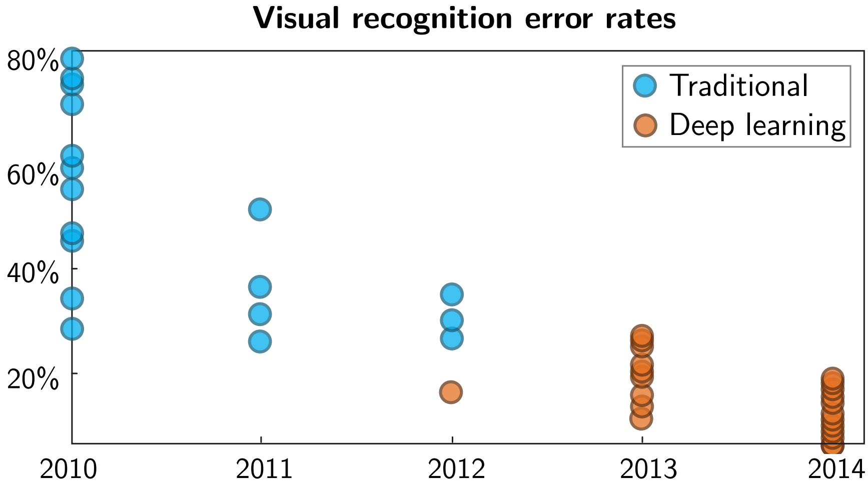
- Element-wise activation function $\sigma(u)$
- Pooling or aggregation operator $\pi(\mathbf{v})$

Impact of deep learning



Source: Clarifi

Impact of deep learning



Source: Clarifi

Why DNNs work?

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Why DNNs work?

- Representation power?
- Role of depth?
- Role of pooling?
- Role of nonlinearity?
- How to train?
- How much training data are needed?

Representation power

- DNNs are universal approximators of any Borel function[1]

[1]Cybenko 1989; Hornik 1991; [2]Barron 1992

Representation power

- DNNs are universal approximators of any Borel function[1]
- Estimation error of a function $\mathbf{f}$ by DNN is[2]

$$\mathcal{O}\left(\frac{C_f}{K}\right) + \mathcal{O}\left(\frac{nK}{T}\log T\right)$$

$C_f =$ smoothness of f
$K = \#$ of degrees of freedom
$n =$ input dimension
$T = \#$ of training samples

[1]Cybenko 1989; Hornik 1991; [2]Barron 1992

Representation power

- DNNs represent restricted Boltzmann machines with number of parameters exponentially greater than the number of degrees of freedom of the network[1]

[1]Montúfar & Morton, 2014; [2]Montúfar *et al.*, 2014

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- **Depth is important!**

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Role of pooling

- Pooling provides shift invariance[1]

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Training

- Supervised training by back propagation (chain rule) results in non-convex optimization problem

Saxe, McClelland & Ganguli, 2014; Dauphin *et al.*, 2014; Choromanska *et al.*, 2015; Haeffele & Vidal, 2015

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Training

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- Random initialization works well
- Extreme learning strategies rely only on randomization

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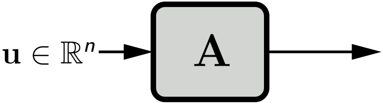
Assumptions



$\mathbf{A} \in \mathbb{R}^{m \times n}$

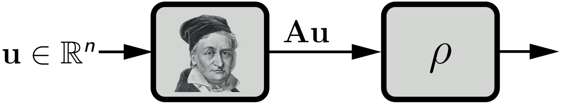
- Fully-connected linear layers

Assumptions

$$a_{ij} \sim \mathcal{N}(0, \tfrac{1}{m})$$

- Fully-connected linear layers with random Gaussian weights

Assumptions



$a_{ij} \sim \mathcal{N}(0, \frac{1}{m})$

- Fully-connected linear layers with random Gaussian weights
- Element-wise (approximately) truncated linear activation

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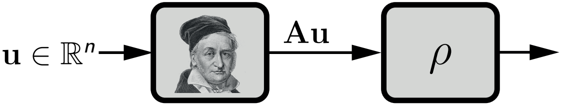

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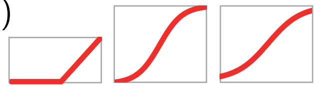

$\max\{v, 0\}$ $\tanh v$ $1/(1+e^{-v})$

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$$\rho|_{[a,b]} \text{ linear} \qquad \rho|_{\mathbb{R}\setminus[a,b]} = \text{const}$$

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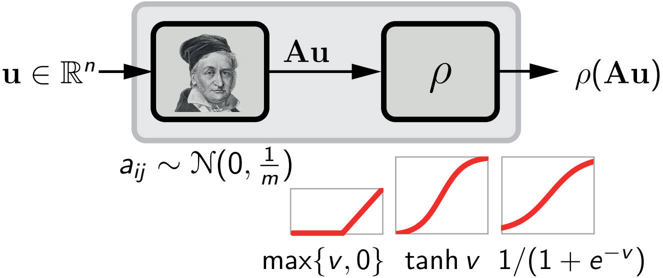

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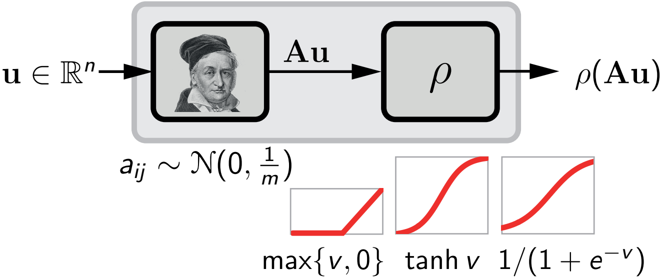
Assumptions



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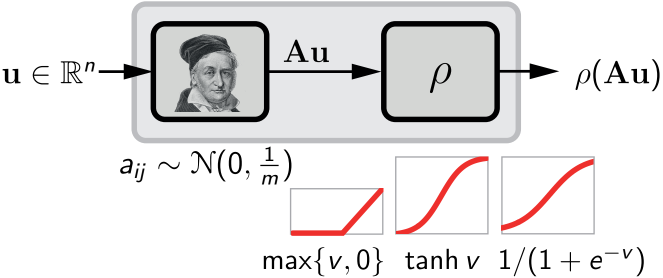
- No pooling (pooling = invariance)

Assumptions



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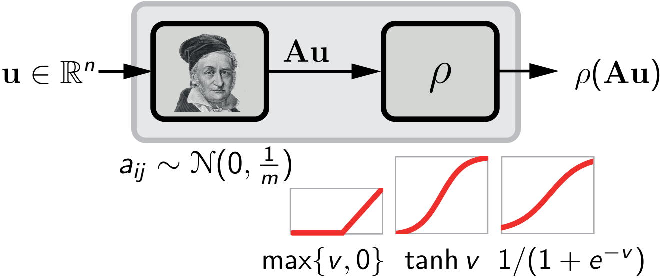

- Fully-connected linear layers with random Gaussian weights
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$$\rho|_{[a,b]} \text{ linear} \qquad \rho|_{\mathbb{R}\setminus[a,b]} = \mathrm{const}$$

- No pooling: input already invariant
- Low dimensional input data

Random Gaussian weights

5406 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 52, NO. 12, DECEMBER 2006

Near-Optimal Signal Recovery From Random Projections: Universal Encoding Strategies?

Emmanuel J. Candes and Terence Tao

Candés, Romberg, Tao, 2006; Candés & Tao, 2006

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A k**-sparse vector** $\mathrm{u} \in \mathbb{R}^n$ **can be reconstructed from** $m = \mathcal{O}(k \log(n/k))$ **random projections**

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Random projection is universally good

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Low-dimensional input

Input data have a small number of degrees of freedom

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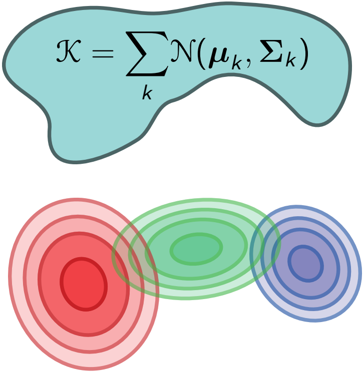

Gaussian mixture model

Low-dimensional input

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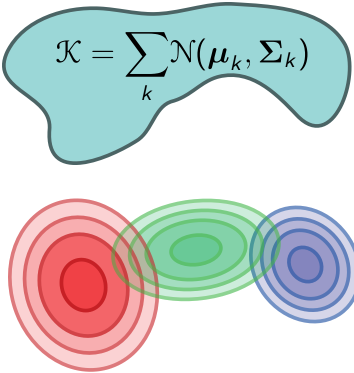

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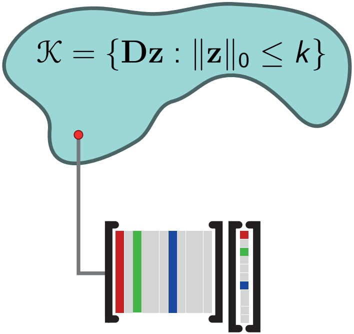
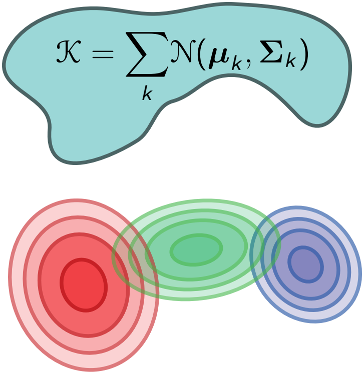


Sparse representation

Low-dimensional input

Violated at the output due to DNN nonlinearity!



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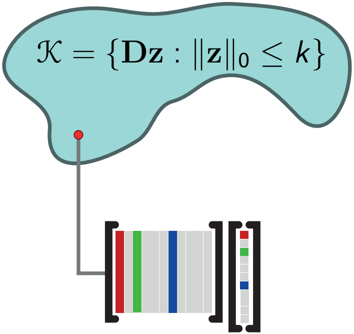
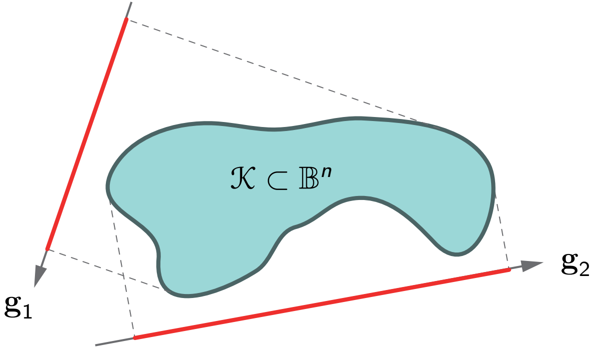


Sparse representation

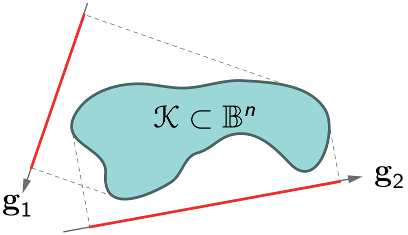
Gaussian mean width



$$\omega(\mathcal{K}) = \mathbb{E} \sup_{\mathbf{u},\mathbf{v} \in \mathcal{K}} \langle \mathbf{u} - \mathbf{v}, \mathbf{g} \rangle \qquad \mathbf{g} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

Plan & Vershynin, 2012

Gaussian mean width



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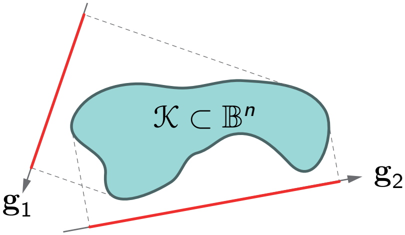

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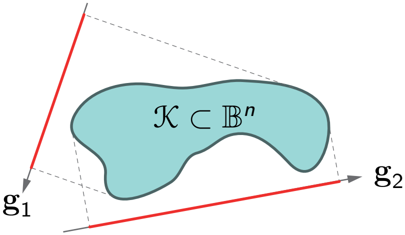

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- $\mathcal{K}$ is k-sparsely representable: $\omega^2(\mathcal{K}) = \mathcal{O}(k \log(n/k))$

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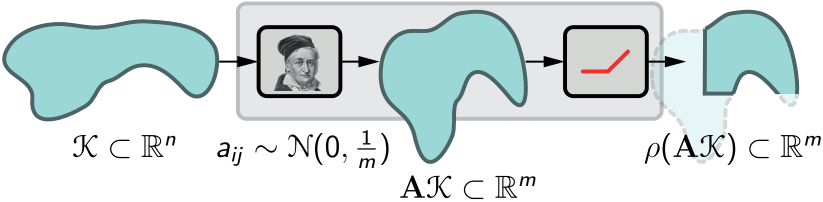
Low dimension in – low dimension out



Giryes, Sapiro, B, 2015

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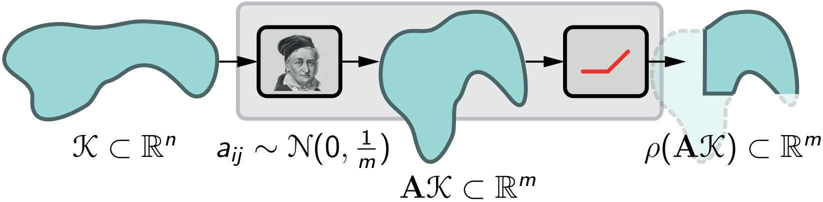

Theorem: if $\omega^2(\mathcal{K}) \ll m$ **then** $\omega^2(\rho(\mathbf{A}\mathcal{K})) \approx \omega^2(\mathcal{K})$

Giryes, Sapiro, B, 2015

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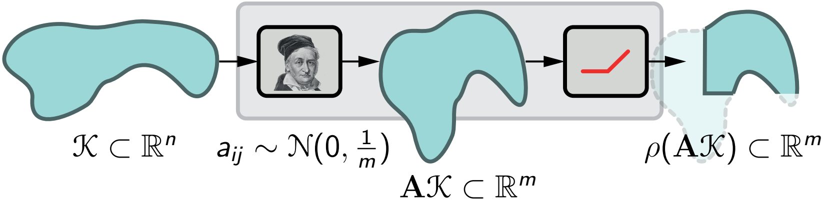

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Proof: covering argument

Giryes, Sapiro, B, 2015

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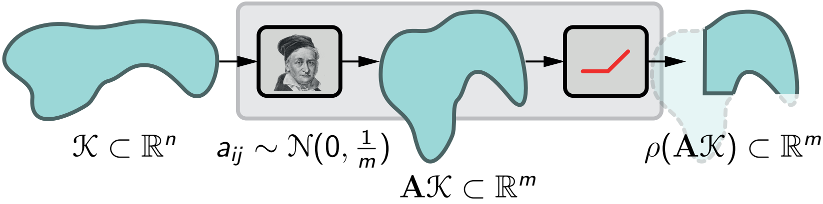

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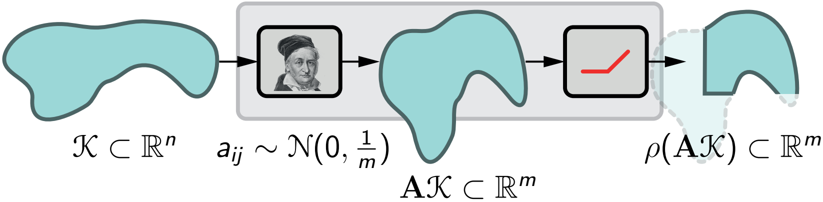

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Proof: covering argument

- Intrinsic data dimension does not grow significantly through the network
- It is sufficient to analyze a single layer in DNN

Giryes, Sapiro, B, 2015

Stable embedding

Theorem: the map $\mathbf{h} : (\mathcal{K} \subset \mathbb{S}^{n-1}, d_{\mathbb{S}^{n-1}}) \mapsto (\mathbf{h}(\mathcal{K}), d_{\mathbb{H}^m})$ defined by $\mathbf{h}(\mathbf{u}) = \mathrm{sign}(\rho(\mathbf{A}\mathbf{u}))$ is a δ-isometry with $\delta = c\, m^{-1/6}\, \omega^{1/3}(\mathcal{K})$

Plan & Vershynin, 2013; Giryes, Sapiro, B, 2015

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$$|d_{\mathbb{S}^{n-1}}(\mathbf{u}, \mathbf{v}) - d_{\mathbb{H}^m}(\mathbf{h}(\mathbf{u}), \mathbf{h}(\mathbf{v}))| \leq \delta \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{K}$$

and every $\mathbf{w} \in \mathrm{h}(\mathcal{K})$ **has some** $\mathbf{u} \in \mathcal{K}$ **such that** $d_{\mathbb{H}^m}(\mathbf{h}(\mathbf{u}, \mathbf{w}) \leq \delta$ **with** $\delta = c\, m^{-1/6}\, \omega^{1/3}(\mathcal{K})$

Plan & Vershynin, 2013; Giryes, Sapiro, B, 2015

Stable embedding

Theorem: the map $\mathbf{h} : (\mathcal{K} \subset \mathbb{S}^{n-1}, d_{\mathbb{S}^{n-1}}) \mapsto (\mathbf{h}(\mathcal{K}), d_{\mathbb{H}^m})$ **defined by** $\mathrm{h}(\mathbf{u}) = \mathrm{sign}(\rho(\mathbf{Au}))$ **is a** δ**-isometry, i.e.,**

$$|d_{\mathbb{S}^{n-1}}(\mathbf{u}, \mathbf{v}) - d_{\mathbb{H}^m}(\mathbf{h}(\mathbf{u}), \mathbf{h}(\mathbf{v}))| \leq \delta \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{K}$$

and every $\mathbf{w} \in \mathrm{h}(\mathcal{K})$ **has some** $\mathbf{u} \in \mathcal{K}$ **such that** $d_{\mathbb{H}^m}(\mathbf{h}(\mathbf{u}, \mathbf{w}) \leq \delta$ **with** $\delta = c\, m^{-1/6}\, \omega^{1/3}(\mathcal{K})$

Proof: follows Plan & Vershynin, 2013

Plan & Vershynin, 2013; Giryes, Sapiro, B, 2015

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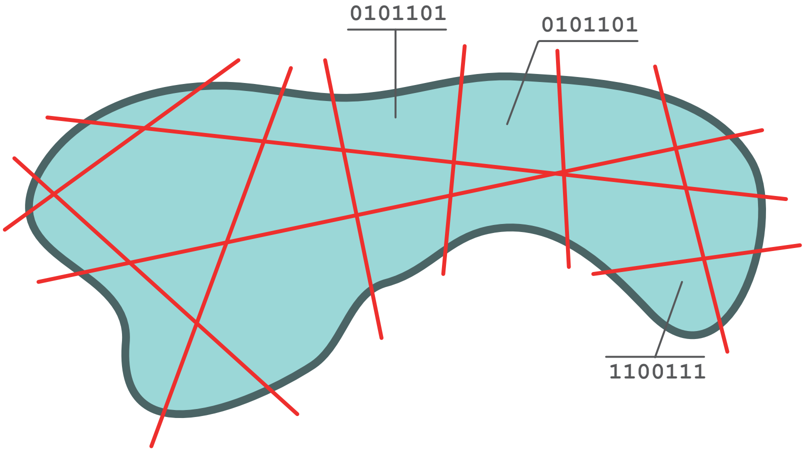
DNN layer performs stable embedding in the Gromov-Hausdorff sense

Plan & Vershynin, 2013; Giryes, Sapiro, B, 2015

Tessellation of the input space



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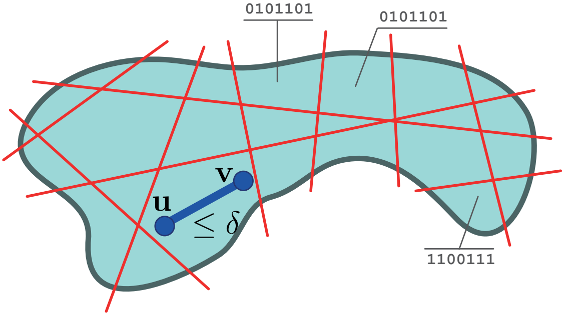

- Cell diameter $\leq \delta$

Giryes, Sapiro, B, 2015

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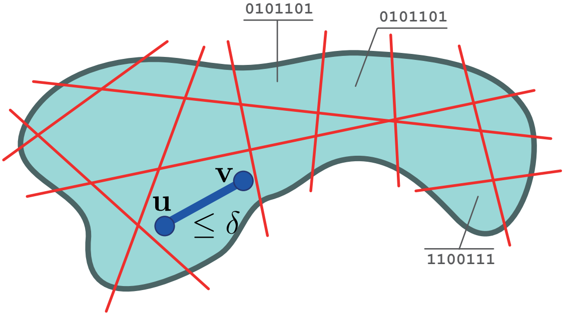

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- If $\mathbf{h}(\mathbf{u}) = \mathbf{h}(\mathbf{v})$ then $d_{\mathbb{S}^{n-1}}(\mathbf{u}, \mathbf{v}) \leq \delta$

Giryes, Sapiro, B, 2015

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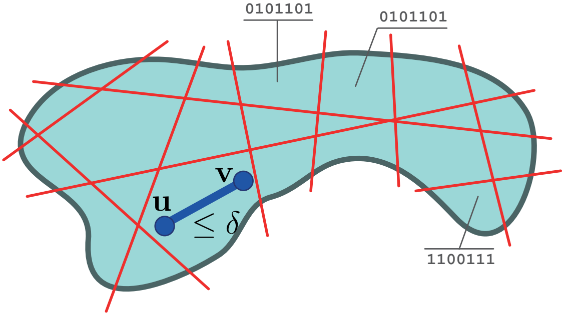

- Cell diameter $\leq \delta$
- If $\mathbf{h}(\mathbf{u}) = \mathbf{h}(\mathbf{v})$ then $d_{\mathbb{S}^{n-1}}(\mathbf{u}, \mathbf{v}) \leq \delta$
- Input metric can be recovered up to a small distortion

Giryes, Sapiro, B, 2015

Stable embedding

Theorem: **there exists a procedure $\mathcal{P}$ such that**

$$\| \mathcal{K} - \mathcal{P}(\rho(\mathbf{A}\mathcal{K})) \| < \mathcal{O}\left(\frac{\omega(\mathcal{K})}{\sqrt{m}}\right) = \mathcal{O}(\delta^3)$$

Plan & Vershynin, 2013; Giryes, Sapiro, B, 2015

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Plan & Vershynin, 2013; Giryes, Sapiro, B, 2015

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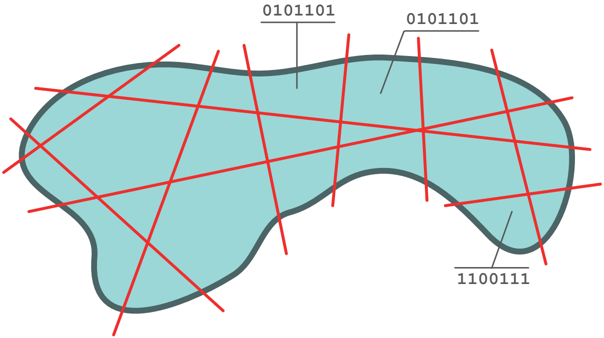
- After N layers the error grows as $\mathcal{O}(N\delta^3)$
- DNNs keep important information of the data
- Input can be recovered from output if output dimension m is big enough

Plan & Vershynin, 2013; Giryes, Sapiro, B, 2015

Inverting a CNN

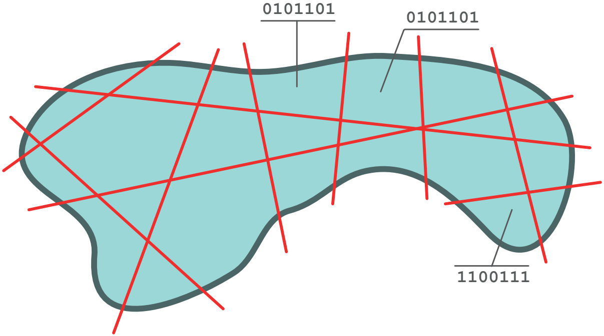
Mahendran & Vedaldi, 2015

Similarity-preserving hashing



Giryes, Sapiro, B, 2015

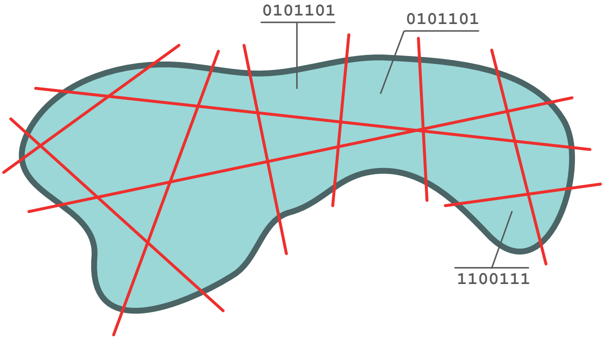
Similarity-preserving hashing



- Single layer = **locality sensitive hashing** **(LSH)**

Giryes, Sapiro, B, 2015

Similarity-preserving hashing



- Single layer = **locality sensitive hashing (LSH)**
- Random weights perform well universally

Giryes, Sapiro, B, 2015

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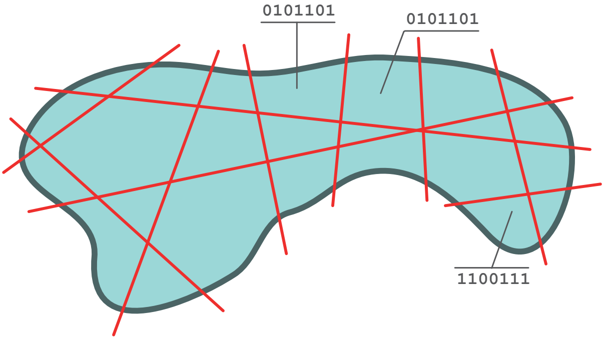
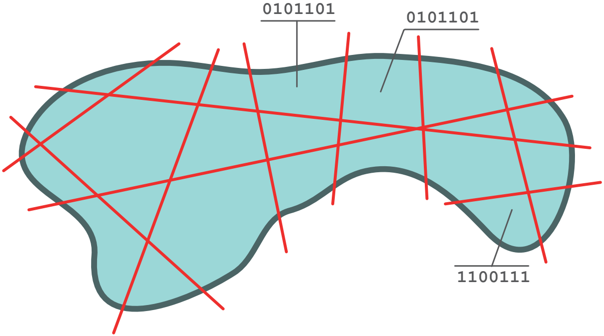


- Single layer = **locality sensitive hashing** **(LSH)**
- Random weights perform well universally
- Can be tuned to specific data by training

Giryes, Sapiro, B, 2015

Similarity-preserving hashing



For deep networks, number of cells in the tessellation is exponentially greater than the number of degrees of freedom

Montúfar *et al.*, 2014

Image retrieval with similarity-preserving hashing

60K images from 10 different classes taken from Tiny images
Represented using 384-dimensional GIST descriptor
Training: 200 images per class; Testing: 59K images

Method / *m*		**12**	**24**	**48**
	Raw		19.16	
	DiffHash	14.72	13.35	12.85
	SSH	15.42	16.75	17.06
	AGH	15.46	15.29	15.15
	KSH	25.79	29.01	30.84
NN	**1 layer**	31.48	35.41	36.79
	2 layer	**45.42**	**49.88**	**50.46**

Performance (mAP in %)

Data: Torralba et al. 2008, Krizhevsky 2009; Methods: Strecha et al. 2011 (diff-hash); Shakhnarovich 2005 (SSH); Liu et al. 2011 (AGH); Liu et al. 2012 (KSH); Masci, B^2, Schmidhuber 2012 (NN)

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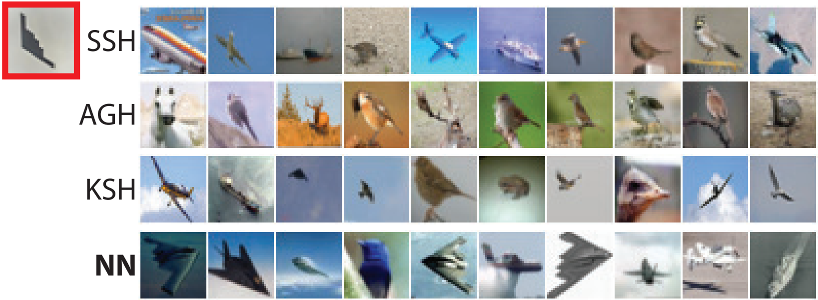

Ranking using 48-bit hashes

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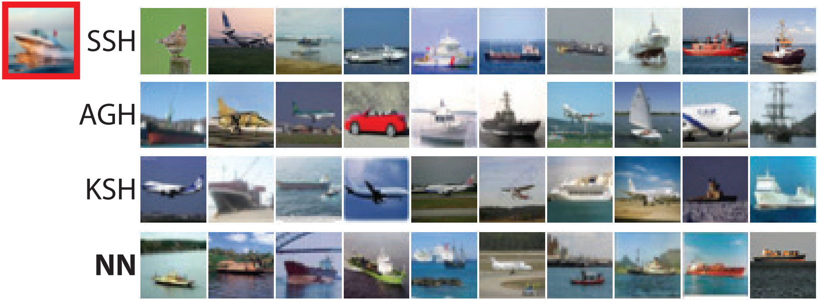

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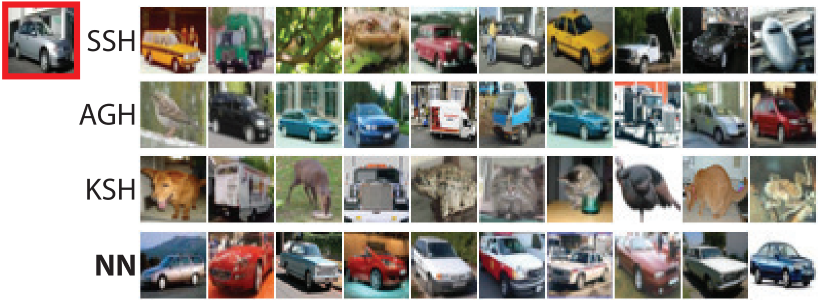

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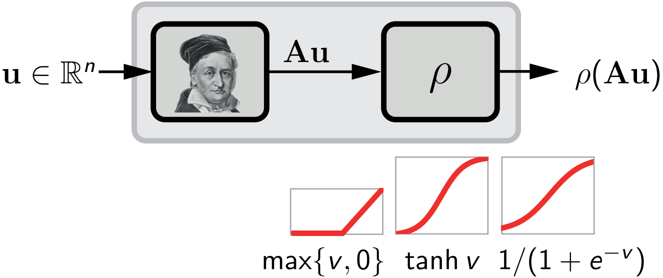
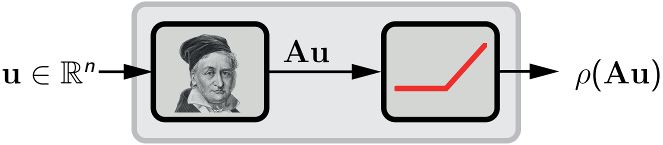
Assumptions



- Fully-connected linear layers with random Gaussian weights
- Element-wise approximately truncated linear activation
- No pooling
- Low dimensional input data

Assumptions



- Fully-connected linear layers with random Gaussian weights
- ReLU activation
- No pooling
- Low dimensional input data

Angle distortion

Theorem (concentration of output angle): **for** $\mathbf{u}, \mathbf{v} \in \mathcal{K}$

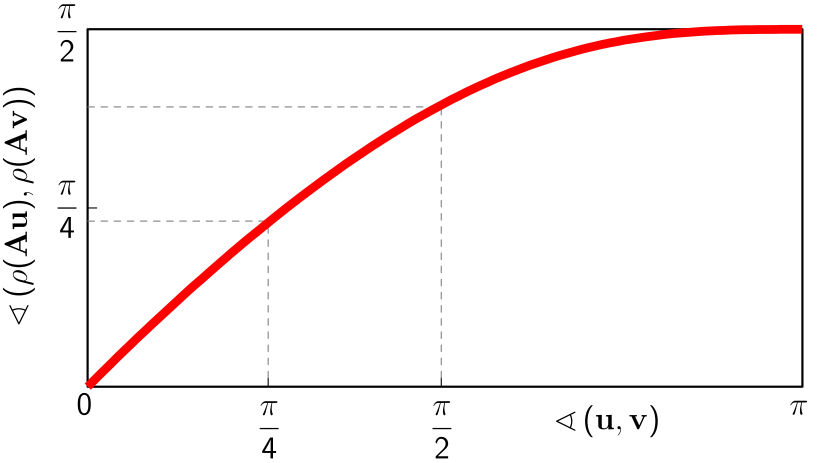
$$\cos \sphericalangle (\rho(\mathbf{A}\mathbf{u}), \rho(\mathbf{A}\mathbf{v})) \;\approx\; \cos \sphericalangle (\mathbf{u}, \mathbf{v}) + \psi(\sphericalangle (\mathbf{u}, \mathbf{v}))$$

where $\psi(\alpha) = \frac{1}{\pi}(\sin \alpha - \alpha \cos \alpha)$

Giryes, Sapiro, B, 2015

Angle distortion

$$\cos\sphericalangle(\rho(\mathbf{Au}),\rho(\mathbf{Av})) \approx \cos\sphericalangle(\mathbf{u},\mathbf{v}) + \psi(\sphericalangle(\mathbf{u},\mathbf{v}))$$



Giryes, Sapiro, B, 2015

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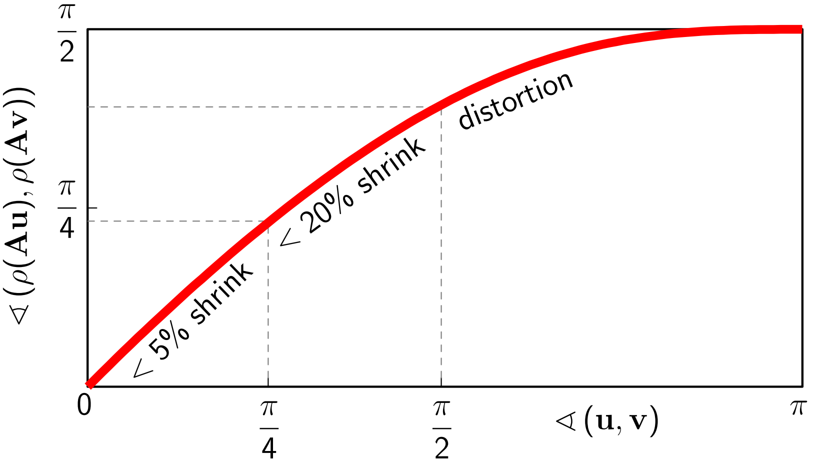


Giryes, Sapiro, B, 2015

Distance distortion

Theorem (concentration of output distance):

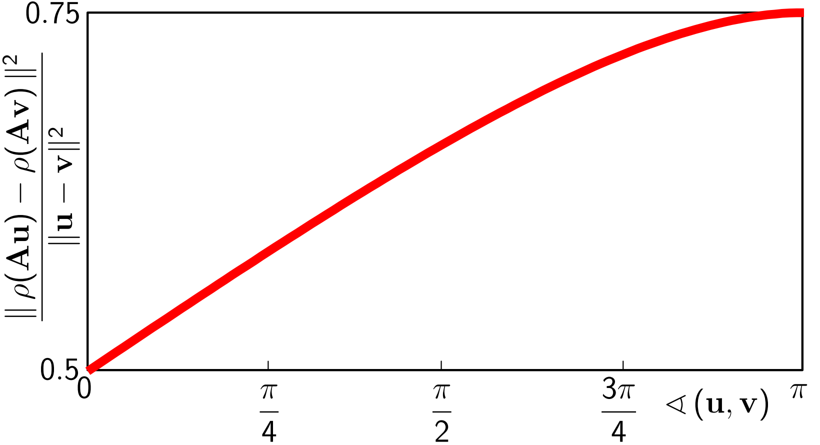
$$\| \rho(\mathbf{Au}) - \rho(\mathbf{Av}) \|^2 \approx \frac{1}{2}\|\mathbf{u} - \mathbf{v}\|^2 + \|\mathbf{u}\|\|\mathbf{v}\| \, \psi(\sphericalangle(\mathbf{u}, \mathbf{v}))$$

for $\mathbf{u}, \mathbf{v} \in \mathcal{K}$**, where** $\psi(\alpha) = \frac{1}{\pi}(\sin\alpha - \alpha\cos\alpha)$

Giryes, Sapiro, B, 2015

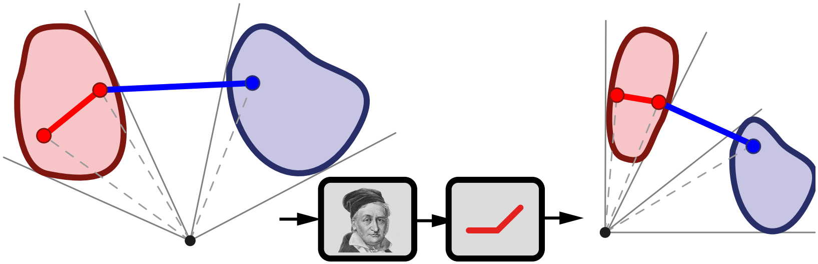
Distance distortion

$$\|\,\rho(\mathbf{Au})-\rho(\mathbf{Av})\,\|^2 \;\approx\; \frac{1}{2}\|\mathbf{u}-\mathbf{v}\|^2+\|\mathbf{u}\|\|\mathbf{v}\|\,\psi(\sphericalangle\,(\mathbf{u},\mathbf{v}))$$



Giryes, Sapiro, B, 2015

Angle and distance distortion



Giryes, Sapiro, B, 2015

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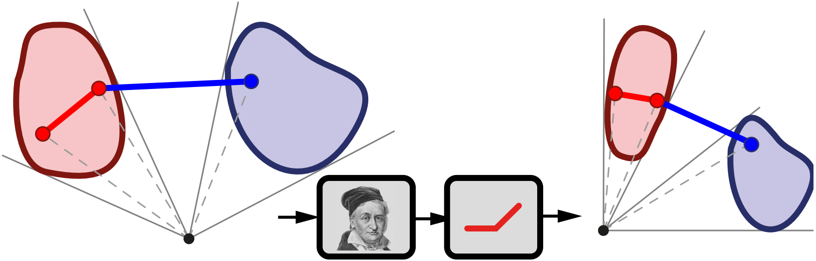
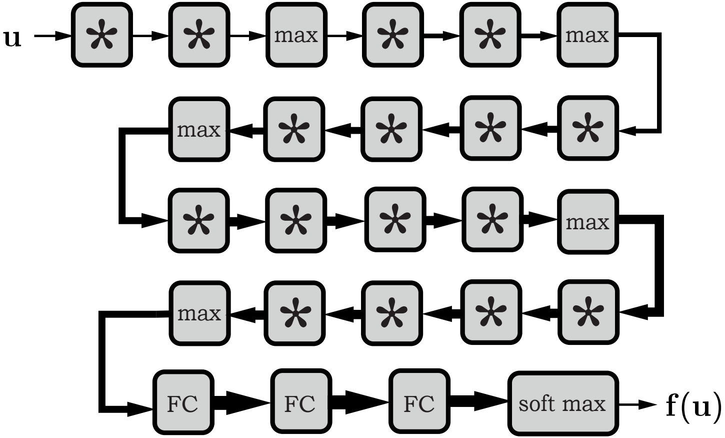


- Points with small angles between them become closer than points with large angles between them

Giryes, Sapiro, B, 2015

Inside a real network

State-of-the-art 19-layer CNN trained on ImageNet



Simonyan & Zisserman, 2014

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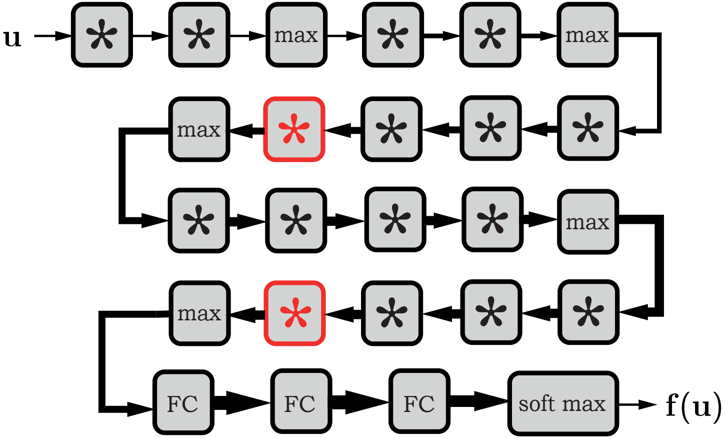
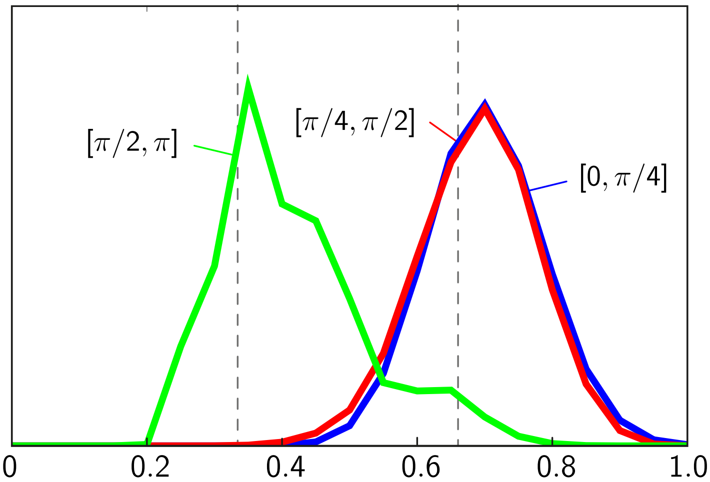


Simonyan & Zisserman, 2014

Angle distortion at 8-th layer

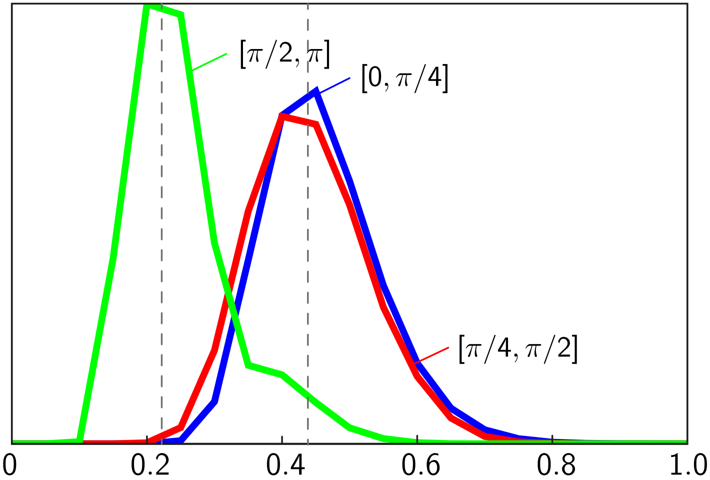
Distribution of $\sphericalangle\left(\mathbf{f}(\mathbf{u}), \mathbf{f}(\mathbf{v})\right) / \sphericalangle(\mathbf{u}, \mathbf{v})$



Giryes, Sapiro, B, 2015

Angle distortion at 16-th layer

Distribution of $\sphericalangle\left(\mathbf{f(u)},\mathbf{f(v)}\right)/\sphericalangle(\mathbf{u},\mathbf{v})$



Giryes, Sapiro, B, 2015

Training set size

- DNNs are stable: close points in the input are close in the output

Giryes, Sapiro, B, 2015

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- Network performing well on an ϵ-net $\mathcal{K}_\epsilon$ in $\mathcal{K}$ performs well on entire $\mathcal{K}$

Giryes, Sapiro, B, 2015

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- **Sudakov's minoration:**

$$\log |\mathcal{K}_\epsilon| \leq \frac{c\, \omega^2(\mathcal{K})}{\epsilon^2}$$

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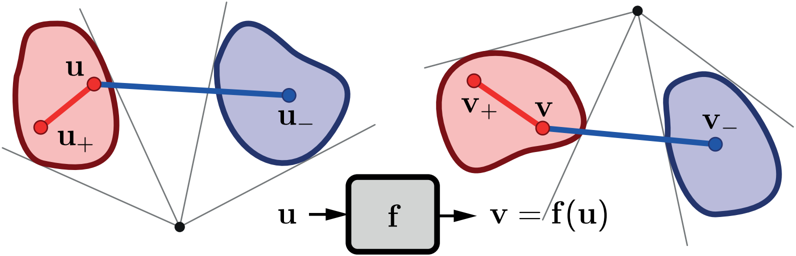
- Not tight! ...but introduces Gaussian mean width $\omega(\mathcal{K})$ as the measure of data complexity
- Situation is much better in practice

Giryes, Sapiro, B, 2015

Random data points

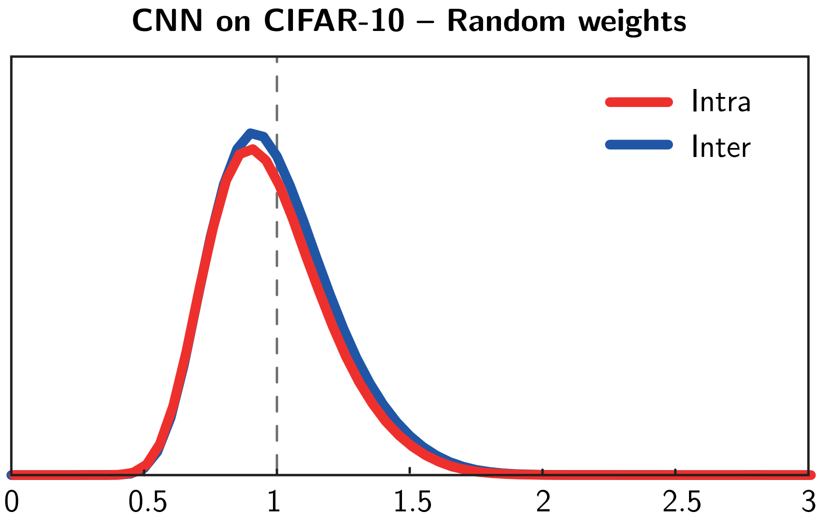


Distance ratios between random triplets $(\mathbf{u}, \mathbf{u}_+, \mathbf{u}_-)$

Intra-class $= \dfrac{\|\mathbf{v}_+ - \mathbf{v}\|}{\|\mathbf{u}_+ - \mathbf{u}\|}$ **Inter-class** $= \dfrac{\|\mathbf{v}_- - \mathbf{v}\|}{\|\mathbf{u}_- - \mathbf{u}\|}$

Giryes, Sapiro, B, 2015

Random data points



Giryes, Sapiro, B, 2015

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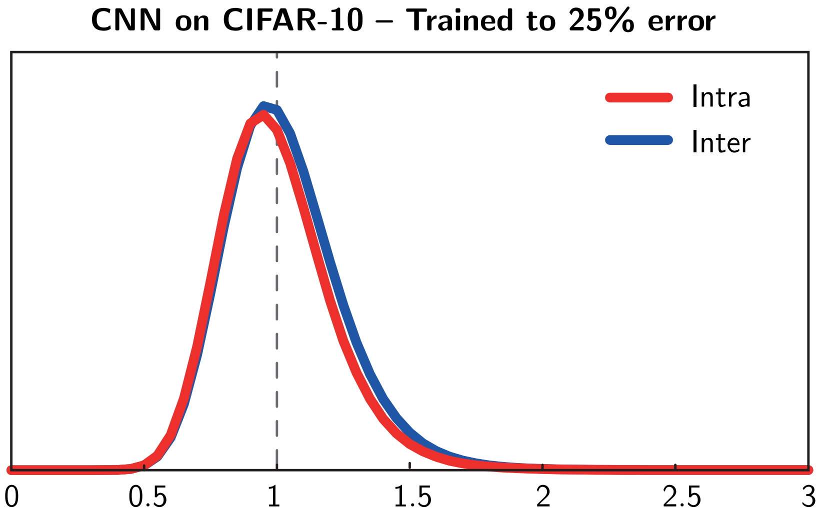
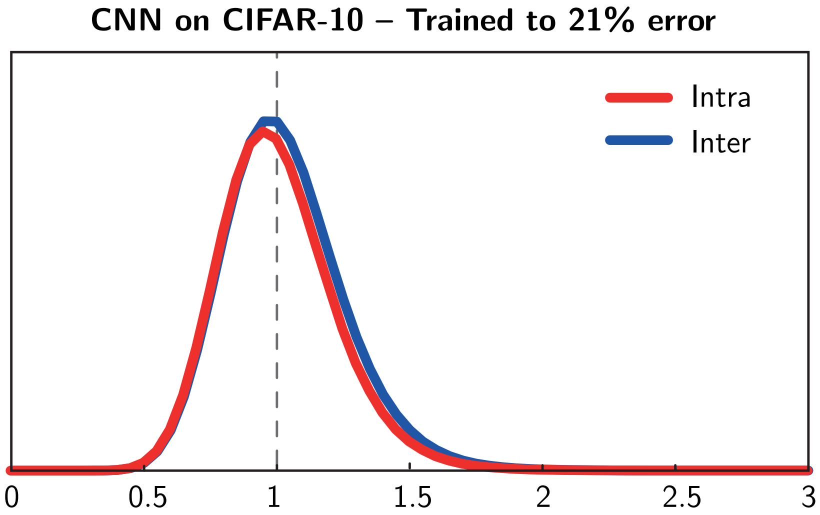


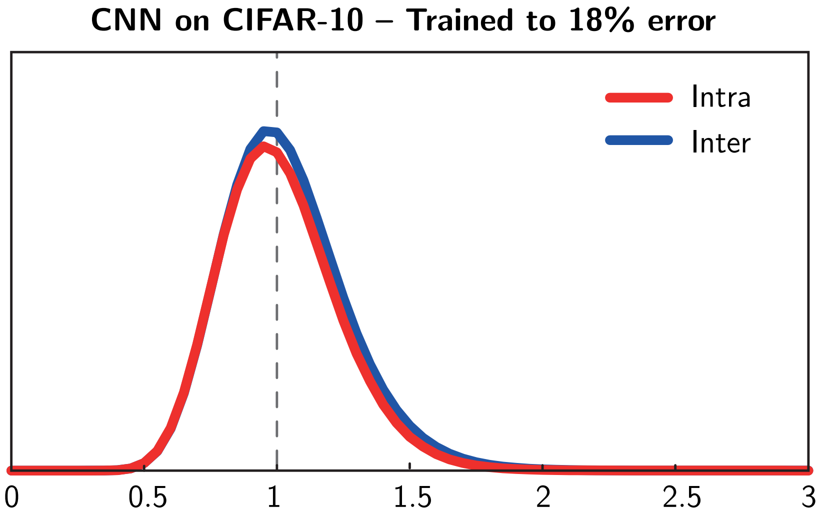
Giryes, Sapiro, B, 2015

Random data points



Giryes, Sapiro, B, 2015

Random data points



Giryes, Sapiro, B, 2015

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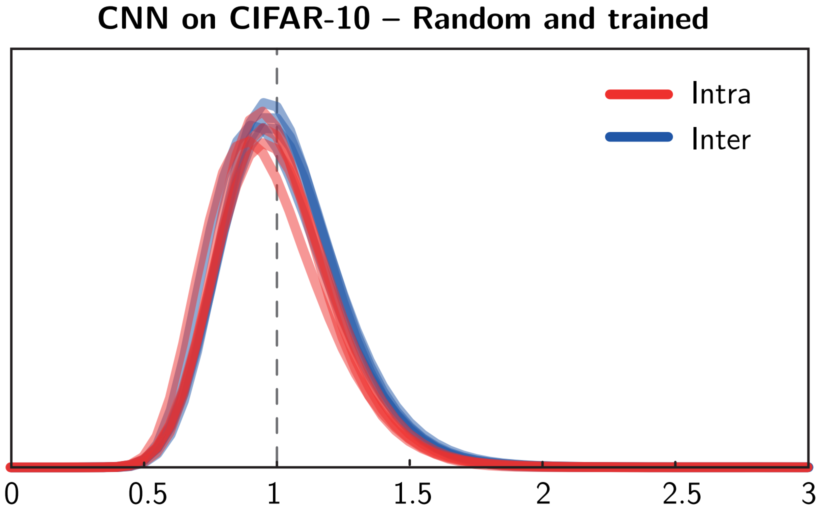
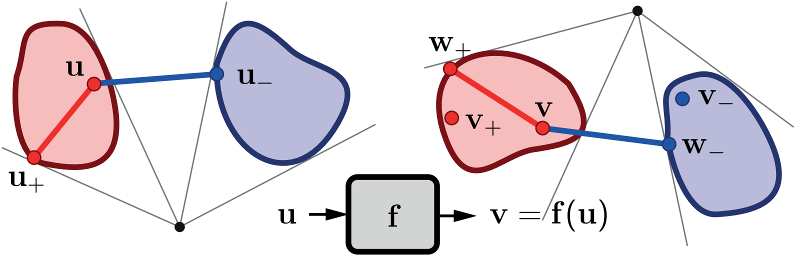


Giryes, Sapiro, B, 2015

Class boundary points

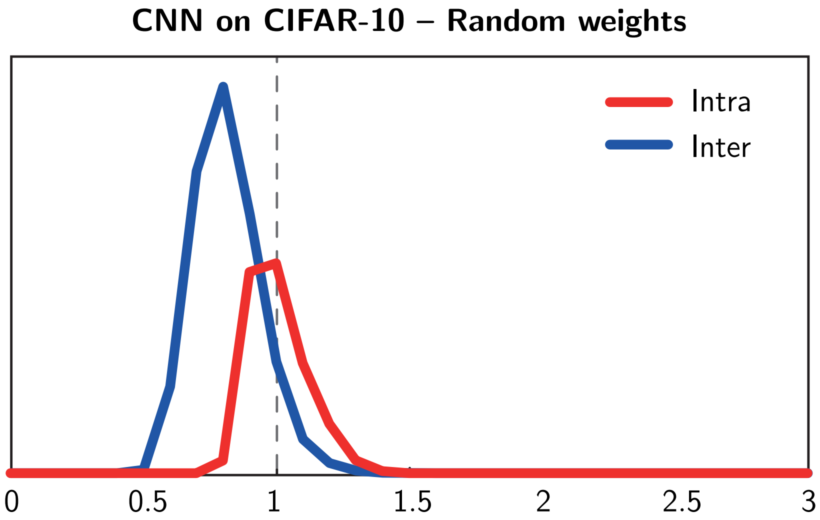


$\mathbf{u}$ random, $\mathbf{u}_+$ farthest in class, $\mathbf{u}_-$ closest not in class

$$\textbf{Intra-class} = \frac{\|\mathbf{w}_+ - \mathbf{v}\|}{\|\mathbf{u}_+ - \mathbf{u}\|} \qquad \textbf{Inter-class} = \frac{\|\mathbf{w}_- - \mathbf{v}\|}{\|\mathbf{u}_- - \mathbf{u}\|}$$

Giryes, Sapiro, B, 2015

Class boundary points



Giryes, Sapiro, B, 2015

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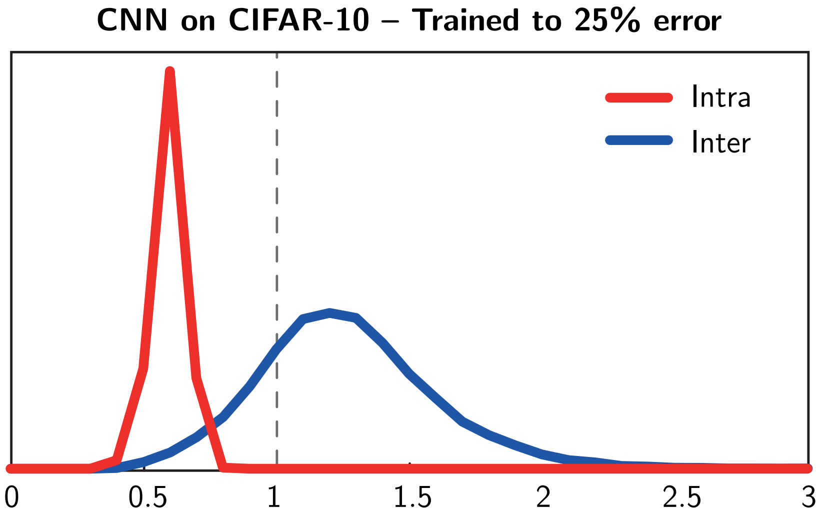

Giryes, Sapiro, B, 2015

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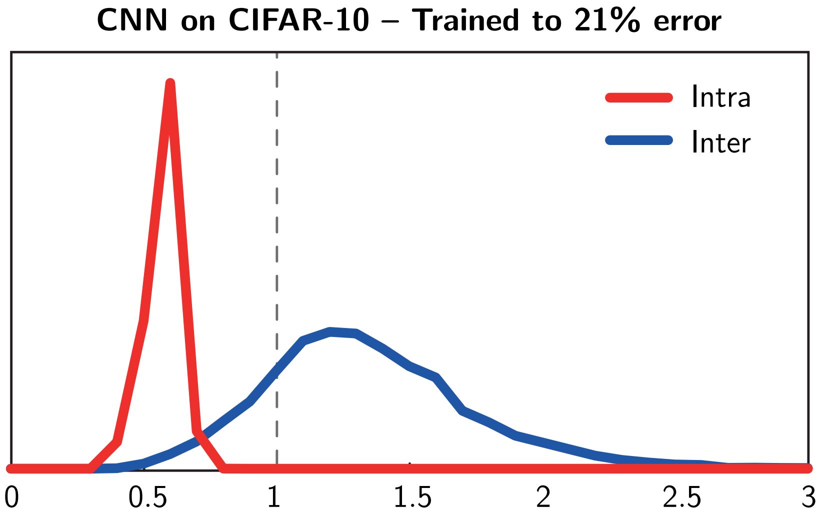

Giryes, Sapiro, B, 2015

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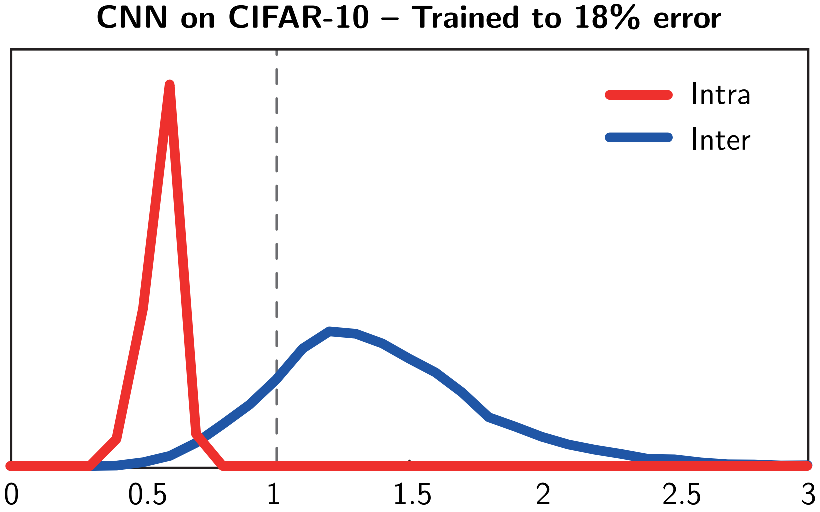

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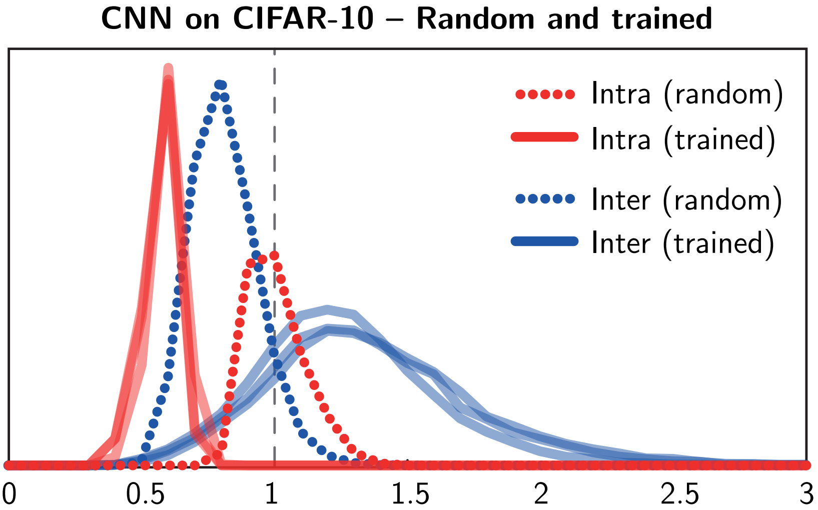

Giryes, Sapiro, B, 2015

Role of training

- Negligible effect on random data points

Giryes, Sapiro, B, 2015

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Giryes, Sapiro, B, 2015

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  Intra-class distances shrink

Giryes, Sapiro, B, 2015

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- Random weights perform well universally
- Major effect on class boundary points

  Intra-class distances shrink

  Inter-class distances grow

Giryes, Sapiro, B, 2015

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- Major effect on class boundary points

  Intra-class distances shrink

  Inter-class distances grow
- Only a small subset of $\mathcal{K}_\epsilon$ is required for training

Giryes, Sapiro, B, 2015

DNNs as metric learners

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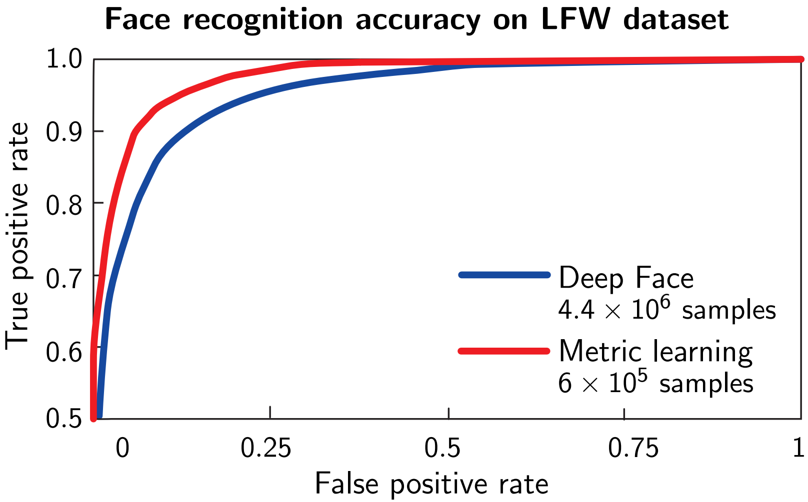
- Inject metric learning criterion into training objective to reduce the amount of labeled data

DNNs as metric learners



Huang *et al.*

Compressed discrimination

Compressed sensing: **reconstruct signal $\mathbf{x} \in \mathbb{R}^n$ given $m \ll n$ measurements**

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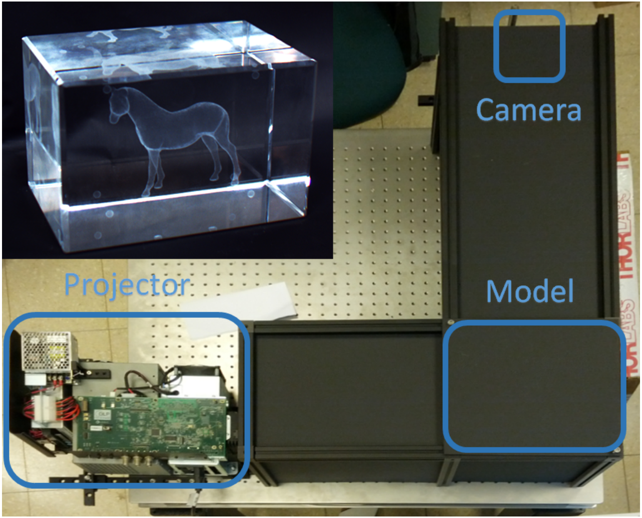
m' is insufficient to reconstruct the signal!

Compressed scattering tomography



Menashe & B, 2013

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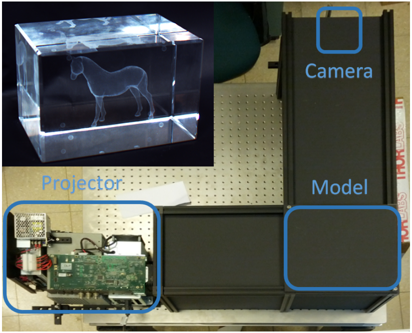

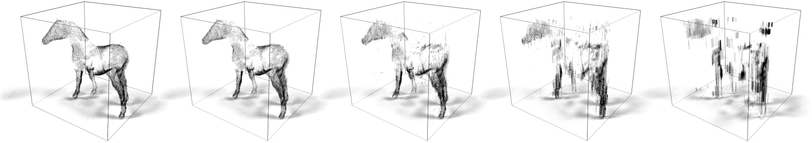

$m : n = 1 : 2$ $\quad 1 : 4 \quad 1 : 8 \quad 1 : 16 \quad 1 : 32$

Menashe & B, 2013

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- Deep learning can be viewed as metric learning