Potentially Singular Behavior of the 3D Incompressible Navier-Stokes Equations

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Motivation

The 3D Navier-Stokes equations describe the motion of viscous fluid:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0. \tag{1}$$

The special case of $\nu = 0$ corresponds to the 3D Euler equations. Define vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, then $\boldsymbol{\omega}$ is governed by

$$\boldsymbol{\omega}_t + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = \nabla \mathbf{u} \cdot \boldsymbol{\omega} + \nu \Delta \boldsymbol{\omega}. \tag{2}$$

Note that $\nabla \mathbf{u}$ is related to $\boldsymbol{\omega}$ by a Riesz transform K: $\nabla \mathbf{u} = K(\boldsymbol{\omega})$, and the nonocal vortex stretching term $\nabla \mathbf{u} \cdot \boldsymbol{\omega}$ is formally of the order $\boldsymbol{\omega}^2$.

The cancellation between the transport and vortex stretching could lead to dynamic depletion of nonlinearity, thus prevent blowup.

A brief review for 3D Euler equations

- (**Beale-Kato-Majda criterion**, 1984) A smooth solution develops a singularity at T if and only if $\int_0^T \|\boldsymbol{\omega}(t)\|_\infty \, dt = \infty$.
- **Geometry regularity of direction field** of $\boldsymbol{\omega}$: Constantin, Fefferman and Majda (1996). Let $\boldsymbol{\omega} = |\boldsymbol{\omega}|\xi$. The solution is smooth if $\|\mathbf{u}(t)\|_{L^\infty(\Omega_t)}$ and $\int_0^t \|\nabla\xi\|^2_{L^\infty(\Omega_\tau)} d\tau$ are bounded.
- **Localized non-blow-up criteria**, Deng-Hou-Yu (2005). Let L_t be a vortex line segment around maximum vorticity. There is no blowup if $\int_0^T \max_{L_t} |\mathbf{u}| dt < \infty$ and $\int_{L_t} (|\kappa| + |\nabla \cdot \xi|) ds < \infty$.
- Elgindi (2021): 3D Euler singularity with C^α vorticity for axisymmetric Euler with no swirl ($u^\theta \equiv 0$) and very small α.

A brief review for the NSE

- Global existence for small data (Leray, Ladyzhenskaya, Kato, etc). If $\|\mathbf{u}_0\|_{L^p}$ $(p \geq 3)$ or $\|\mathbf{u}_0\|_{L^2}\|\nabla \mathbf{u}_0\|_{L^2}$ is small, then the 3D Navier-Stokes equations have a globally smooth solution.
- Non-blowup criteria due to G. Prodi 59, J. Serrin 63. A weak solution $\mathbf{u}$ of the 3D Navier-Stokes equations is smooth on $[0,T] \times \mathbb{R}^3$ provided that $\|\mathbf{u}\|_{L_t^q L_x^p([0,T]\times\mathbb{R}^3)} < \infty$ for some p, q satisfying $\frac{3}{p} + \frac{2}{q} \leq 1$ with $3 < p \leq \infty$ and $2 \leq q < \infty$.
- The critical case of $p = 3,\ q = \infty$ was proved by L. Escauriaza, G. Seregin, and V. Sverak in 2003.
- **Partial regularity theory** (Caffarelli-Kohn-Nirenberg 82, F. Lin 98) For any suitable weak solution of the 3D Navier-Stokes equations on an open set in space-time, the one-dimensional Hausdorff measure of the associated singular set is zero.

A brief review for the NSE – continued

- For the axisymmetric Navier-Stokes equations, Chen-Strain-Tsai-Yau (2007) and Koch-Nadirashvili-Seregin-Sverak (2007) proved that if $|\mathbf{u}(\mathbf{x},t)| \leq C|T-t|^{-1/2}$ and $|\mathbf{u}(\mathbf{x},t)| \leq \frac{C}{r}$ for $r \geq r_0$, then $\mathbf{u}$ is regular up to T.

- T. Tao (2014) introduced an averaged NSE in the Fourier space:

$$u_t(x,t) + \tilde{B}(u,u) = \Delta_x u(x,t),$$

and showed that there exists a symmetric averaged bilinear operator $\tilde{B}$ such that the solutions to the averaged NSE with a divergence-free initial data u_0 blow up in finite time.

Stabilizing effect of advection (Hou-Li, CPAM 2008)

The 3D axisymmetric Euler (Hou-Li, 2008) can be reformulated as

$$u_{1,t} + u^r u_{1,r} + u^z u_{1,z} = 2u_1\psi_{1,z},$$
$$\omega_{1,t} + u^r \omega_{1,r} + u^z \omega_{1,z} = (u_1^2)_z,$$
$$-\left[\partial_r^2 + \tfrac{3}{r}\partial_r + \partial_z^2\right]\psi_1 = \omega_1, \quad u^r = -r\psi_{1,z},\ u^z = 2\psi_1 + r\psi_{1,r},$$

for smooth initial data, where $u_1 = u^\theta/r,\ \omega_1 = \omega^\theta/r,\ \psi_1 = \psi^\theta/r.$

In [Hou-Li,2008], we studied the stabilizing effect of advection by considering a class of initial data $u_0^\theta(r,z) = ru_0(z),\ \omega_0^\theta(r,z) = rw_0(z).$

We constructed a Lyapunov function $(u_{1,z})^2 + \omega_1^2$ and showed that

$$((u_{1,z})^2 + \omega_1^2)_t + (u^r, u^z)\cdot\nabla_{r,z}((u_{1,z})^2 + \omega_1^2) = 0.$$

Advection cancels vortex stretching exactly, leading to global regularity.

Review of numerical study of 3D Euler singularity

- On the numerical search for the Euler singularity:
  - Grauer and Sideris (1991): first numerical study of axisymmetric flows with swirl, blowup reported away from the axis;
  - Pumir and Siggia (1992): axisymmetric flows with swirl; blowup reported away from the axis;
  - Kerr (1993): antiparallel vortex tubes; blowup reported;
  - E and Shu (1994): 2D Boussinesq; no blowup observed;
  - Boratav and Pelz (1994): viscous simulations using Kida's high-symmetry initial condition; blowup reported;
  - Grauer et al. (1998): perturbed vortex tube; blowup reported;
  - Hou and Li (2006): use Kerr's two anti-parallel vortex tube initial data; observed only double exponential growth of max vorticity.
  - Charles Doering (UMich), Bartosz Protas (McMaster) used numerical optimization in space/time to search for potential blowup of Navier-Stokes, maximum vorticity grows less than a factor of 2.
  - Wang-Lai-Gomez-Serrano-Buckmaster (2022), used PINN to construct self-similar profiles of Hou-Luo blowup scenario.

The 3D Euler blowup on the boundary, Luo-Hou 2014

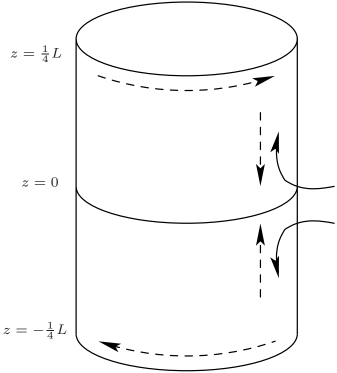


Figure: Vorticity kinematics of the 3D Euler singularity; solid: vortex lines; straight dashed lines: axial flow; curved dash lines: vortical circulation.

Applying the Ideas: Computing the Line Fitting

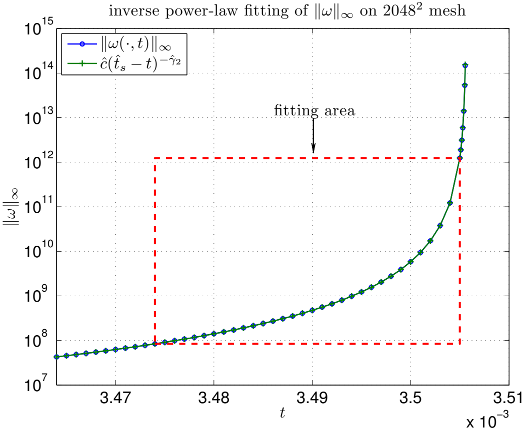


Figure: Maximum vorticity $\|\omega\|_\infty$ and its inverse power-law fitting $\hat{c}(T-t)^{-\hat{\gamma}_2}$ with $\gamma_2 = 2.4579$, $\omega \sim \frac{1}{(T-t)^{\gamma_2}} W\left(\frac{r-1,z}{(T-t)^{c_l}}\right)$ with $c_l \approx 2.9215$.

Theorem 1 (Chen-Hou, 2022, 2023). There is a family of smooth initial data (θ_0, ω_0) with finite energy and boundary, such that the 2D Boussinesq and 3D Euler equations develop a stable and nearly self-similar finite time singularity.

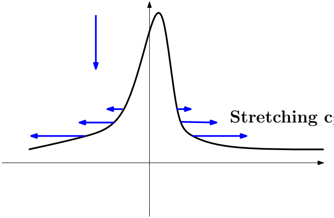
- Compared with the nonlinear Schrödinger equation or the Keller-Segel system, we do not have an analytic ground state, Q.
- Computer assistance is used to construct an approximate self-similar profile $\overline{W}$ with a rigorously justified small residual $O(10^{-7})$ in some singularly weighted L^∞ and $C^{1/2}$ norm.
- We also need computer assistance to obtain sharp upper bounds for our stability constants involving singular integrals with singular kernels, singular weights, and the approximate profile.

The Dynamic Rescaling Formulation

- Adding scaling terms to the 2D Boussinesq equation in $\mathbb{R}^2_+$:

$$\omega_\tau + (c_l(\tau)x + u)\cdot\nabla\omega = \theta_{x_1} + c_\omega(\tau)\omega, \quad \theta_\tau + (c_l(\tau)x + u)\cdot\nabla\theta = c_\theta(\tau)\theta.$$

- The $c_l(\tau)x\cdot\nabla$ terms stretch the solutions in space.
  The $c_\omega(\tau)\omega$ and $c_\theta(\tau)\theta$ terms scale the solutions in amplitude.



- Using $c_\theta(\tau) = c_l(\tau) - 2c_\omega(\tau)$, the dynamic rescaling equations are equivalent to the original Boussinesq up to rescaling.

The Equivalence Relation for Dynamic Rescaling

Let $\theta(x,t)$, $\omega(x,t)$ be solutions to the original 2D Boussinesq system,

$$C_\theta(\tau)\theta\left(C_l(\tau)x, t(\tau)\right), \quad C_\omega(\tau)\omega\left(C_l(\tau)x, t(\tau)\right),$$

are solutions to the dynamic rescaling equations, where

$$C_\theta(\tau) = e^{\int_0^\tau c_\theta(s)\mathrm{d}s},\ C_w(\tau) = e^{\int_0^\tau c_\omega(s)\mathrm{d}s},\ C_l(\tau) = e^{\int_0^\tau -c_l(s)\mathrm{d}s}$$

and

$$t(\tau) = \int_0^\tau \exp\left(\int_0^s c_\omega(y)\mathrm{d}y\right)\mathrm{d}s.$$

Remark 1: if $c_\omega(\tau) \to c_\omega < 0$, $t(+\infty) = T < +\infty$ is the blowup time.

Remark 2: The dynamic rescaling formulation has been used by McLaughlin, Papanicolaou, Merle, Raphael, Martel, Zaag and others.

A general approach to nonlinear stability

- Denote $v = (\omega, \theta)^T$, and write the dynamic rescaling equations as

$$\frac{\mathrm{d}}{\mathrm{d}\tau} v = F(v).$$

- Let $\bar{v}$ be an approximate steady state of the dynamic rescaling equations:

$$\|F(\bar{v})\| = \varepsilon \ll 1.$$

- Decompose the solution as $v = \bar{v} + \tilde{v}$, then the equation for the perturbation $\tilde{v}$ is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{v} = F(\bar{v} + \tilde{v}) = \underbrace{\nabla F(\bar{v})\,\tilde{v}}_{\text{Linear: } \mathcal{L}(\tilde{v})} + \underbrace{\tilde{v}^T \nabla^2 F(\bar{v}) \tilde{v}}_{\text{Nonlinear: } N(\tilde{v})} + F(\bar{v}).$$

Note that F is bilinear in our problem.

A general approach to nonlinear stability

- The perturbation solution $\tilde{v}$ can be written as

$$\tilde{v}(\tau) = \int_0^\tau \mathrm{e}^{(\tau-s)\mathcal{L}}\big(N(\tilde{v}) + F(\bar{v})\big)\,\mathrm{d}s.$$

- Assume that the linearized operator around the approximate steady state enjoys certain stability with a suitable norm:

$$\|\mathrm{e}^{\tau\mathcal{L}}\| \le \mathrm{e}^{-\lambda\tau}, \quad \lambda > 0,$$

and assume that the nonlinear term can be bounded as

$$\|N(\tilde{v})\| \le C\|\tilde{v}\|^2.$$

Then we have

$$\|\tilde{v}(\tau)\| \le \int_0^\tau \mathrm{e}^{-\lambda(\tau-s)}(C\|\tilde{v}(s)\|^2 + \varepsilon)\,\mathrm{d}s.$$

A general approach to nonlinear stability

- If $4\varepsilon C < \lambda^2$, there exists some $\delta > 0$ such that

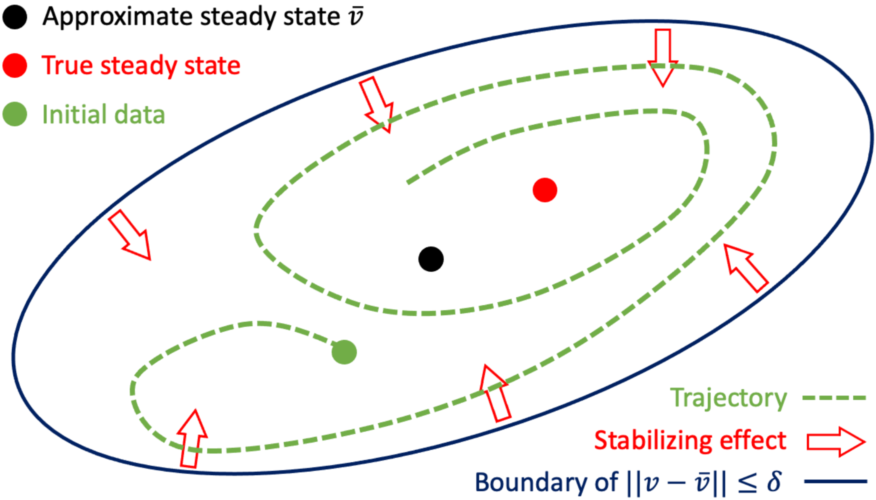
$$\frac{C\delta^2}{\lambda} + \frac{\varepsilon}{\lambda} \leq \delta.$$

- Bootstrapping: suppose that $\|\tilde{v}(s)\| \leq \delta$ for all $s \in [0, \tau)$, then

$$\|\tilde{v}(\tau)\| \leq \int_0^\tau \mathrm{e}^{-\lambda(\tau - s)}(C\delta^2 + \varepsilon)\,\mathrm{d}s < \frac{C\delta^2}{\lambda} + \frac{\varepsilon}{\lambda} \leq \delta,$$

  which means that $\|\tilde{v}(\tau)\| \leq \delta$ for all time given $\|\tilde{v}(0)\| \leq \delta$.

- The dynamic rescaling solution $v(\tau) = \bar{v} + \tilde{v}(\tau)$ will not escape a norm ball of radius δ around the approximate steady state $\bar{v}$.
- We turn the problem of proving finite time singularity into the problem of long time stability of an approximate blowup profile.


Approximate steady state $\bar{v}$
True steady state
Initial data
Trajectory
Stabilizing effect
Boundary of $||v - \bar{v}|| \leq \delta$

- We study potential singularity of the 3D axisymmetric Euler and potentially singular behavior of the 3D Navier–Stokes equations:

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{3}$$

- Introduce a change of variables: $u_1 = u^\theta / r$, $\omega_1 = \omega^\theta / r$, and $\psi_1 = \psi^\theta / r$. The equations (3) can be rewritten in an equivalent form:

$$\begin{cases} u_{1,t} + u^r u_{1,r} + u^z u_{1,z} = 2u_1 \psi_{1,z} + \nu \Delta u_1, \\ \omega_{1,t} + u^r \omega_{1,r} + u^z \omega_{1,z} = 2u_1 u_{1,z} + \nu \Delta \omega_1, \\ -\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right) \psi_1 = \omega_1, \\ u^r = -r\psi_{1,z}, \quad u^z = 2\psi_1 + r\psi_{1,r}. \end{cases} \tag{4}$$

- Our smooth initial condition has a very simple form:

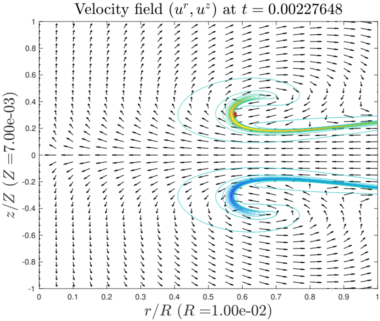
$$u^\theta(0,r,z) = \frac{12000r(1-r^2)^{18}\sin(2\pi z)}{1+12.5(\sin(\pi z))^2}, \quad \omega^\theta(0,r,z) = 0. \qquad (5)$$

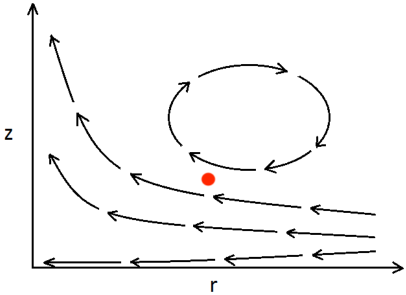
  An important feature of this initial data is that it produces nearly self-similar scaling properties compatible with those of the NSE.

- Our new initial condition shares several features with the two-scale traveling wave singularity using variable diffusion coefficients $\nu(r,z) = O(r^2+z^2)$ with De Huang (Physica D, 2022, MMS, 2023).

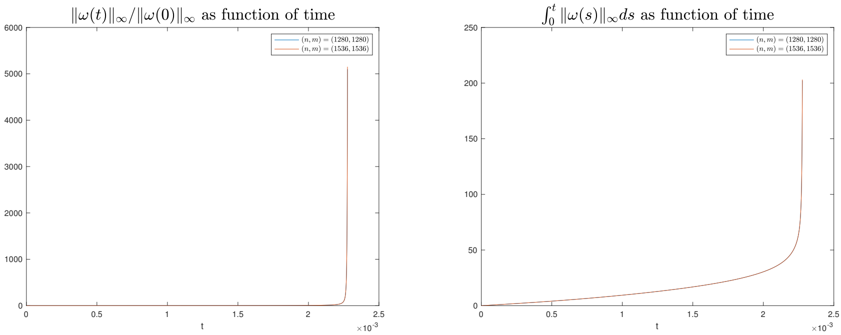
- However, the two-scale travel wave solution does not survive viscous regularization since the small scale $Z(t) \sim (T-t)$ and $R(t) \sim \sqrt{T-t}$. The maximum vorticity with $\nu = 10^{-5}$ increases less than a factor of 2. The solution also suffers from tail instability.

Dipole structure of ω^θ





- The odd symmetry of ω^θ (in z) induces a hyperbolic flow structure and a vortex dipole, which generate a negative radial velocity field and push the solution near $z = 0$ towards to symmetry axis $r = 0$.



- Left plot: the amplification of maximum vorticity relative to its initial maximum vorticity, $\|\omega(t)\|_{L^\infty}/\|\omega(0)\|_{L^\infty}$ as a function of time. Right plot: the time integral of maximum vorticity, $\int_0^t \|\omega(s)\|_{L^\infty} ds$ as a function of time.

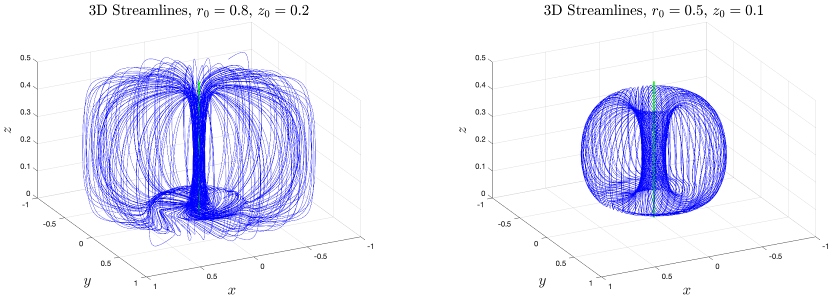
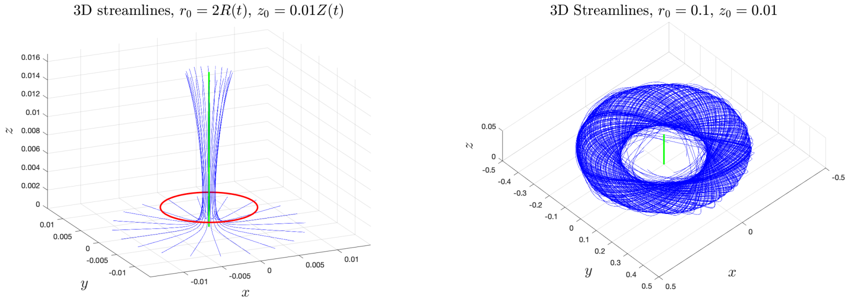


Figure: The 3D streamlines of at time $t = 0.00227648$ with different initial points. The 3D velocity field (u^r, u^z, u^θ) is a tornado solution with a quiet wind eye.



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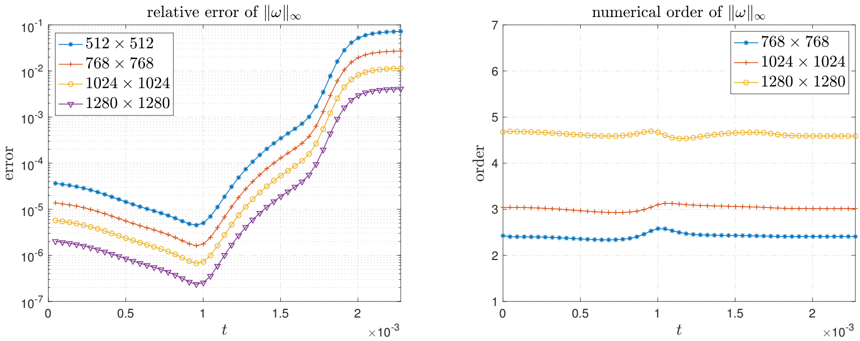
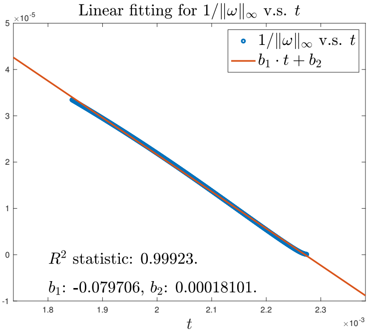
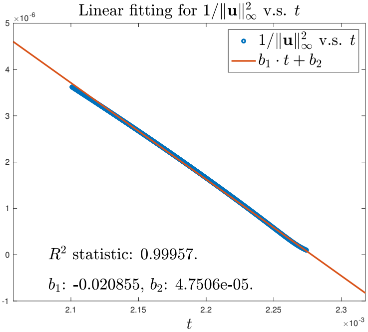


- Relative error and numerical order of $\|\omega(t)\|_{L^\infty}$. The last time instant shown in the figure is $t = 0.00227648$.

Linear fitting of maximum vorticity and velocity

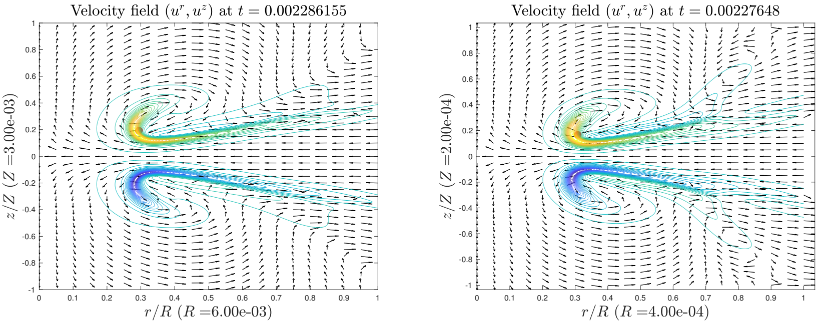




- The linear regression of (a) $\|\omega(t)\|_{L^\infty}^{-1}$ vs t, (b) $\|\mathbf{u}(t)\|_{L^\infty}^{-2}$ vs t. This implies that $\|\omega(t)\|_{L^\infty} \sim \frac{1}{T-t}$ and $\|\mathbf{u}(t)\|_{L^\infty} \sim \frac{1}{(T-t)^{1/2}}$. This suggests that the vorticity blows up like $\omega \sim \frac{1}{(T-t)}\Omega(\frac{(r-R(t),z-Z(t))}{(T-t)^{1/2}}, t)$.

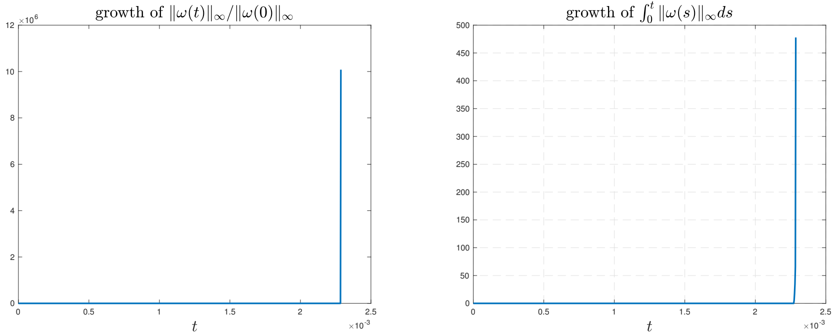
- Given that the scaling properties of the 3D Euler equations are compatible with those of the 3D Navier-Stokes equations, it is natural to investigate whether the Navier-Stokes equations with the same initial condition would develop a singularity.
- It turns out that the choice of the viscosity is important in producing a nearly singular behavior of the NSE.
- We will first use $\nu = 5 \cdot 10^{-4}$ from $t = 0$ to $t_1 = 0.00227375$, then increase ν to $5 \cdot 10^{-3}$ for $t \geq t_1$.
- This relatively large viscosity enhances the nonlinear alignment of vortex stretching, producing a relative long stable phase of strong nonlinear alignment and a nearly singular solution at the origin.

Dipole structure of ω_1



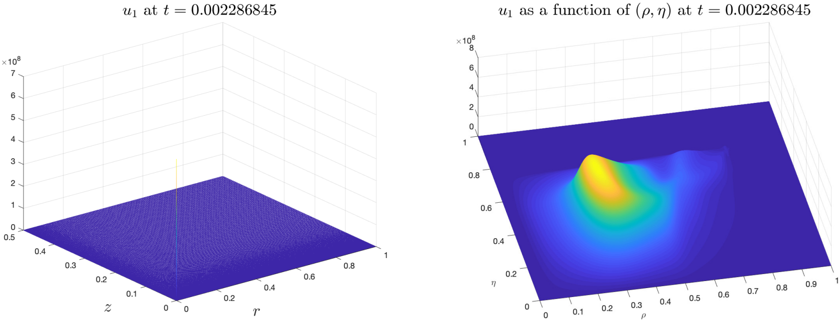
- The odd symmetry of ω_1 (in z) induces a flow structure that has the desirable property of pushing the solution near $z = 0$ towards to symmetry axis $r = 0$.

Rapid growth of maximum vorticity



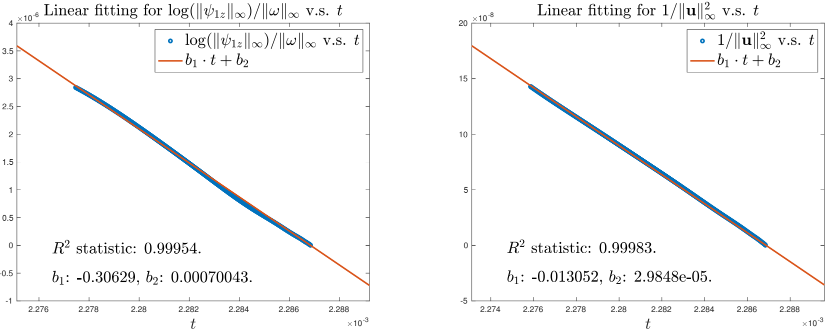
- Left plot: the amplification of maximum vorticity relative to its initial maximum vorticity, $\|\omega(t)\|_{L^\infty} / \|\omega(0)\|_{L^\infty}$ as a function of time, growing by a factor of 10^7. Right plot: the time integral of maximum vorticity, $\int_0^t \|\omega(s)\|_{L^\infty} ds$ as a function of time.

Resolution study

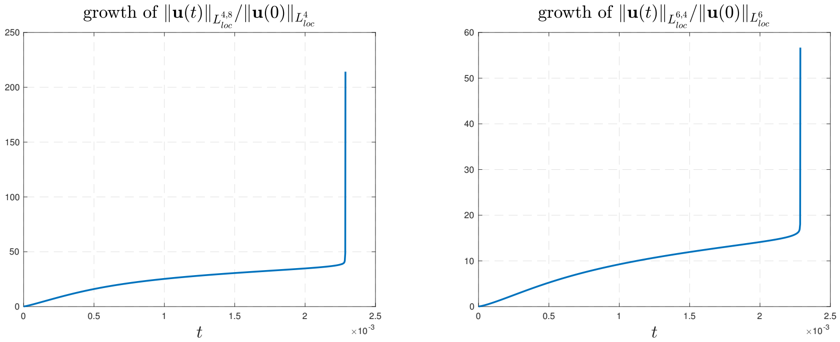


- (a): u_1 in the whole physical coordinates r, z.
- (b): u_1 in the transformed coordinates (ρ, η).

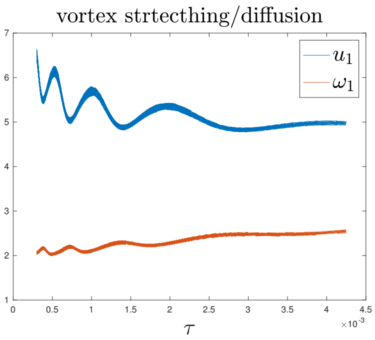
Linear fitting of maximum vorticity and velocity

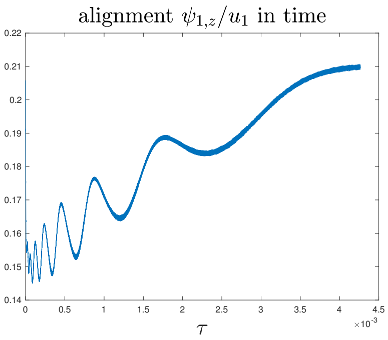


- The linear regression of (a) $\log(\|\psi_{1z}(t)\|_{L^\infty})\|\omega(t)\|_{L^\infty}^{-1}$ vs t, (b) $\|\mathbf{u}(t)\|_{L^\infty}^{-2}$ vs t. This implies that $\|\omega(t)\|_{L^\infty} \sim \frac{|\log(T-t)|}{T-t}$ and $\|\mathbf{u}(t)\|_{L^\infty} \sim \frac{1}{(T-t)^{1/2}}$. We observe that $Z(t) \sim (T-t)^{1/2}$ with a logarithmic correction.



Left subplot: $\|\mathbf{u}\|_{L^{4,8}_{loc}} = (\int_0^t \|\mathbf{u}(s)\|^8_{L^4_{loc}} \, ds)^{1/8}$. Right subplot: $\|\mathbf{u}\|_{L^{6,4}_{loc}}$.





Left plot: Vortex stretching/diffusion. Right plot: Alignment between $\psi_{1,z}$ and u_1 at $(R(\tau), Z(\tau))$. We solve $(u_1)_t + \mathbf{u} \cdot \nabla u_1 = 2\psi_{1,z} u_1 + \nu \Delta u_1$ and $(\omega_1)_t + \mathbf{u} \cdot \nabla \omega_1 = 2u_1 u_{1,z} + \nu \Delta \omega_1$.

We consider a generalized axisymmetric Navier–Stokes equation in n dimensions with $n = 1 + 2R(t)/Z(t)$ and $(R(t), Z(t)) = \arg\max(u_1)$:

$$\begin{aligned} &u_{1,t} + u^r u_{1,r} + u^z u_{1,z} = 2\psi_{1,z} u_1 + \nu \Delta_{n+2} u_1 \\ &\omega_{1,t} + u^r \omega_{1,r} + u^z \omega_{1,z} = (u_1^2)_z - (n-3)\psi_{1,z}\omega_1 + \nu \Delta_{n+2}\omega_1 \\ &-\Delta_{n+2}\psi_1 = \omega_1, \quad \Delta_{n+2} = \partial_r^2 + \frac{n}{r}\partial_r + \partial_z^2 \ , \end{aligned}$$

where $u^r = -(r^{n-2}\psi^\theta)_z / r^{n-2}, \ u^z = (r^{n-2}\psi^\theta)_r / r^{n-2}$. It satisfies a generalized incompressibility condition $(r^{n-2}u^r)_r + (r^{n-2}u^z)_z = 0$, the conservation of total circulation $\Gamma = ru^\theta$, and the energy conservation

$$\frac{d}{dt}\int |\mathbf{u}|^2 r^{n-2} dr dz = -\nu \int |\nabla \mathbf{u}|^2 r^{n-2} dr dz.$$

We also consider a generalized axisymmetric Boussinesq system in n dimensions by treating Γ as density and removing $(n-3)\psi_{1,z}\omega_1$:

$$\Gamma_t + u^r\Gamma_r + u^z\Gamma_z = \nu_1\left(\Gamma_{rr} + \frac{(n-4)}{r}\Gamma_r + \frac{(6-2n)}{r^2}\Gamma + \Gamma_{zz}\right), \tag{6a}$$

$$\omega_{1,t} + u^r\omega_{1,r} + u^z\omega_{1,z} = \left(\frac{\Gamma^2}{r^4}\right)_z + \nu_2\left(\omega_{1,rr} + \frac{n}{r}\omega_{1,r} + \omega_{1,zz}\right), \tag{6b}$$

$$-\left(\partial_r^2 + \frac{n}{r}\partial_r + \partial_z^2\right)\psi_1 = \omega_1, \tag{6c}$$

where $u^r = -(r^{m-2}\psi^\theta)_z/r^{m-2}$, $u^z = (r^{m-2}\psi^\theta)_r/r^{m-2}$, $n = 2m-3$, $m = 1 + 2R(t)/Z(t)$. It satisfies the incompressibility condition $(r^{m-2}u^r)_r + (r^{m-2}u^z)_z = 0$, the conservation of total circulation. If $n < 7$, the energy $\int(|u^\theta|^2 + \frac{(7-n)}{4}(|u^r|^2 + |u^z|^2)r^{n-2}drdz$ is conserved.

In the case of $\nu_1 = \nu_2 = \nu_0 \|u_1\|_\infty Z(t)^2$, we have $\nu_i C_\psi / C_{lz} = \nu_0$, and $\widetilde{\psi}_1$, $\widetilde{u}_1$, $\widetilde{\omega}_1$ satisfy the following dynamic rescaling equations

$$\widetilde{\Gamma}_\tau + c_{lr}\xi\widetilde{\Gamma}_\xi + c_{lz}\eta\widetilde{\Gamma}_\eta + \widetilde{\mathbf{u}}\cdot\nabla_{(\xi,\eta)}\widetilde{\Gamma} = c_\Gamma\widetilde{\Gamma} + \nu_0\widetilde{\Delta}\widetilde{\Gamma},$$

$$\widetilde{\omega}_{1,\tau} + c_{lr}\xi\widetilde{\omega}_{1,\xi} + c_{lz}\eta\widetilde{\omega}_{1,\eta} + \widetilde{\mathbf{u}}\cdot\nabla_{(\xi,\eta)}\widetilde{\omega}_1 = c_\omega\widetilde{\omega}_1 + \left(\frac{\widetilde{\Gamma}^2}{\xi^4}\right)_\eta + \nu_0\Delta\widetilde{\omega},$$

$$-\Delta\widetilde{\psi}_1 = \widetilde{\omega}_1, \quad \Delta = -\left(\delta^2\partial_\xi^2 + \delta^2\frac{n}{\xi}\partial_\xi + \partial_\eta^2\right),$$

where $\delta = C_{lz}(\tau)/C_{lr}(\tau)$ and $(c_{lz}, c_{lr}, c_\psi, c_u, c_\omega)$ satisfy the relationship

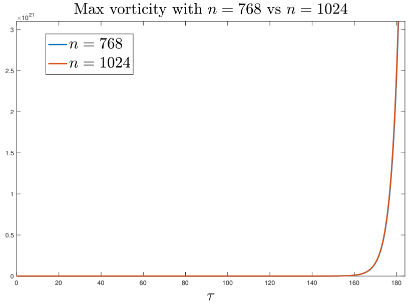
$$c_\psi = c_u + c_{lz}, \quad c_\omega = c_u - c_{lz}, \quad c_\Gamma = c_u + 2c_{lr} = 2(c_{lr} - c_{lz}).$$

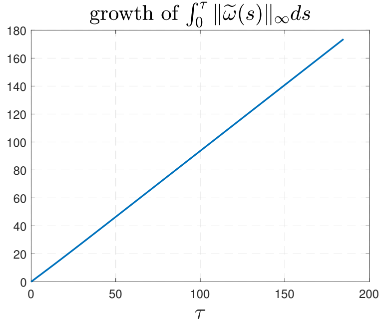
Thus, the self-similar profile satisfies the generalized Navier–Stokes equations with constant viscosity ν_0.

We take $\nu = \nu_0 \|u_1(t)\|_\infty Z(t)^2$ where $(R(t), Z(t))$ is where u_1 achieves its maximum and choose $n = 1 + 2R(t)/Z(t)$ so that u^r and u^z scale like $(R(t)/Z(t))\psi_1$. We obtain a self-similar blowup with $n \approx 3.188$:

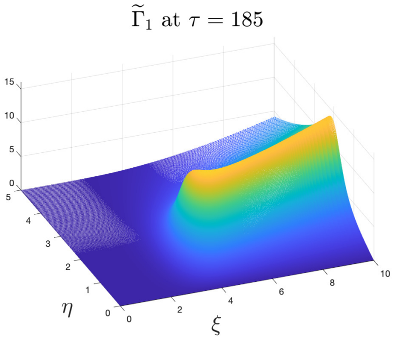
$$\begin{aligned}
\psi_1(t,r,z) &= \frac{\lambda(t)}{(T-t)^{1/2}} \Psi_1\left(\tau, \frac{r}{\lambda(t)\sqrt{(T-t)}}, \frac{z}{\lambda(t)\sqrt{(T-t)}}\right), \\
u_1(t,r,z) &= \frac{1}{(T-t)} V_1\left(\tau, \frac{r}{\lambda(t)\sqrt{(T-t)}}, \frac{z}{\lambda(t)\sqrt{(T-t)}}\right), \\
\omega_1(t,r,z) &= \frac{1}{\lambda(t)(T-t)^{3/2}} \Omega_1\left(\tau, \frac{r}{\lambda(t)\sqrt{(T-t)}}, \frac{z}{\lambda(t)\sqrt{(T-t)}}\right),
\end{aligned}$$

where $\lambda(t) = (T-t)^{0.023}$ and $R(\tau)/Z(\tau) \to 1.09$ as $\tau \to \infty$.

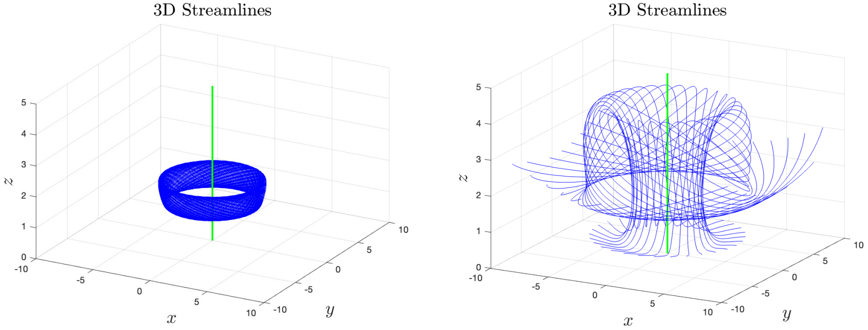




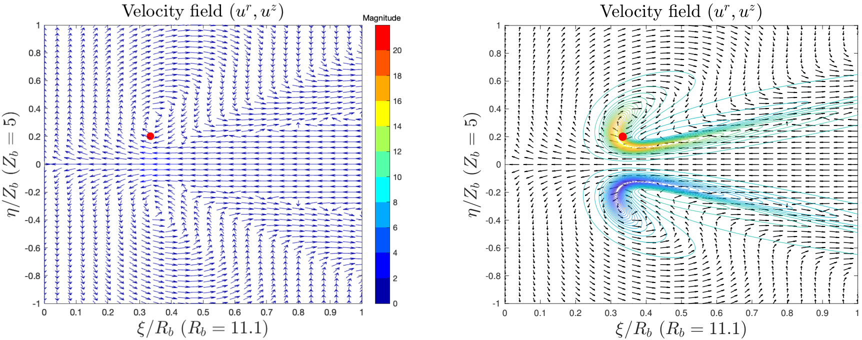
Left plot: Growth of $\|\boldsymbol{\omega}(\tau)\|_{L^\infty}$ in τ using 768×768 vs 1024×1024.
Right plot: Growth of $\int_0^\tau \|\widetilde{\boldsymbol{\omega}}(s)\|_{L^\infty} ds$ in τ.



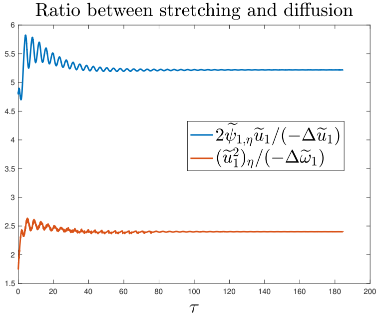
Left subplot: 3D plot of $\widetilde{\Gamma}$ at $\tau = 185$. Right subplot: 3D plot of $\widetilde{\omega}_1$ at $\tau = 185$.

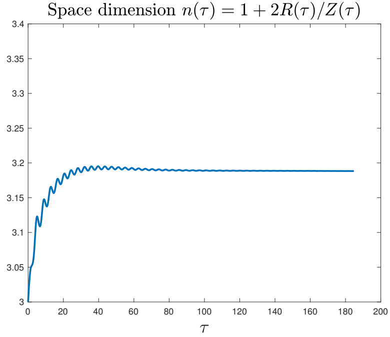


Left subplot: Streamlines at $\tau = 185$ with $(r_0, z_0) = (4, 2)$. Right subplot: Streamlines at $\tau = 185$ with $(r_0, z_0) = (6, 0.5)$.

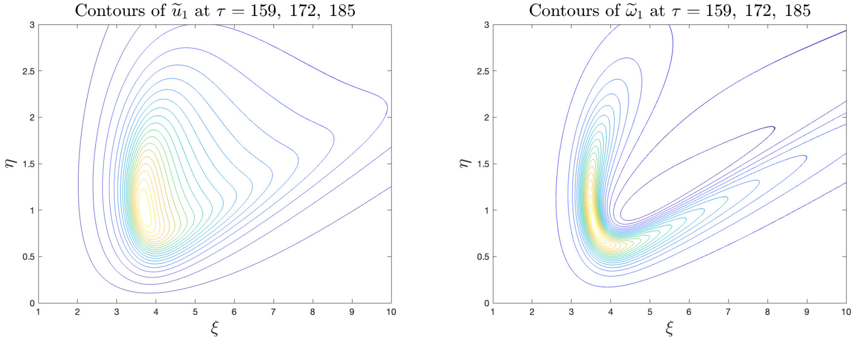


The dipole structure of ω_1 and the induced local velocity field at $\tau = 185$. Left plot: the velocity vector. Right plot: the velocity vector with the ω_1 contour as background.





Left plot: The ratio between vortex stretching and diffusion for $\widetilde{u}_1$ at (R, Z) and for $\widetilde{\omega}_1$ at (R_ω, Z_ω). $u_{1,t} + \mathbf{u}\cdot\nabla u_1 = 2\psi_{1,z}u_1 + \nu(t)\Delta u_1$, $\omega_{1,t} + \mathbf{u}\cdot\nabla\omega_1 = (u_1^2)_z - (n-3)\psi_{1,z}\omega_1 + \nu(t)\Delta\omega_1$. Right plot: The dimension $n(\tau) = 1 + 2R(\tau)/Z(\tau)$ with $n(185) = 3.188$.

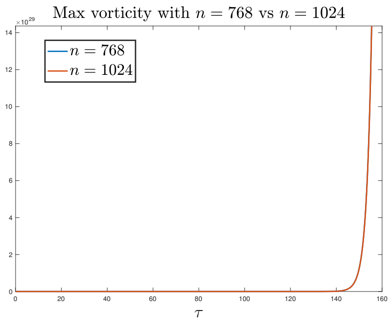


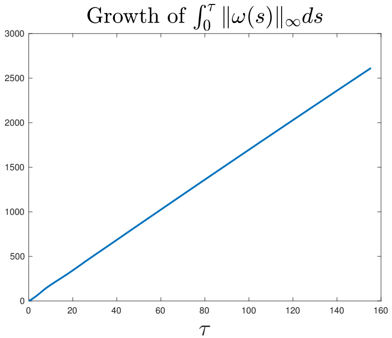
Left plot: Contours of $\widetilde{u}_1$ at $\tau = 159,\ 172,\ 185$ during which $\|\boldsymbol{\omega}\|_\infty$ has increased by 1029. Right plot: Contours of $\widetilde{\omega}_1$ at $\tau = 159,\ 172,\ 185$.

By taking $\nu_1 = 6 \times 10^{-4}$, $\nu_2 = 10\nu_1$, we obtain a new blowup scenario:

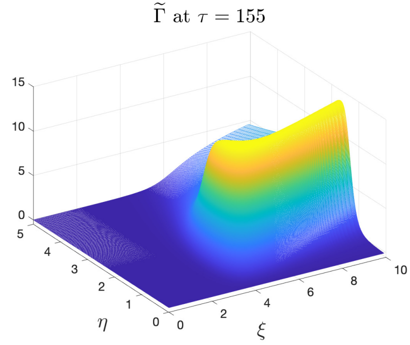
$$\begin{aligned}
\psi_1(t,r,z) &= \frac{\lambda(t)}{(T-t)^{1/2}}\Psi_1\left(\tau, \frac{r}{\lambda(t)\sqrt{(T-t)}}, \frac{z}{\lambda(t)\sqrt{(T-t)}}\right),\\
u_1(t,r,z) &= \frac{1}{(T-t)}V_1\left(\tau, \frac{r}{\lambda(t)\sqrt{(T-t)}}, \frac{z}{\lambda(t)\sqrt{(T-t)}}\right),\\
\omega_1(t,r,z) &= \frac{1}{\lambda(t)(T-t)^{3/2}}\Omega_1\left(\tau, \frac{r}{\lambda(t)\sqrt{(T-t)}}, \frac{z}{\lambda(t)\sqrt{(T-t)}}\right),
\end{aligned}$$

where $\lambda(t) = (1+\epsilon|\log(T-t)|)^{-1/2}$. We vary the dimension continuously by modifying $u^r = -r\psi_{1,z}$ and $u^z = (m-1)\psi_1 + r\psi_{1,r}$ with $(m-1) = 2R(t)/Z(t)$. u^r and u^z now scale like $(R(t)/Z(t))\psi_1$.

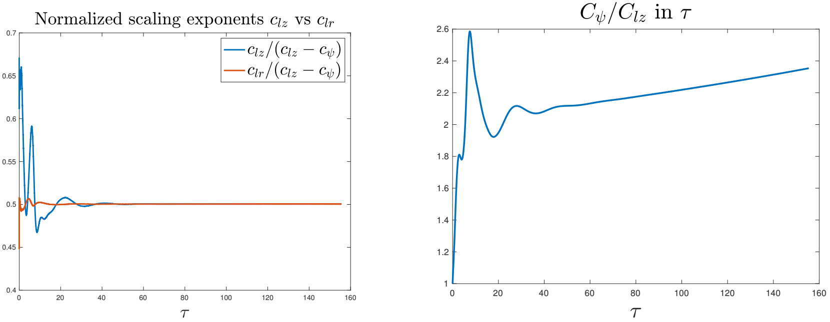




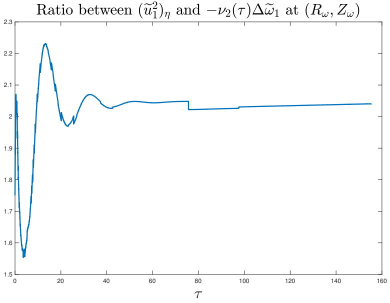
Left plot: Growth of $\|\boldsymbol{\omega}(\tau)\|_{L^\infty}$ in τ using 768×768 vs 1024×1024.
Right plot: Growth of $\int_0^\tau \|\widetilde{\boldsymbol{\omega}}(s)\|_{L^\infty} ds$ in τ.

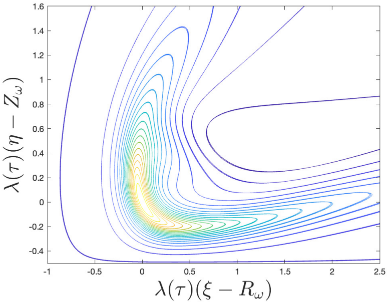


Left subplot: 3D plot of $\widetilde{\Gamma}$ at $\tau = 155$. Right subplot: 3D plot of $\widetilde{\omega}_1$ at $\tau = 155$.

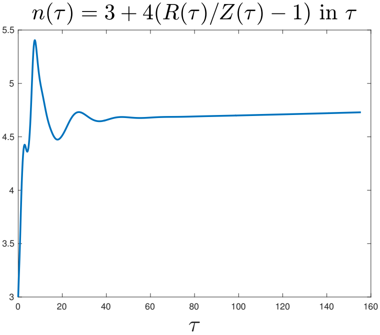


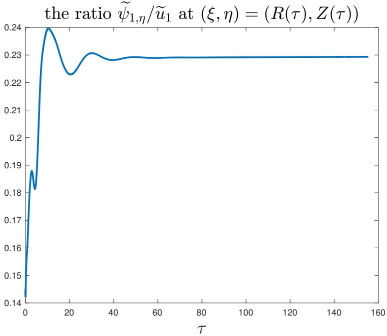
Left plot: The normalized scaling exponent c_{lz} for $Z(t) \sim (T-t)^{c_{lz}}$ and c_{lr} for $R(t) \sim (T-t)^{c_{lr}}$ in τ. Right plot: $C_\psi(\tau)/C_{lz}(\tau) = 1/\lambda(\tau)^2$. The almost linear growth in τ implies that c_{lz} and c_{lr} converge to 1/2 and $\lambda = (1+\epsilon\tau)^{-1/2} = (1+\epsilon|\log(T-t)|)^{-1/2}$.





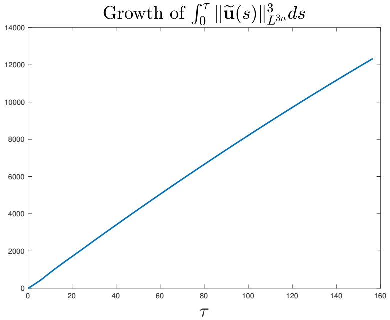
Left plot: The ratio between vortex stretching and diffusion for $\widetilde{\omega}_1$.
Right plot: Contours of $\widetilde{\omega}_1$ with respect to $((\xi - R_\omega)\lambda(\tau), (\eta - Z_\omega)\lambda(\tau))$ at $\tau = 139,\ 147,\ 155$ during which $\|\boldsymbol{\omega}\|_\infty$ has increased by 1554.

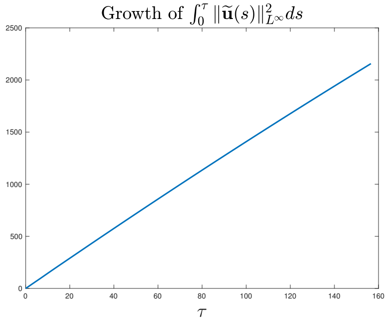




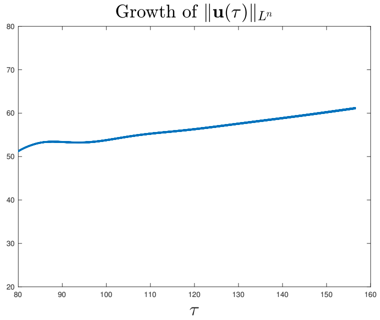
Left plot: The dimension $n(\tau) = 3 + 4(R(\tau)/Z(\tau) - 1)$ with $n(155) = 4.73$. Right plot: The ratio $\widetilde{\psi}_{1,\eta}/\widetilde{u}_1$ at $(R(\tau), Z(\tau))$.

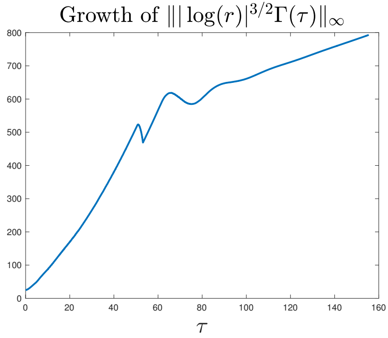
Generalized Boussinesq system with 2 constant viscosity coefficients





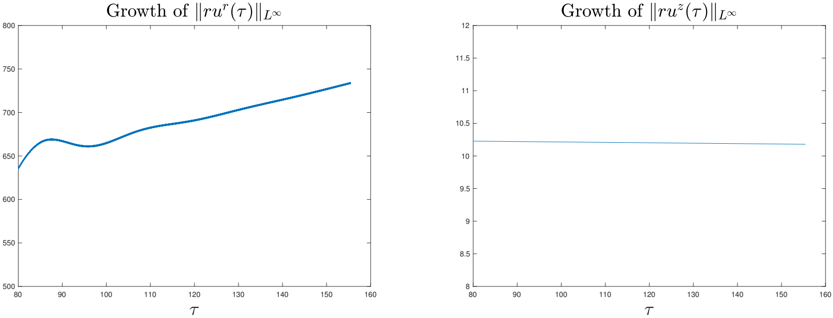
Serrin-Prodi criteria $n/p + 2/q \leq 1$. Left plot: $\int_0^\tau \|\widetilde{\mathbf{u}}(s)\|_{L^{3n}}^3 ds$ in τ. The linear fitting implies $\|\mathbf{u}(\tau)\|_{L^{3n,3}} \sim O(\tau^{1/3})$. Right plot: $\int_0^\tau \|\widetilde{\mathbf{u}}(s)\|_{L^\infty}^2 ds$. The linear fitting implies $\|\mathbf{u}(\tau)\|_{L^\infty} \sim O(\tau^{1/2})$.





Left plot: The end point case with $p = n, q = \infty$: $\|\mathbf{u}\|_{L^n}$. Right plot: $\||\log(r)|^{3/2}\Gamma(\tau, r, z)\|_{L^\infty}$ in τ. Dongyi Wei (2017) showed that the axisymmetric NSE cannot blowup if $\||\log(r)|^{3/2}\Gamma(\tau, r, z)\|_{r \le r_0} \le 1$.

Solving Navier-Stokes using new initial data– continued



Left plot: The dynamic growth of $\|ru^r\|_{L^\infty}$ as a function of τ. Right plot: The dynamic growth of $\|ru^z\|_{L^\infty}$ as a function of τ. This violates the non-blowup criteria by Sverak (2007) et al and Yau et al (2007).

Summary

- We presented a new class of initial data that lead to a potentially singular solution of the axisymmetric Euler at the origin.
- An important feature of this potentially singular solution is that the scaling properties are compatible with those of the NSE.
- We demonstrated the potentially singular behavior of 3D Navier-Stokes with maximum vorticity increased by a factor of 10^7.
- We demonstrated a generalized Naveri–Stokes and a generalized Boussinesq system with constant viscosity develops a nearly self-similar blowup with maximum vorticity increased by 10^{30}.
- We used a novel two-scale dynamic rescaling formulation and a conservative formulation to capture the logarithmic correction.

Happy Birthday, Russ!

References.

- J. Chen and T. Y. Hou, *Stable nearly self-similar blowup of the 2D Boussinesq and 3D Euler equations with smooth data. Part I. Analysis, arXiv:2210.07191v2 [math.AP], Part II: Rigorous Numerics, acepted by MMS. arXiv:2305.05660 [math.AP].*
- *T, Y. Hou,* The Nearly Singular Behavior of the 3D Navier-Stokes Equations, published online on 9/7/2022, DOI: https://doi.org/10.1007/s10208-022-09578-4.
- T, Y. Hou and D. Huang, *A potential two-scale traveling wave singularity for 3D incompressible Euler equations, Physica D, Vol 435, 133257, 2022.*
- *T. Y. Hou,* Nearly self-similar blowup of generalized axisymmetric Navier–Stokes and Boussinesq equations, arXiv:2405.10916 [math.AP], 2024.