Vortex sheet motion following curvature singularity formation

D. I. Pullin[1] and Naijian Shen[2]

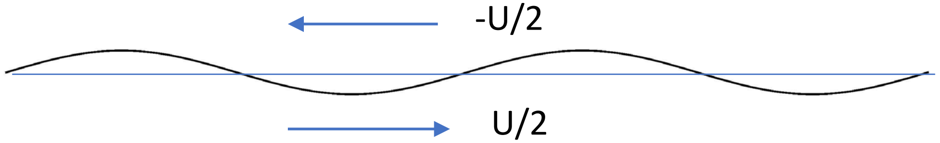
1: California Institute of Technology
2: Massachusetts Institute of Technology

International Conference on Multi-Scale Modeling and Simulation based on Physics and Data: RUSS2024

Happy 70th birthday Russel

Do vortex sheets roll up (Birkhoff & Fisher, 1959)?

- Spatially periodic vortex sheet
- Helmholtz (1868) : discussed prototype of vortex-sheet instability
- Thompson (1871): quantitative theory of Kelvin-Helholtz instability



k= 2π/λ

- Growth of initial disturbance ε proportional to ε *exp[kUt] : ill posed?*
- Rosehhead (1931): nonlinear calculation with 12 vortices per wavelength

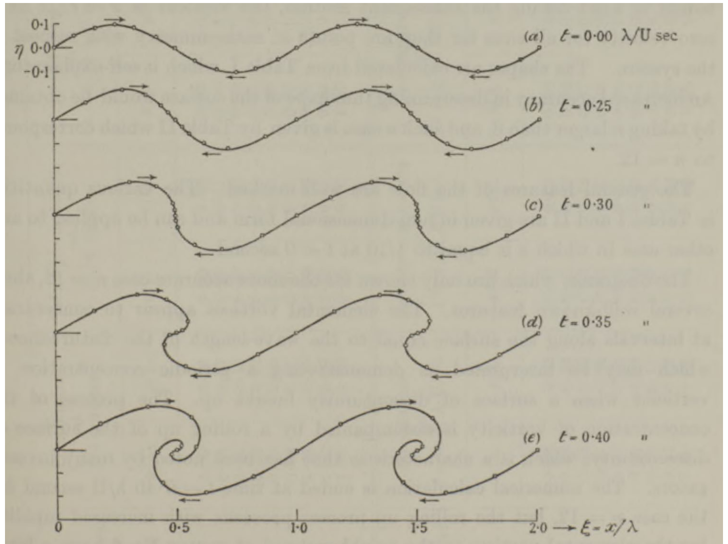


FIG. 4.

Do vortex sheets roll up?

- Birkhoff (1962) used N = 20 vortices and found chaotic motion. Verified by Krasny (1986) and others.
- Vortex sheet evolution governed by Birkhoff-Rott equation (periodic 2π)

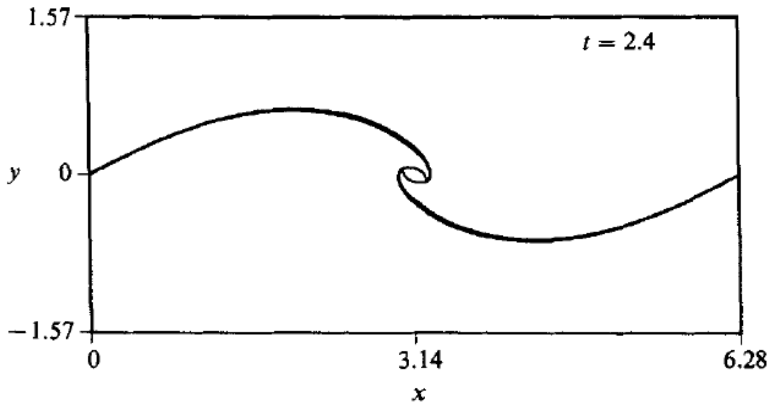
$$\frac{\partial \bar{z}(\Gamma,t)}{\partial t} = \frac{1}{4\pi i}\int_{-\pi}^{\pi} \cot\left(\frac{z(\Gamma,t)-z(\hat{\Gamma},t)}{2}\right) d\hat{\Gamma},$$

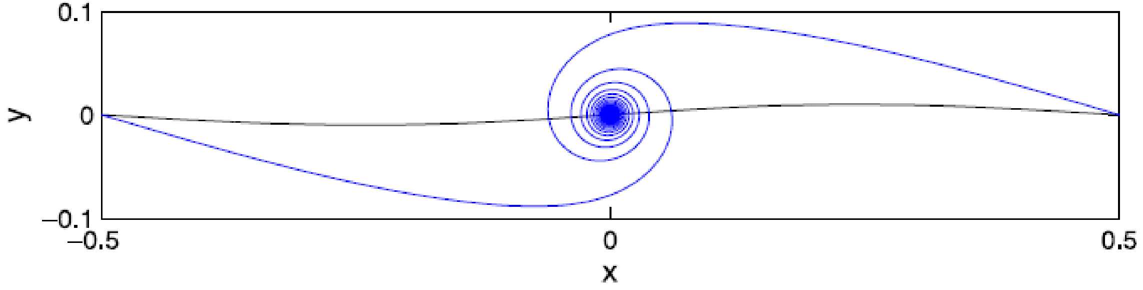
- Moore (1979) resolved issue by showing that periodic vortex-sheet evolution produced a sheet-shape, curvature singularity in a finite time t_c = -log(1/ε). All higher derivatives singular.
- Verified by Meiron et al. (1982), Krasny (1986), Shelly (1990)
  - Point vortex method converges with increasing N up to $t = t_c$
  - Singularity formation can be interpreted as movement of singularity in the complex time plane onto the real axis at $t = t_c$. (Cowley, Baker and Tanveer 1990)
- Continuation beyond $t = t_c$ for Birkoff-Rott is an open question ?????????????
- We consider $t > t_c$ but with $t-t_c \ll 1$. Small-time continuation!

Limit δ ⟶ 0

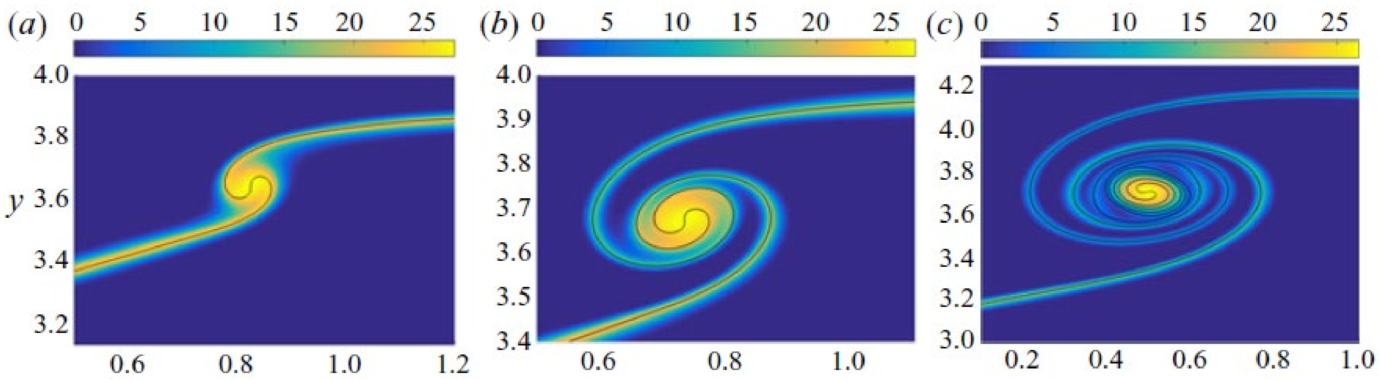
Baker & Shelly (JFM, 1990)
Contour dynamics δ = 0.025

Sohn (PoF, 2016)
Krasny regularization δ = 0.02



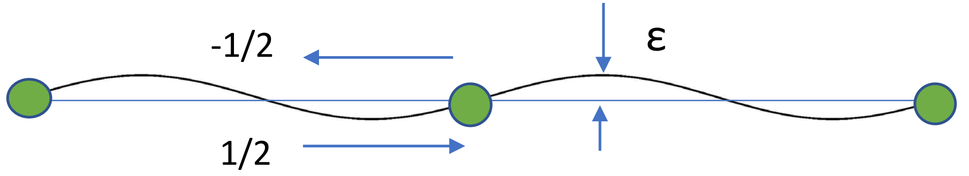


Caflisch *et al.* (JFM, 2022). Navier Stokes, Re = $2*10^4$, $\delta = O(Re^{-1/2})$



Moore singularity (Proc. R. Soc.,1979)

- Periodic vortex sheet; Birkhoff-Rott equation



Γ is the circulation: Lagrangian marker

$$\frac{\partial \bar{z}(\Gamma,t)}{\partial t} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\hat{\Gamma}}{z(\Gamma,t) - z(\hat{\Gamma},t)}, \qquad z(\Gamma,t) = x(\Gamma,t) + iy(\Gamma,t)$$

$$z(\Gamma + 2\pi, t) = 2\pi + z(\Gamma,t), \qquad z(-\Gamma,t) = -z(\Gamma,t).$$

- Initial perturbation: $z(\Gamma,0) = \Gamma + e^{i\phi}\epsilon\sin(\Gamma)$.

- Approximate solution using Fourier sine series (Moore 1979) $\lambda_1 = \sin\phi - \cos\phi,$ $\lambda_1 \neq 0$

$$z(\Gamma,t) = \Gamma + 2i\sum_{n=1}^{\infty}\mathcal{A}_n(t)\sin(n\Gamma) \simeq \Gamma + 2i\sum_{n=1}^{\infty}\mathcal{A}_{n,0}(t)\epsilon^n\sin(n\Gamma).$$

$$\mathcal{A}_n(t(s)) \sim \frac{(1+i)[\mathrm{sign}(\lambda_1)]^n}{\sqrt{2\pi}n^{5/2}s}\exp\left\{n\left[1+\frac{s}{2}+\ln\left(\frac{|\lambda_1|\epsilon s}{4}\right)\right]\right\} \qquad s = t + \frac{\alpha_1}{t} + O\left(t^{-2}\right)$$

- Fourier series loses exponential convergence at a critical time: curvature singularity

$$t_c = 2W_0\left(\frac{2}{e|\lambda_1|\epsilon}\right), \qquad \lambda_1 = 1$$

Lambert W (Product-log) function

Moore solution up to $t = t_c$

- Summed Fourier sine series (polylog functions) gives asymptotic sheet shape up to $t = t_c$

$$z(\Gamma, t) \simeq \Gamma - \frac{(1+\mathrm{i})\left[\mathrm{Li}_{5/2}\left(\frac{\operatorname{sign}(\lambda_1)t|\lambda_1|\epsilon}{4}\mathrm{e}^{-\mathrm{i}\Gamma+1+(t/2)}\right) - \mathrm{Li}_{5/2}\left(\frac{\operatorname{sign}(\lambda_1)t|\lambda_1|\epsilon}{4}\mathrm{e}^{\mathrm{i}\Gamma+1+(t/2)}\right)\right]}{\sqrt{2\pi t}},$$

$$\mathrm{Li}_n(s) \equiv \sum_{p=1}^{\infty} s^p/p^n$$

$$z(\Gamma, t_c) \simeq \Gamma - \frac{(1+\mathrm{i})\left[\mathrm{Li}_{5/2}\left(\operatorname{sign}(\lambda_1)\mathrm{e}^{-\mathrm{i}\Gamma}\right) - \mathrm{Li}_{5/2}\left(\operatorname{sign}(\lambda_1)\mathrm{e}^{\mathrm{i}\Gamma}\right)\right]}{\sqrt{2\pi t_c}}$$

- Expansion near $0 \leq \Gamma \ll 1$

$$z_c(\Gamma) \sim b\left(\Gamma + A\Gamma^{3/2}\right) + O\left(\Gamma^2\right)$$

$$b(\lambda_1\epsilon) = 1 - \frac{(1-\mathrm{i})\,\zeta\left(\frac{3}{2}\right)}{\sqrt{2\pi}\,W_0\left(\frac{2}{\mathrm{e}\,\lambda_1\epsilon}\right)}, \quad A(\lambda_1\epsilon) = \frac{1}{\frac{3}{4}(1+\mathrm{i})\,W_0\left(\frac{2}{\mathrm{e}\,\lambda_1\epsilon}\right) - \frac{3\zeta\left(\frac{3}{2}\right)}{2\sqrt{2\pi}}}$$

- For ϵ << 1

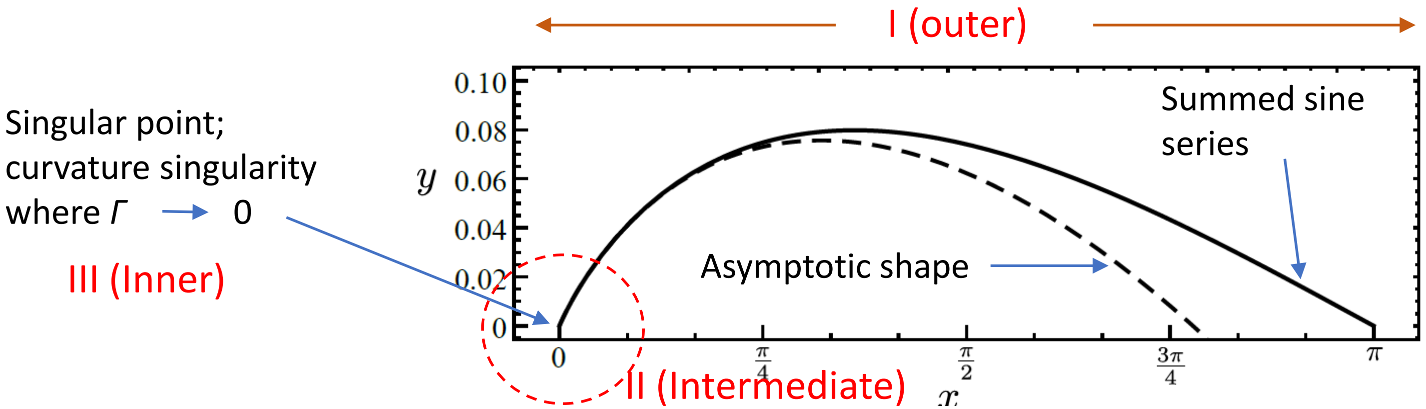
$$b = 1 + O(\hat{\varepsilon})$$
$$A = \frac{2}{3}(1-\mathrm{i})\hat{\varepsilon} + O(\hat{\varepsilon})^2$$
$$\hat{\varepsilon} = \frac{1}{W_0\left(\frac{2}{\mathrm{e}\lambda_1\epsilon}\right)}$$

Moore solution up to $t = t_c$

- Summed Fourier sine series (polylog functions) gives asymptotic sheet shape a $t = t_c$

$$z_c(\Gamma) \sim b\left(\Gamma + A\Gamma^{3/2}\right) + O\left(\Gamma^2\right)$$

Constants (b(ε),A(ε)) known



$$b = 1 + O(\hat{\varepsilon})$$

$$A = \frac{2}{3}(1 - \mathrm{i})\hat{\varepsilon} + O(\hat{\varepsilon})^2$$

$$\hat{\varepsilon} = \frac{1}{W_0\left(\frac{2}{e\lambda_1\epsilon}\right)}$$

- At $t = t_c$:
  - Sheet curvature, angular velocity, acceleration and higher derivatives all singular

Region II solution:

- Use extended Taylor series for $t>t_c$ that fits initial condition: C_0 = A at $t = t_c$

$$z_{\mathrm{II}}(\Gamma,\tau) = b\left(\Gamma + \Gamma^{3/2}\sum_{n=0}^{\infty} C_n\left(\frac{\tau}{\Gamma}\right)^n\right), \qquad \tau = t - t_c$$

- Constants C_n, n > 0 need to be determined

Intermediate solution (Region II)

- Taylor series in $\tau = t - t_c$

$$z_{II}(\Gamma,\tau) = b\left(\Gamma + \Gamma^{3/2}\sum_{n=0}^{\infty} C_n\left(\frac{\tau}{\Gamma}\right)^n\right),$$

- Coefficients C_n determined by iterative differentiation of the BR equation

$$\left.\frac{\partial^n \overline{z}}{\partial \tau^n}\right|_{\tau=0} = -\frac{1}{2\pi i}\int_0^{\infty}\left(\frac{\frac{\partial^{n-1} z}{\partial \tau^{n-1}}|_{\tau=0} - \frac{\partial^{n-1}\hat{z}}{\partial \tau^{n-1}}|_{\tau=0}}{\left(z(\Gamma,0)-z(\hat{\Gamma},0)\right)^2} + \frac{\frac{\partial^{n-1} z}{\partial \tau^{n-1}}|_{\tau=0} + \frac{\partial^{n-1}\hat{z}}{\partial \tau^{n-1}}|_{\tau=0}}{\left(z(\Gamma,0)+z(\hat{\Gamma},0)\right)^2}\right) d\hat{\Gamma}.$$

- Gives recursion relation for complex constants C_n (using asymptotic IC)

$$\overline{C}_n = -\frac{1}{2\pi i\,|b|^2}\frac{I(n)}{n}C_{n-1}, \qquad I(n) = \int_0^{\infty}\left(\frac{1-x^{5/2-n}}{(1-x)^2} + \frac{1+x^{5/2-n}}{(1+x)^2}\right) dx$$

- Recursion relation can be solved giving explicit C_n and series solution obtained.

$$z(\Gamma,\tau) = b\left(\Gamma + \Gamma^{3/2}\left(A\,S_1 + i\,\overline{A}\,S_2\right)\right),$$

$$S_1 = \sum_{n=0}^{\infty}\hat{K}_{2n}\eta^{-2n}, \quad S_2 = \sum_{n=0}^{\infty}\hat{K}_{2n+1}\eta^{-(2n+1)}$$

$$\hat{K}_n = \frac{3\,i^n\,2^{-(n+2)}\,\hat{\Gamma}(-\frac{3}{2}+n)}{\pi^{1/2}\,\hat{\Gamma}(1+n)}, \quad \hat{K}_n = \frac{3\,i^{n+1}\,2^{-(n+2)}\,\hat{\Gamma}(-\frac{3}{2}+n)}{\pi^{1/2}\,\hat{\Gamma}(1+n)}$$

$\eta = \frac{\Gamma\,|b|^2}{\tau}$: similarity variable

Convergent for:

$$\begin{cases} \eta > 1/2 \\ \Gamma > \tau/(2\,|b|^2) \end{cases}$$

- All $\frac{\partial^n z}{\partial \tau^n}(\Gamma \to 0, \tau = 0) \sim \Gamma^{3/2-n}$ and are singular for $n \geq 2$

Intermediate solution (Region II)

- Series can be summed to give a closed form solution (+ algebra)
- Can be analytically continued to $\eta \to 0$ ($\Gamma \to 0$).

$$z_{\rm II}(\eta,\tau) = \frac{\tau}{|b|}\,\Omega_{\rm II}(\eta,\tau) \qquad \Omega_{\rm II}(\eta,\tau) = e^{i\,\theta}\left(w_0(\eta) + \frac{\tau^{\frac{1}{2}}}{|b|}\,w_1(\eta)\right),$$

$$w_0(\eta) = \eta, \qquad w_1(\eta) = A\,Q_1(\eta) + i\,\overline{A}\,Q_2(\eta),$$

$$Q_1(\eta) \equiv \eta^{3/2}\,S_1 = \frac{1}{2\sqrt{2}}\left(4\eta^2+1\right)^{3/4}\cos\left[\tfrac{3}{2}\,\mathrm{ArcCot}\,(2\,\eta)\right],$$

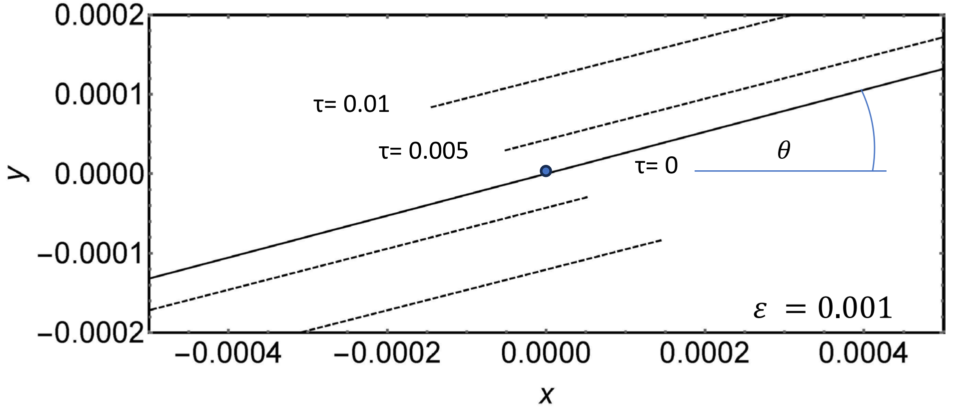
$$Q_2(\eta) \equiv \eta^{3/2}\,S_2 = \frac{1}{2\sqrt{2}}\left(4\eta^2+1\right)^{3/4}\sin\left[\tfrac{3}{2}\,\mathrm{ArcCot}\,(2\,\eta)\right]$$

$$\tau = t - t_c$$

$$\eta = \frac{\Gamma\,|b|^2}{\tau}$$

η is a similarity variable

θ is initial sheet angle at origin



$$z_{II}(\Gamma \to 0^+,\tau) = -\frac{b}{4}\left(\frac{\tau}{|b|^2}\right)^{3/2}(A_r - A_i)(1-i))$$

- Matches analytically-continued Moore solution in region I.
- z_{II} shows jump formation (shock?) at the singularity point
- Not acceptable near singularity
- Unphysical without vortex sheet end rollup

GALCIT

Inner solution (Region III)

$$\tau = t - t_c$$
$$\eta = \frac{\Gamma |b|^2}{\tau}$$
$$\Gamma < O(\tau/(2|b|^2))$$

- Suggests the existence of an inner solution of same form

$$z_{\mathrm{III}}(\eta, t) = \frac{\tau}{|b|}\,\Omega_{\mathrm{III}}(\eta, \tau),$$

$$\Omega_{\mathrm{III}}(\eta, \tau) = \mathrm{e}^{i\,\theta}\left(\omega_0(\eta) + \frac{\tau^{\frac{1}{2}}}{|b|}\,\omega_1(\eta) + \frac{\tau}{|b|^2}\,\omega_2(\eta) + \dots\right)$$

- Form is expansion in powers of $\tau^{\frac{1}{2}}$ Must satisfy BR equation and match intermediate region II solution at $\eta \longrightarrow \infty$
- Substitute into Birkhoff-Rott equation expanded in powers of $\tau^{\frac{1}{2}}$ gives to first order

$$\overline{\omega}_0 - \eta\frac{d\overline{\omega}_0}{d\eta} = \frac{1}{2\pi i}\int_0^{\infty}\left(\frac{1}{\omega_0 - \omega_0'} + \frac{1}{\omega_0 + \omega_0'}\right) d\eta',$$

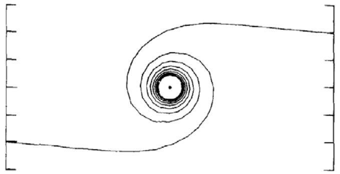
- Zeroth order

$$\frac{3}{2}\,\overline{\omega}_1 - \eta\frac{d\overline{\omega}_1}{d\eta} = -\frac{1}{2\pi i}\int_0^{\infty}\left(\frac{\omega_1 - \omega_1'}{(\omega_0 - \omega_0')^2} + \frac{\omega_1 + \omega_1'}{(\omega_0 + \omega_0')^2}\right) d\eta'$$

- First order

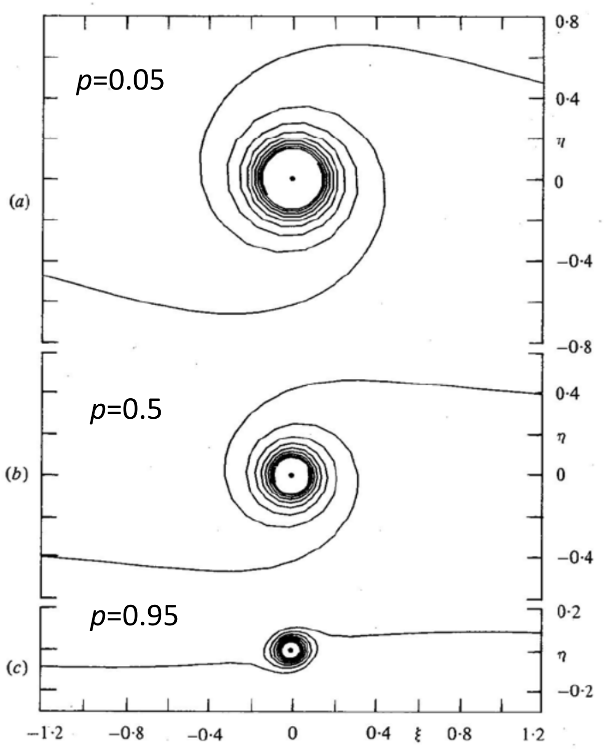
- Solution of zeroth-order equation:

  - $\omega_0 = \eta$. Does not give a solution of first order equation that is <u>bounded at singular point</u>.
  - Double-centered-spiral solution
    Pullin & Phillips (1982)
    Does not exist!

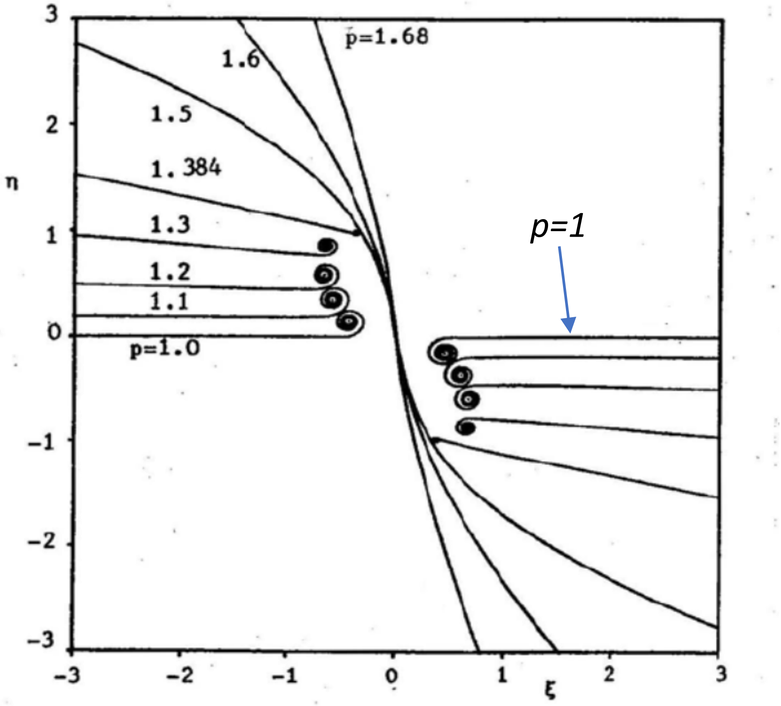


Inner solution (Region III)

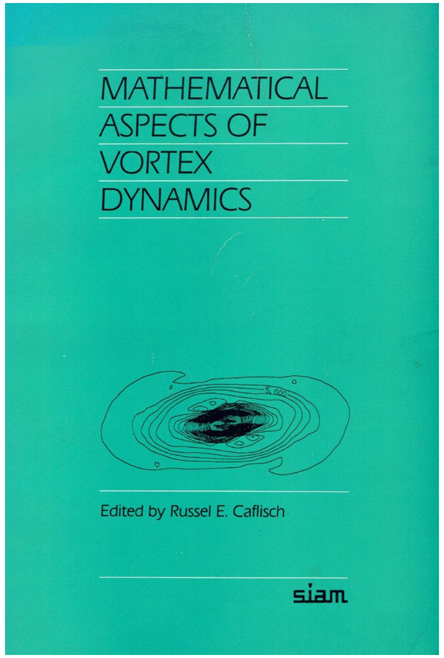
$$\frac{1}{2-p}\left(\overline{\omega}_0 - p\eta\frac{d\overline{\omega}_0}{d\eta}\right) = \frac{1}{2\pi i}\int_0^\infty \left(\frac{1}{\omega_0-\omega_0'}+\frac{1}{\omega_0+\omega_0'}\right) d\eta' , \qquad \omega_0 \to \eta^{1/p}, \quad \eta\to\infty$$



Pullin & Phillips (JFM, 1981)
P =1 solution is flat sheet!



Pullin (Math. Aspects of Vortex Dynamics, 1988)



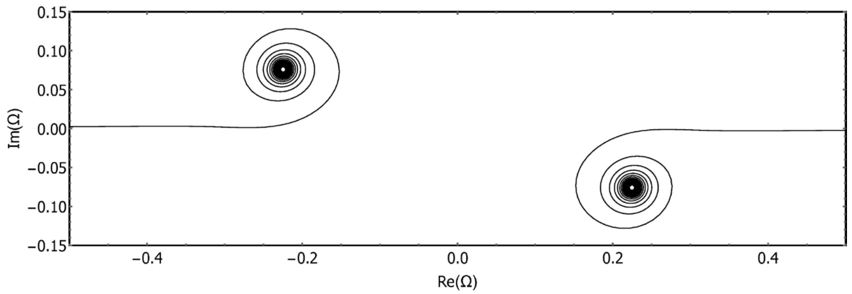
Leesburg Virginia
April 25-27 1988

Inner region III solution

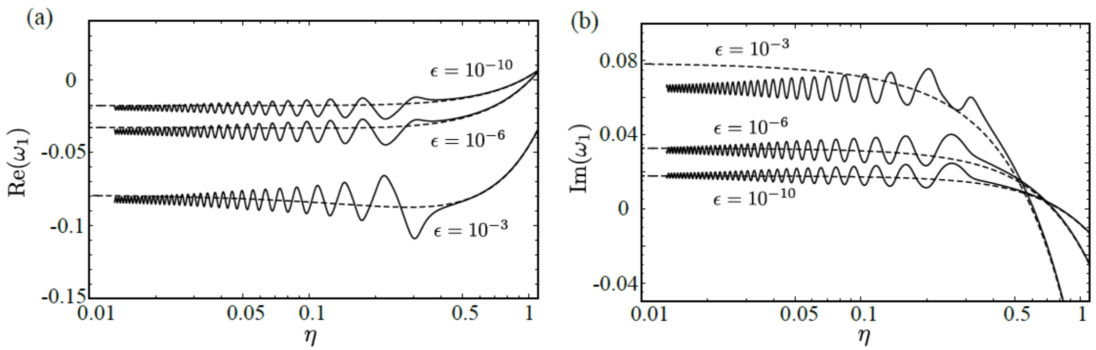
- Zeroth-order: $$\overline{\omega}_0 - \eta \frac{d\overline{\omega}_0}{d\eta} = \frac{1}{2\pi i} \int_0^{\infty} \left(\frac{1}{\omega_0 - \omega_0'} + \frac{1}{\omega_0 + \omega_0'} \right) d\eta'$$

$$\tau = t - t_c$$
$$\eta = \frac{\Gamma |b|^2}{\tau}$$
$$\Gamma < O(\tau/(2|b|^2))$$

- Separated spiral solution can be found numerically (similarity space)



- First order equation:
  Numerical solution: $$\frac{3}{2}\overline{\omega}_1 - \eta \frac{d\overline{\omega}_1}{d\eta} = -\frac{1}{2\pi i} \int_0^{\infty} \left(\frac{\omega_1 - \omega_1'}{(\omega_0 - \omega_0')^2} + \frac{\omega_1 + \omega_1'}{(\omega_0 + \omega_0')^2} \right) d\eta'$$

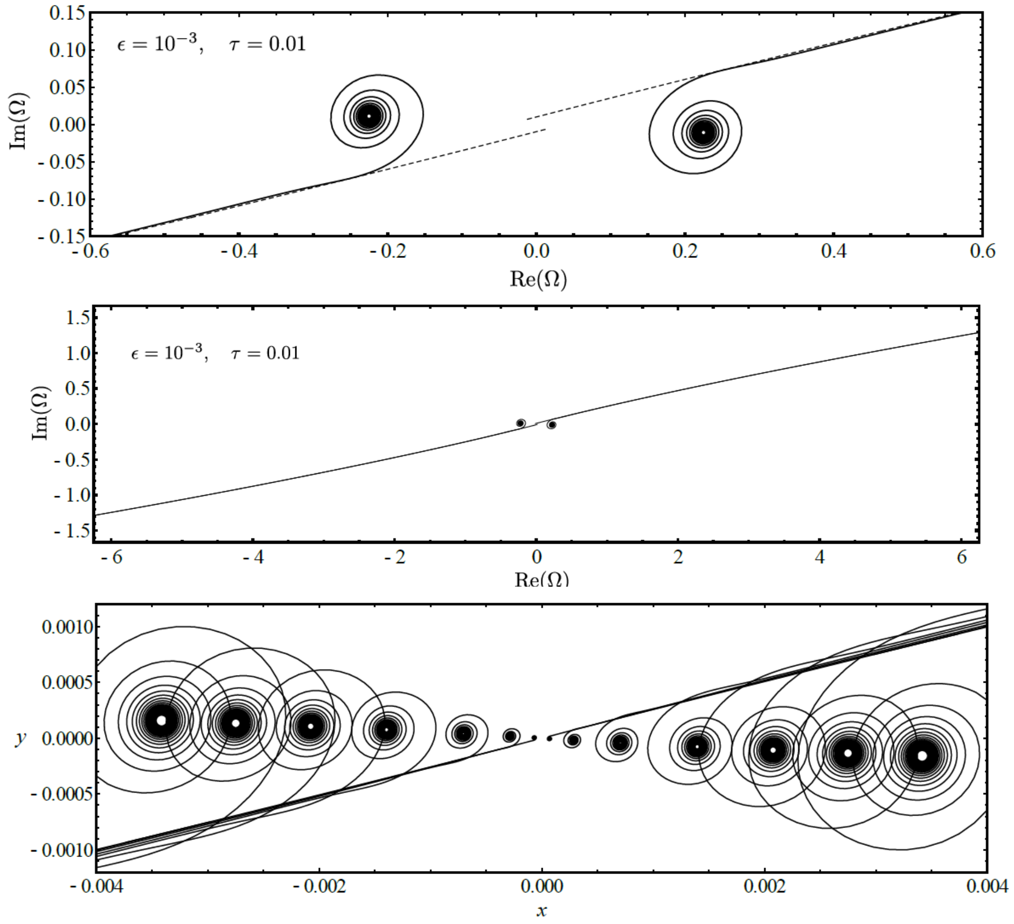


Dependence on ε owing to matching to region II solution with constant A(ε)

- Both zeroth and first order region III solutions are bounded and match intermediate region II solution as $\eta \longrightarrow \infty$

Composite solution

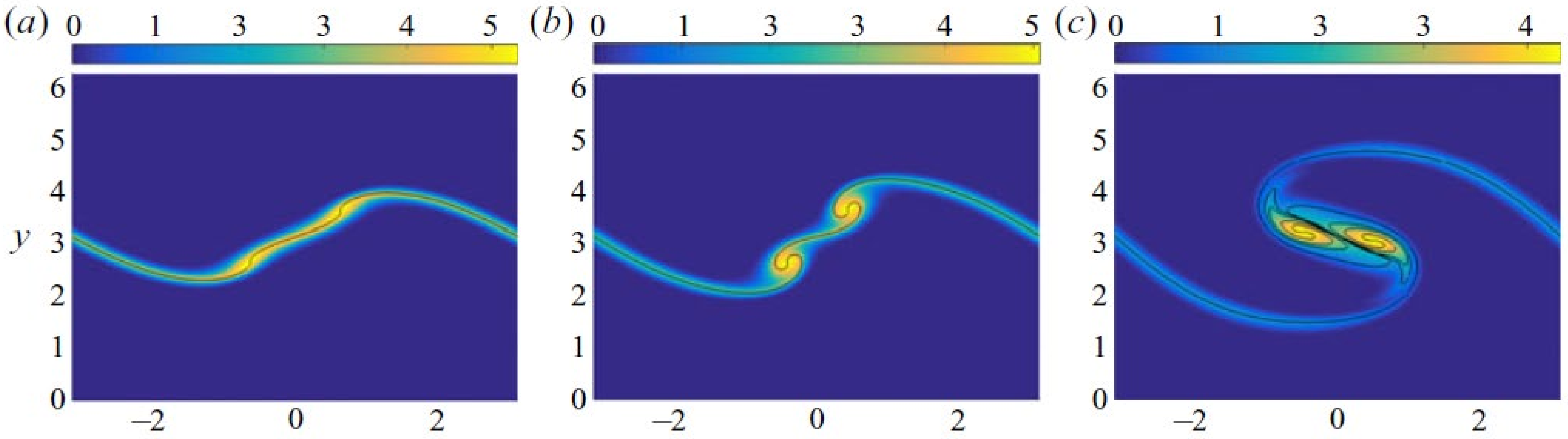
- Use sum of zeroth- and first-order numerical solutions to give timewise evolution:



$\tau = 0.0005, 0.001, 0.0025, 0.005, 0.0075, 0.01, 0.0125$

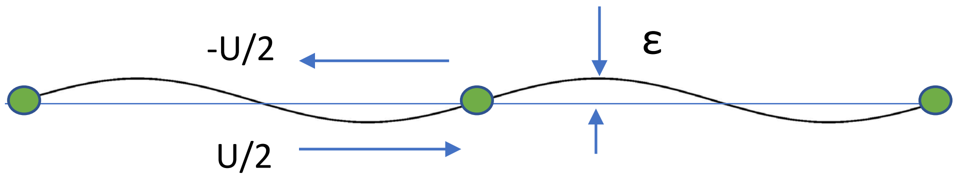
Limit of Navier-Stokes solution??

Caflisch eta l. (JFM, 2022). Navier Stokes, Re = 10^3, $\delta = O(Re^{-1/2})$



Conclusions

- Periodic vortex sheet; Birkhoff-Rott equation. Continuation of motion past $t > t_c$



- Region I : Moore solution gives both t_c and vortex-sheet structure at $t = t_c$. Curvature and all higher derivatives singular. Can be analytically continued for $t > t_c$

- Region II : Taylor series solution for $t > t_c$. Summation gives an intermediate region II solution in terms of expansion in powers of $\tau^{\frac{1}{2}}$ Matches region I Moore solution both analytically and numerically
  - ➢ Solution behaves unphysically at singular point, but form suggests the structure of an inner solution

- Region III : zeroth and first-order solutions found numerically
  - ➢ Zeroth-order solution gives vortex sheet rupture at $t = t_c$ ($\tau = 0$) into two distinct spiral-vortex branches whose centers separate a distance proportional to τ.
  - ➢ First-order solution represents an outer correction
  - ➢ Matches region II solution
  - ➢ Neither ``flat'' sheet nor centered double-spiral solutions are admissible

- Vortex sheet does roll up in KH instability. But in unexpected way

- Uniqueness not established.
  - ➢ Will require demonstration that Navier-Stokes solution with same initial conditions converges to present solution.

- Pullin. D.I. & Shen, N., *J. Fluid Mech.,* **967** (2023)