Global Dynamics for the Kompaneets Equation

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0. Russ Caflisch

- We met in 1975 during his graduate student orientation at NYU.
- We shared a remarkable summer in 1977 that shaped both of us.
- Our work on the Kompaneets equation began in the early 1980s.
- We worked closely together when he was the IPAM Director.
- We have shared in many wonderful drinks, meals, discussions, fetes, and moments of friendship over almost 49 years!

Outline

1. Kompaneets Equation
2. Bose-Einstein Equilibria and a Paradox
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1. Kompaneets Equation

Photons can play a mojor role in the transport of energy within a plasma through the processes of emission, absorption, and scattering. At high enough temperatures the domiant process in a fully ionized plasma will be *Compton scattering* off of free electrons. We will consider a model of only this phenomena for a *spatially uniform plasma at a fixed temperature*.

When a photon is scattered by a free electron its direction of travel typically changes a lot while its energy barely changes. Therefore we describe the photon field at time t by an isotropic density $f(x,t)$ with respect to $x^2\,\mathrm{d}x$, where x is a nondimensional energy variable.

The number and energy densities of the photon field are then

$$N[f] = \int_0^\infty f\,x^2\,\mathrm{d}x\,, \qquad E[f] = \int_0^\infty f\,x^3\,\mathrm{d}x\,.$$

The evolution of $f(x,t)$ is governed by a *quantum kinetic equation* built from the Klein-Nishina cross-section for Compton scattering. Because the energy exchange is usually small, a *Fokker-Planck approximation* can be applied. In *nonrelativistic regimes* this yields the *Kompaneets equation*,

$$\partial_t f = \frac{1}{x^2}\partial_x\Big[x^4(\partial_x f + f + f^2)\Big], \qquad f\Big|_{t=0} = f^{\mathrm{in}}. \tag{1}$$

Because x is a radial variable, here the divergence has the form $x^{-2}\partial_x x^2$, while the diffusion coefficient is x^2. Notice that the diffusion coefficent becomes degenerate (vanishes) at the origin!

The question arises as to whether or not a boundary condition needs to be imposed either at $x = 0$ or as $x \to \infty$. The answer to this question will depend upon the space in which one wants to establish well-posedness! For now we will proceed formally.

When we consider the evolution of $N[f]$ we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}N[f] &= \int_0^\infty \partial_t f \, x^2 \, \mathrm{d}x \\ &= \int_0^\infty \partial_x\Big[x^4(\partial_x f + f + f^2)\Big] \, \mathrm{d}x \\ &= \Big[x^4(\partial_x f + f + f^2)\Big]\Big|_0^\infty .\end{aligned}$$

Because the Kompaneets equation (1) describes the evolution of f due to only a scattering process, the number of photons should be conserved. Therefore we might expect that f should satisfy

$$\Big[x^4(\partial_x f + f + f^2)\Big]\Big|_{x=0} = \Big[x^4(\partial_x f + f + f^2)\Big]\Big|_{x\to\infty} = 0 \, ,$$

which yields the *expected number density conservation*

$$\frac{\mathrm{d}}{\mathrm{d}t}N[f] = 0 \, . \tag{2}$$

2. Bose-Einstein Equilibria and a Paradox

Stationary solutions of the Kompaneets equation (1) satisfy

$$\partial_x \Big[x^4 (\partial_x f + f + f^2) \Big] = 0 \,.$$

The boundary behavior as $x \to \infty$ that we have already assumed implies

$$\partial_x f + f + f^2 = 0 \,.$$

Stationary solutions of the Kompaneets equation thereby have the form

$$f = f_\mu(x) = \frac{1}{e^{x+\mu} - 1} \quad \text{for some } \mu \geq 0 \,.$$

This is the family of *Bose-Einstein equilibria*. The *Planckian*, which is the usual equilibrium for an interacting photon field, is recovered when $\mu = 0$. The others arise because we are considering only a scattering process.

It is natural to ask whether every solution of the Kompaneets equation approaches one of these equilibria as $t \to \infty$. If so, because $N[f]$ is conserved, it would be natural to think that the solution associated with initial data f^{in} would approach f_μ determined by

$$N[f^{\mathrm{in}}] = N[f_\mu] = \int_0^\infty \frac{1}{e^{x+\mu} - 1}\, x^2 \mathrm{d}x\,.$$

However $N[f_\mu]$ is a strictly decreasing function of μ bounded above by

$$\begin{aligned} N[f_0] = \int_0^\infty \frac{1}{e^{x} - 1}\, x^2 \mathrm{d}x &= \sum_{k=1}^{\infty} \int_0^\infty e^{-kx}\, x^2 \mathrm{d}x \\ &= \sum_{k=1}^{\infty} \frac{2}{k^3} = 2\zeta(3) \approx 2.404\,. \end{aligned}$$

Therefore this picture is clearly **WRONG** for any f^{in} with $N[f^{\mathrm{in}}] > N[f_0]$, and might even be wrong for some f^{in} with $N[f^{\mathrm{in}}] \leq N[f_0]$.

What went wrong?

Do some f not approach an equilibrium? If so, do they have dynamic long-time behavior or do they become singular?

Are our assumptions regarding the Kompaneets boundary conditions wrong? If so, is $N[f]$ conserved?

This same paradox arises for the underlying quantum kinetic equation. Therefore the question arises as to how these two models of the same physical process resolve the paradox in a consistent fashion.

In fact, the same paradox arises for most kinetic models that describe the evolution of a boson field by a scattering process.

3. Quantum Entropy Structure

Another physical quantity is the quantum entropy (*Helmholtz free energy*)

$$H[f] = \int_0^\infty h(f, x)\, x^2\, \mathrm{d}x\,,$$

where the entropy density is given by

$$h(f, x) = f \log(f) - (1 + f) \log(1 + f) + xf\,.$$

It can be shown that $H[f]$ is defined for every $f \geq 0$ with $E[f] < \infty$ with

$$H[f_0] \leq H[f] < E[f]\,,$$

and that $H[f]$ is a strictly convex function of f.

Moreover, if $H[f] < \infty$ then $E[f] < \infty$.

When we consider the evolution of $H[f]$ we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H[f] &= \int_0^\infty h_f\,\partial_t f\,x^2\,\mathrm{d}x \\ &= \int_0^\infty h_f\,\partial_x\Big[x^4 f(1+f)\partial_x h_f\Big]\,\mathrm{d}x \\ &= \Big(h_f\Big[x^4 f(1+f)\partial_x h_f\Big]\Big)\Big|_0^\infty - \int_0^\infty x^4 f(1+f)(\partial_x h_f)^2\,\mathrm{d}x\,.\end{aligned}$$

We might expect that

$$h_f\Big[x^4 f(1+f)\partial_x h_f\Big]\Big|_{x=0} = h_f\Big[x^4 f(1+f)\partial_x h_f\Big]\Big|_{x\to\infty} = 0\,.$$

When that is the case we obtain the *entropy dissipation relation*

$$\frac{\mathrm{d}}{\mathrm{d}t}H[f] = -\int_0^\infty x^4 f(1+f)(\partial_x h_f)^2\,\mathrm{d}x\,. \tag{3}$$

Moreover, for $f > 0$ we see that the dissipation vanishes if and only if

$$\partial_x h_f = \partial_x \log\left(\frac{e^x f}{1+f}\right) = 0\,,$$

which is if and only if

$$f = f_\mu(x) = \frac{1}{e^{x+\mu} - 1} \quad \text{for some } \mu \geq 0\,.$$

This is an analog of the Boltzmann H-Theorem.

This fact along with the number density conservation (2) rules out long-time behavior that is either periodic, quasi-periodic, or almost periodic.

However, these assertions can be called into question if the above formal calculation is not justified or, more precisely, if $H[f]$ does not dissipate.

4. Bose-Einstein Condensation

The problem clearly arises from the quadratic quantum term. To see what is happening, let's keep just that term. Then the Kompaneets equation (1) becomes

$$\partial_t f = \frac{1}{x^2}\partial_x\left[x^4 f^2\right], \qquad f\Big|_{t=0} = f^{\mathrm{in}}.$$

Introducing $n = x^2 f$, this becomes the inviscid Burgers equation

$$\partial_t n = \partial_x\left[n^2\right] = 2n\partial_x n, \qquad n\Big|_{t=0} = n^{\mathrm{in}},$$

where $n^{\mathrm{in}} = x^2 f^{\mathrm{in}}$. Its characteristic equations are

$$\dot{x} = -2n, \qquad \dot{n} = 0.$$

Because $n \geq 0$, no boundary condition is needed at the origin. Clearly any nontrivial viscosity solution will develop a nonzero flux of photons into the origin in finite time. These photons cannot disappear! Rather, they produce a “delta function” at the origin — a *Bose-Einstein condensate*.

5. Some Earlier Work

Early studies of the Kompaneets equation largely examined its solutions before the onset of any condensate:

- E.W. Larsen, C.D.L., G.C. Pomraning, J.G. Sanderson (1985)
- R.E. Caflisch and C.D.L. (1986)
- C.D.L. and O. Kavian (1990, unpublished)
- M. Escobedo, M.A. Herrero and J.J.L. Velazquez (1998)
- M. Escobedo and S. Mischler (2001)

Mathematics Remark. Analytic approaches that treat the nonlinear term as a perturbation of the linear ones can show that certain solutions will "blow up" out of spaces associated with the linear problem, but usually cannot capture the condensate because functions in those spaces satisfy the "expected" boundary conditions at the origin — which breaks down.

Later works analyzed aspects of Bose-Einstein condensation:

- M. Escobedo, M.A. Herrero and J.J.L. Velazquez (1998)
- M. Escobedo, S. Mischler and J.J.L. Velazquez (2004)
- M. Escobedo, S. Mischler and M.A. Valle (2004, 2005)
- X. Lu and C. Mouhot (2012, 2015)
- X. Lu (2013, 2018)
- H. Spohn (2015)
- M. Escobedo and J.J.L. Velazquez (2015)
- H. Liu, R.L. Pego and C.D.L. (2016)
- J. Ballew, G. Iyer and R.L. Pego (2016)

This brings us to the work featured in this talk.

- J. Ballew, G. Iyer, C.D.L., H. Liu and R.L. Pego (2023)

6. Main Results

Again we let $n = x^2 f$ and $n^{\mathrm{in}} = x^2 f^{\mathrm{in}}$. In this setting we redefine

$$N[n] = \int_0^\infty n \, \mathrm{d}x \,, \qquad E[n] = \int_0^\infty xn \, \mathrm{d}x \,.$$

The initial-value problem for the Kompaneets equation (1) becomes

$$\partial_t n = \partial_x J[n] \,, \qquad n\Big|_{t=0} = n^{\mathrm{in}} \,, \tag{4a}$$

where the flux $J[n]$ is given by

$$J[n] = x^2 \partial_x n + (x^2 - 2x)n + n^2 \,. \tag{4b}$$

We impose the boundary condition

$$J[n](x,t) \to 0 \quad \text{as } x \to \infty \,. \tag{4c}$$

We do not impose a boundary condition at $x = 0$!

We assume that $n^{\text{in}} \in L^\infty(\mathbb{R}_+)$ with $n^{\text{in}} \geq 0$ decaying so that

$$x^2 n^{\text{in}}(x) \to 0 \quad \text{as } x \to \infty\,. \tag{5}$$

(This implies that $N[n^{\text{in}}] < \infty$ but not that $E[f] < \infty$.)

We show that for each n^{in} there exists a unique $n \in C^\infty(\mathbb{R}^2_+)$ that is a global solution of (4). Each such solution satisfies

$$0 \leq (1 + x^2)e^{-4t} n(x,t) \leq C \quad \text{uniformly over } \mathbb{R}^2_+\,,$$

$$x^2 n(x,t) \to 0 \quad \text{as } x \to \infty \text{ uniformly over } t \in \mathbb{R}_+\,,$$

and an *Oleinik inequality* in the form

$$\partial_x n \geq -\left(\frac{1}{2t} + \frac{5x}{2} + \alpha\right)\,, \tag{6}$$

where $\alpha > 0$ depends on an L^∞ norm of n.

To control the flux at $x = 0$ we use the Oleinik inequality (6) to show that

$$n(0,t) \equiv \lim_{x \to 0^+} n(x,t) \quad \text{exists for every } t \in \mathbb{R}_+ \,.$$

Remark. *This pointwise limit over $\mathbb{R}_+$ may not be locally uniform over $\mathbb{R}_+$! This extension is not defined at $t = 0$ and may not be continuous in t!*

We then show for every $[s,t] \subset \mathbb{R}_+$ that

$$\lim_{x \to 0^+} x^2 \int_s^t |\partial_x n(x,\tau)| \, \mathrm{d}\tau = 0 \,,$$

$$\lim_{x \to 0^+} \int_s^t \left| J[n](x,\tau) - n(0,\tau)^2 \right| \mathrm{d}\tau = 0 \,,$$

and for every $[s,t] \subset [0,\infty)$ that

$$\lim_{x \to 0^+} \int_s^t J[n](x,\tau) \, \mathrm{d}\tau = \int_s^t n(0,\tau)^2 \, \mathrm{d}\tau \,. \tag{7}$$

To control the flux at ∞ we show for every $[s,t] \subset \mathbb{R}_+$ that

$$\lim_{x\to\infty} \int_s^t J[n](x,\tau)\,\mathrm{d}\tau = 0\,. \tag{8}$$

By combining the flux controls (7) and (8) we can show that *we do not get the expected number density conservation (2)*, but rather

$$\frac{\mathrm{d}}{\mathrm{d}t} N[n] = -n(0,t)^2\,,$$

or more precisely, we show for every $[s,t] \subset [0,\infty)$ that

$$N[n](t) + \int_s^t n(0,\tau)^2\,\mathrm{d}\tau = N[n](s)\,,$$

whereby the “mass” of the Bose-Einstein condensate is

$$\int_0^t n(0,\tau)^2\,\mathrm{d}\tau\,.$$

Therefore the condensate can only grow in time!

More is true! There exists an *onset time* $t_* \in [0, \infty]$ such that

$$n(0,t) = 0 \qquad \text{for every } t \in (0, t_*)\,,$$
$$n(0,t) > 0 \qquad \text{for every } t > t_*\,.$$

Therefore if $t_* < \infty$ then condensate growth persists over $t > t_*$!

If $n^{\mathrm{in}} \leq n_0 = x^2 f_0$ then $t_* = \infty$, so no condensate forms.

If $N[n^{\mathrm{in}}] > N[n_0]$ then $t_* \in (0, \infty)$, so a condensate forms in finite time.

If $\partial_x n^{\mathrm{in}}(0) > 1 = \partial_x n_0(0^+)$ then t_* satisfies

$$t_* \leq \frac{1}{2} \log\left(\frac{\partial_x n^{\mathrm{in}}(0)}{\partial_x n^{\mathrm{in}}(0) - 1} \right)\,.$$

Moreover, if n and m are any two such solutions then for every $t > s \geq 0$ they satisfy the *L^1-contraction inequality*

$$\int_0^\infty |n(x,t) - m(x,t)| \, \mathrm{d}x + \int_s^t |n(0,\tau) - m(0,\tau)| \, \mathrm{d}\tau \leq \int_0^\infty |n(x,s) - m(x,s)| \, \mathrm{d}x \,.$$

Finally, we address the *long-time behavior of n*. For every $\mu \geq 0$ define n_μ by $n_\mu = x^2 f_\mu$. Then there exists a unique $\mu = \mu_* \geq 0$ that solves

$$N[n_\mu] + \int_0^\infty n(0,\tau)^2 \, \mathrm{d}\tau = N[n^{\mathrm{in}}] \,. \tag{9}$$

For this μ_* we adapt the *LaSalle invariance principle* to show

$$\lim_{t\to\infty} \int_0^\infty |n(x,t) - n_{\mu_*}(x)| \, \mathrm{d}x = 0 \,.$$

A convergence rate of n to n_{μ_*} can be bounded by using the entropy. We redefine

$$H[n] = \int_0^\infty n \log(n) - (x^2 + n) \log(x^2 + n) + x^2 \log(x^2) - xn \, \mathrm{d}x \,.$$

We replace the decay assumption (5) with the stronger assumption

$$n^{\mathrm{in}}(x) \le C^{\mathrm{in}}(1 + x^2)e^{-x} \quad \text{for every } x \in \mathbb{R}_+ \,. \tag{10}$$

(This implies that $H[n^{\mathrm{in}}] \le E[n^{\mathrm{in}}] < \infty$.) Then a similar bound holds for n uniformly over $t \in \mathbb{R}_+$ and n *satisfies expected relations like*

$$\frac{\mathrm{d}}{\mathrm{d}t} E[n] = -\int_0^\infty J[n] \, \mathrm{d}x \,, \qquad \frac{\mathrm{d}}{\mathrm{d}t} H[n] = -\int_0^\infty \frac{J[n]^2}{n(x^2 + n)} \, \mathrm{d}x \,.$$

From $H[n]$ and $N[n]$ we can obtain the bound

$$\left(\int_0^\infty |n - n_{\mu_*}| \, x^2 \mathrm{d}x \right)^2 \le C \Big(H[n] - H[n_{\mu_*}] + \mu_* \big(N[n] - N[n_{\mu_*}] \big) \Big) \,,$$

where C depends only on C^{in} and μ_* is determined by (9).

If $n^{\text{in}} \geq n_0$ then $\mu_* = 0$.

If $n^{\text{in}} \leq n_0$ then $N[n_{\mu_*}] = N[n^{\text{in}}]$.

We also prove the general bound

$$N[n_{\mu_*}] \geq \int_0^\infty \min\left\{n^{\text{in}}\,,\, n_0\right\}\, \mathrm{d}x\,.$$

Remark. The foregoing uniform bounds are obtained with the aid of the *family of stationary super solutions* to the Kompaneets equation (4) that is

$$s_\gamma = x^2\Big(f_0 + \gamma f_0(1 + f_0)\Big) \quad \text{over } \gamma > 0\,.$$

Remark. Our improved understanding of the behavior of its solutions led to a *new numerical scheme* for the Kompaneets equation that captures the condensate behavior. This scheme is better than the one put forward in the 1985 paper that started this story!

7. Some Open Questions

- Can the hypotheses BC at ∞ (4c) or initial bound (10) be improved?

- Are there other criteria for the onset of a Bose-Einstein condensate?

- Can we characterize the μ_* associated with any n^{in} by equation (9)? Can we find the convergence rate of n to n_{μ_*} as $t \to \infty$?

- Do analogous results hold for other equations that govern bosons, such as the Nordheim-Uehling-Uhlenbeck quantum kinetic equations? For example, does condensate growth persist for all such models?

Thank You!
Happy 70$^{\mathrm{th}}$ Russ!