Enhanced transmission in random media with mirror symmetry

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Motivation

• Wave scattering in disordered media is an impediment for applications like imaging and free space communications.

• At distance $\lesssim$ scattering mean free path, the wave retains coherence that can be enhanced by: filtering incoherent components (optical coherence tomography, imaging in waveguides with random boundary); correcting wavefront distortion (adaptive optics); coeherent interferometry (CINT).

• Beyond a few scattering mean free paths the wave is basically incoherent and one uses radiative transfer or diffusion theories.

These do not account for wave interference between multiple scattering paths that lead to phenomena like enhanced backscattering a.k.a weak localization and also Anderson localization.

• Experimentalists have been exploiting interference to enhance wave transmission through strongly disordered media.

This requires accurate measurements of the complex wave-field (frequency by frequency) to get the transmission matrix.

"Open channels" i.e., eigenvectors corresponding to eigenvalues ≈ 1 can be identified and used to deliver waves deep inside strongly scattering media.

• **Our motivation:** Computations and experiments of Le Mans group (Chéron, Félix, Pagneaux) show wave transmission enhancement in diffusive slabs (waveguides) and cavities filled with random media that are symmetric about a wave barrier.

What have we done?

"Enhanced wave transmission in random media with mirror symmetry" by L Borcea, J Garnier, preprint arXiv:2401.15757 to appear in Proceedings of the Royal Society A

• We proved enhancement of the mean transmitted intensity in two setups where the propagation is along a preferred direction:

1. Randomly layered media, where the problem is 1-D. We quantify explicitly the effect in terms of the opacity of the barrier.

2. Random waveguides. The analysis is more complex due to complicated expressions of the moments of the reflection and transmission matrices for the propagating modes.

Setup in randomly layered media

- One dimensional wave propagation along $z-$ axis

$$\left[\begin{pmatrix}\rho(z) & 0\\ 0 & K^{-1}(z)\end{pmatrix}\partial_t + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\partial_z\right]\begin{pmatrix}u(t,z)\\ p(t,z)\end{pmatrix} = \mathbf{0}, \quad t\in\mathbb{R}, \quad z\in\mathbb{R}$$

- Medium contains barrier at $|z| < d/2$ and is modeled by density

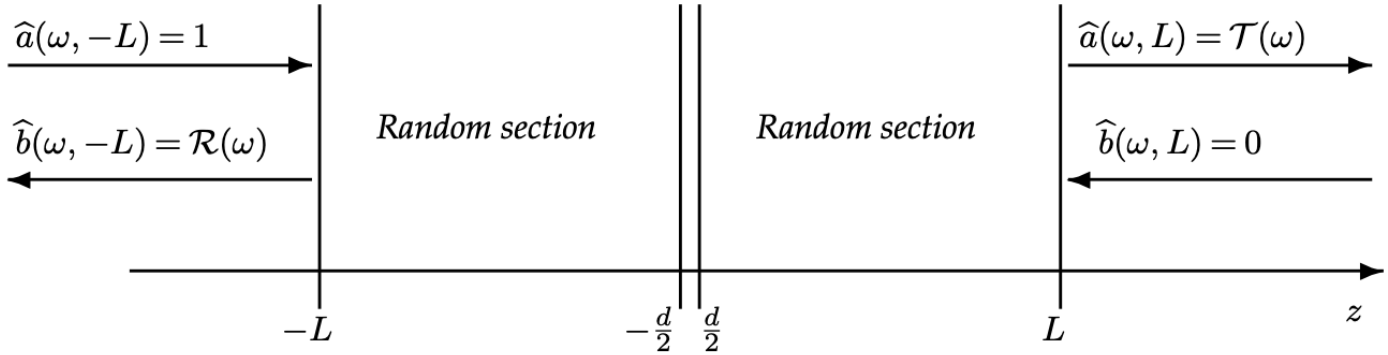
$$\rho(z) = \begin{cases}\rho_0 & \text{if } |z| \geq d/2,\\ \rho_1 & \text{if } |z| < d/2,\end{cases}$$

and bulk modulus

$$\frac{1}{K(z)} = \begin{cases}\frac{1}{K_0}\left[1 + 1_{[d/2,L]}(|z|)\mu(|z|)\right] & \text{if } |z| \geq d/2,\\ \frac{1}{K_1} & \text{if } |z| < d/2,\end{cases}$$

Wave speed is $c(z) = \sqrt{K(z)/\rho(z)}$ and impedance $\zeta(z) = c(z)\rho(z)$

Formulation of problem



- Decomposition in forward/back going waves $\widehat{a}$ and $\widehat{b}$ at $|z| \geq d/2$

$$p(t,z) = \frac{\zeta_0^{1/2}}{4\pi} \int_{\mathbb{R}} d\omega \, e^{-i\omega t} \left[\widehat{a}(\omega,z) e^{i\omega \frac{z}{c_0}} - \widehat{b}(\omega,z) e^{-i\omega \frac{z}{c_0}} \right]$$

$$u(t,z) = \frac{\zeta_0^{-1/2}}{4\pi} \int_{\mathbb{R}} d\omega \, e^{-i\omega t} \left[\widehat{a}(\omega,z) e^{i\omega \frac{z}{c_0}} + \widehat{b}(\omega,z) e^{-i\omega \frac{z}{c_0}} \right]$$

- **Goal**: Analyze the transmitted wave $\mathcal{T}(\omega) = \widehat{a}(\omega, L)$

Model of the barrier

- From continuity of the pressure and velocity at $z = \pm d/2 \quad \rightsquigarrow$

Lemma 1: $\begin{pmatrix} \widehat{a}(\omega, d/2) \\ \widehat{b}(\omega, d/2) \end{pmatrix} = \mathbf{P}_1(\omega) \begin{pmatrix} \widehat{a}(\omega, -d/2) \\ \widehat{b}(\omega, -d/2) \end{pmatrix}$ with propagator matrix

$$\mathbf{P}_1(\omega) = \begin{pmatrix} \alpha(\omega) & \frac{\overline{\gamma(\omega)}}{\alpha(\omega)} \\ \gamma(\omega) & \overline{\alpha(\omega)} \end{pmatrix}$$

where the bar denotes complex conjugate and

$$\alpha(\omega) = \left[\cos\left(\frac{\omega d}{c_1}\right) + \frac{i}{2}\left(\frac{\zeta_1}{\zeta_0} + \frac{\zeta_0}{\zeta_1}\right) \sin\left(\frac{\omega d}{c_1}\right) \right] e^{-i\omega d/c_0}$$

$$\gamma(\omega) = \frac{i}{2}\left(\frac{\zeta_0}{\zeta_1} - \frac{\zeta_1}{\zeta_0}\right) \sin\left(\frac{\omega d}{c_1}\right)$$

Model of the barrier

- Scattering matrix $\mathbf{S}_1$ maps incident waves to the outgoing ones

$$\begin{pmatrix} \widehat{a}(\omega, d/2) \\ \widehat{b}(\omega, -d/2) \end{pmatrix} = \mathbf{S}_1(\omega) \begin{pmatrix} \widehat{a}(\omega, -d/2) \\ \widehat{b}(\omega, d/2) \end{pmatrix}$$

Its entries are the transmission and reflection coefficients

$$\mathbf{S}_1(\omega) = \begin{pmatrix} T_1(\omega) & R_1(\omega) \\ R_1(\omega) & T_1(\omega) \end{pmatrix}, \quad R_1(\omega) = -\frac{\overline{\gamma(\omega)}}{\overline{\alpha(\omega)}}, \quad T_1(\omega) = \frac{1}{\overline{\alpha(\omega)}}$$

- Asymptotic regime

$$\frac{\zeta_0}{\zeta_1} \gg 1 \quad \text{and} \quad \frac{\omega d}{c_1} \ll 1 \quad \text{s.t.} \quad \frac{\zeta_0}{\zeta_1}\frac{\omega d}{2c_1} = q(\omega) = O(1)$$

This gives: $T_1(\omega) = \dfrac{i}{i + q(\omega)}$ and $R_1(\omega) = \dfrac{q(\omega)}{i + q(\omega)}$

- Propagator $\mathbf{P}_+$ defined by

$$\begin{pmatrix} \widehat{a}(\omega,z) \\ \widehat{b}(\omega,z) \end{pmatrix} = \mathbf{P}_+(\omega,z) \begin{pmatrix} \widehat{a}(\omega,d/2) \\ \widehat{b}(\omega,d/2) \end{pmatrix}, \qquad z \in \Big(d/2, L\Big]$$

and equals $\mathbf{I}$ at $z = d/2$

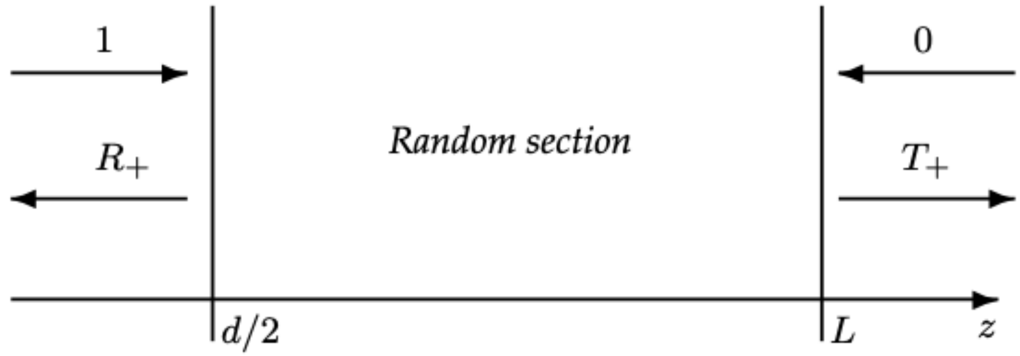
- It is known that $\mathbf{P}_+(\omega,z) = \begin{pmatrix} \alpha_+(\omega,z) & \overline{\gamma_+(\omega,z)} \\ \gamma_+(\omega,z) & \overline{\alpha_+(\omega,z)} \end{pmatrix}$ where

$$\frac{d}{dz}\begin{pmatrix} \alpha_+(\omega,z) \\ \gamma_+(\omega,z) \end{pmatrix} = \frac{i\omega}{2c_0}\mu(z) \begin{pmatrix} 1 & -e^{-2i\omega z/c_0} \\ e^{2i\omega z/c_0} & -1 \end{pmatrix} \begin{pmatrix} \alpha_+(\omega,z) \\ \gamma_+(\omega,z) \end{pmatrix}$$

- It satisfies the energy conservation relation

$$\partial_z \det[\mathbf{P}_+(\omega,z)] = 0 \quad \rightsquigarrow \quad |\alpha_+(\omega,z)|^2 - |\gamma_+(\omega,z)|^2 = 1.$$

Reflection and transmission coefficients for right section



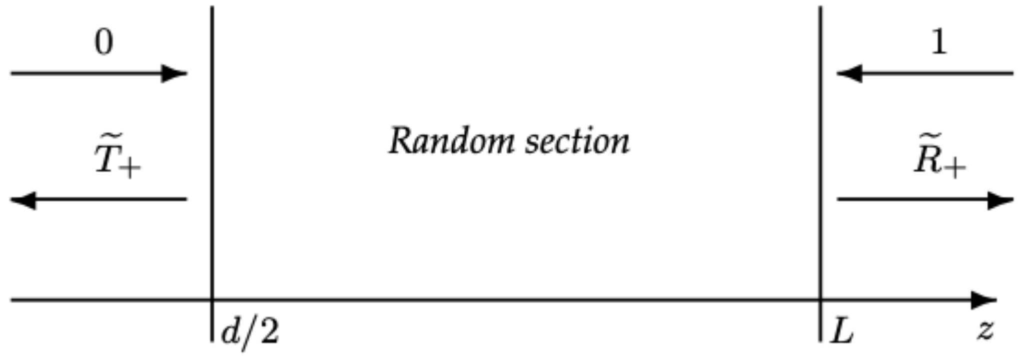
- Schematic shows: $\begin{pmatrix} T_+(\omega) \\ 0 \end{pmatrix} = \mathbf{P}_+(\omega, L) \begin{pmatrix} 1 \\ R_+(\omega) \end{pmatrix}$ where

$$T_+(\omega) = \frac{1}{\overline{\overline{\alpha_+(\omega, L)}}}, \qquad R_+(\omega) = -\frac{\gamma_+(\omega, L)}{\overline{\overline{\alpha_+(\omega, L)}}}$$

- Conservation of energy:

$$1 = \frac{1}{|\alpha_+(\omega, L)|^2} + \frac{|\gamma_+(\omega, L)|^2}{|\alpha_+(\omega, L)|^2} = |T_+(\omega)|^2 + |R_+(\omega)|^2$$

Adjoint setup - needed for mirror symmetry



- Schematic shows: $\begin{pmatrix} \widetilde{R}_+(\omega) \\ 1 \end{pmatrix} = \mathbf{P}_+(\omega, L) \begin{pmatrix} 0 \\ \widetilde{T}_+(\omega) \end{pmatrix}$ where

$$\widetilde{T}_+(\omega) = T_+(\omega) = \frac{1}{\overline{\alpha_+(\omega, L)}}, \qquad \widetilde{R}_+(\omega) = \frac{\overline{\gamma_+(\omega, L)}}{\overline{\alpha_+(\omega, L)}}$$

- Energy conservation

$$|\widetilde{T}_+(\omega)|^2 + |\widetilde{R}_+(\omega)|^2 = 1$$

Lemma 2: Scattering matrices of random sections defined by

$$\begin{pmatrix} \widehat{a}(\omega,-d/2) \\ \widehat{b}(\omega,-L) \end{pmatrix} = \mathbf{S}_-(\omega) \begin{pmatrix} \widehat{a}(\omega,-L) \\ \widehat{b}(\omega,-d/2) \end{pmatrix}$$

and

$$\begin{pmatrix} \widehat{a}(\omega,L) \\ \widehat{b}(\omega,d/2) \end{pmatrix} = \mathbf{S}_+(\omega) \begin{pmatrix} \widehat{a}(\omega,d/2) \\ \widehat{b}(\omega,L) \end{pmatrix}$$

Their algebraic structure is

$$\mathbf{S}_\pm(\omega) = \begin{pmatrix} T_\pm(\omega) & \widetilde{R}_\pm(\omega) \\ R_\pm(\omega) & T_\pm(\omega) \end{pmatrix}$$

and due to the symmetry

$$T_-(\omega) = T_+(\omega), \qquad R_-(\omega) = \widetilde{R}_+(\omega), \qquad \widetilde{R}_-(\omega) = R_+(\omega).$$

Transmission through the system

- Propagator $\mathcal{P}(\omega)$ of the system satisfies $\begin{pmatrix} 1 \\ \mathcal{R}(\omega) \end{pmatrix} = \mathcal{P}(\omega) \begin{pmatrix} \mathcal{T}(\omega) \\ 0 \end{pmatrix}$

- $\mathcal{P}(\omega) =$ product of the propagators of the three sections

- We can solve for the transmission and reflection coefficients

We are interested in $\mathcal{T}(\omega)$

Theorem:* The transmission coefficient of the system is

$$\mathcal{T}(\omega) = T_+^2(\omega) T_1(\omega) [1 - R_+(\omega)]^{-1} \Big[1 - \Big(2R_1(\omega) - 1\Big) R_+(\omega)\Big]^{-1}$$

and the expression of the mean transmitted intensity is

$$\mathbb{E}\Big[\Big|\mathcal{T}(\omega)\Big|^2\Big] = \sum_{k=0}^{\infty} \tau_k(\omega) \mathbb{E}\left[|T_+(\omega)|^4 \Big(1 - |T_+(\omega)|^2\Big)^k\right],$$

where

$$\tau_k(\omega) = \frac{1}{4}\Big|1 - (2R_1(\omega) - 1)^{k+1}\Big|^2$$

*Proof uses Neumann series and known moment formulas

Interpretation of result

• For barrier: $|2R_1(\omega) - 1| = \left|\dfrac{q(\omega) - i}{q(\omega) + i}\right| = 1 \rightsquigarrow |\tau_k(\omega)| \leq 1$

This implies

$$\mathbb{E}\Big[\big|\mathcal{T}(\omega)\big|^2\Big] \leq \mathbb{E}\left[|T_+(\omega)|^4 \sum_{k=0}^{\infty} \left(1 - |T_+(\omega)|^2\right)^k\right] = \mathbb{E}\left[|T_+(\omega)|^2\right]$$

• This says that independent of the barrier, the mean transmitted intensity cannot exceed that of a single random section

The latter depends on the covariance of the random fluctuations $\mu(z)$, the distance L of propagation and the wavelength

- For any $n \in \mathbb{N}$

$$\mathbb{E}\Big[|T_+(\omega)|^{2n}\Big] = \exp\left(-\frac{L}{4L_{\mathrm{loc}}(\omega)}\right)\int_0^\infty e^{-\frac{Ls^2}{L_{\mathrm{loc}}(\omega)}}\frac{2\pi s\sinh(\pi s)}{\cosh^2(\pi s)}\phi_n(s)ds$$

where

$$\phi_1(s) = 1, \qquad \phi_n(s) = \prod_{j=1}^{n-1}\frac{s^2+(j-\frac{1}{2})^2}{j^2}, \quad n \geq 2,$$

and

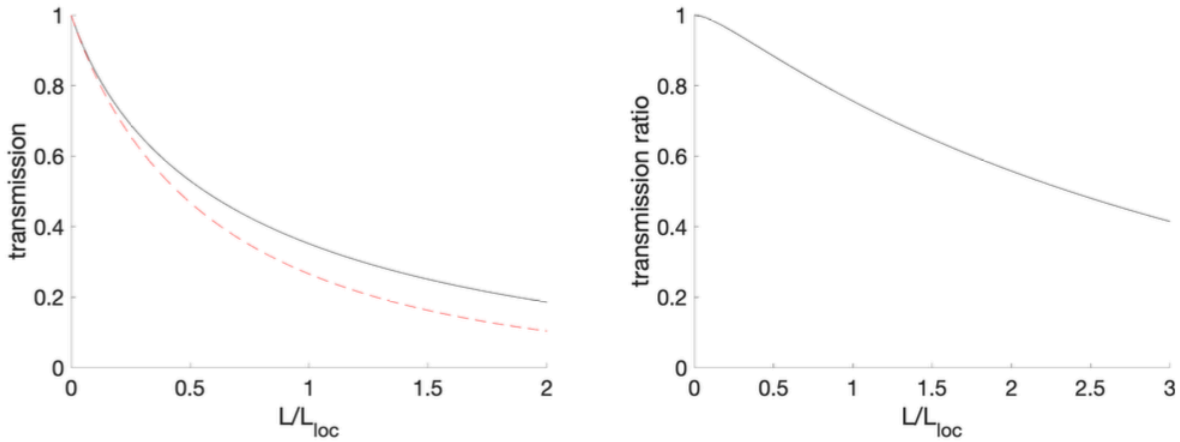
$$\frac{1}{L_{\mathrm{loc}}(\omega)} = \frac{\omega^2}{4c_0^2}\int_{\mathbb{R}}\mathbb{E}[\mu(0)\mu(z)]dz$$

*In strongly heterogeneous white-noise regime

- If $R_1(\omega) = 0$, coefficients $\tau_k = 0$ for odd k and 1 for even k $\rightsquigarrow$

$$\mathbb{E}\Big[\big|\mathcal{T}(\omega)\big|^2\Big] = \mathbb{E}\left[|T_+(\omega)|^4 \sum_{k=0}^{\infty} \Big(1 - |T_+(\omega)|^2\Big)^{2k}\right]$$
$$= \mathbb{E}\left[\frac{|T_+(\omega)|^2}{2 - |T_+(\omega)|^2}\right] < \mathbb{E}\Big[|T_+(\omega)|^2\Big]$$

- Comparison with transmission through independent sections (red dotted line) shows that symmetry helps



Effect of barrier opacity

• If the random medium is weak ($L \ll L_{\text{loc}}$) we can show that

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] \approx |T_1(\omega)|^2 \left\{1 + 2\left(1 - 2|T_1(\omega)|^2\right)\mathbb{E}[|R_+(\omega)|^2]\right\}$$

where due to conservation, $\mathbb{E}[|R_+(\omega)|^2] = 1 - \mathbb{E}[|T_+(\omega)|^2]$

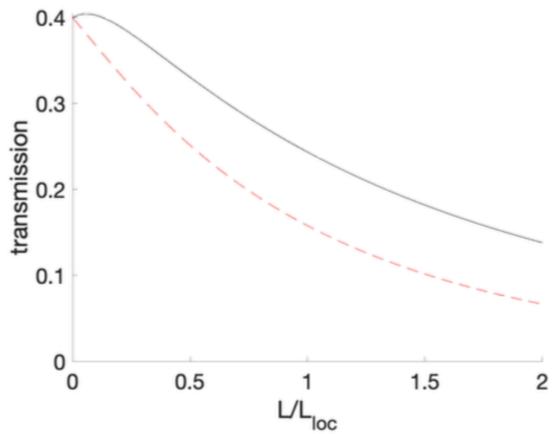
• This is larger than $|T_1(\omega)|^2$ if the barrier is opaque enough i.e., for $|T_1(\omega)| < 1/\sqrt{2}$.

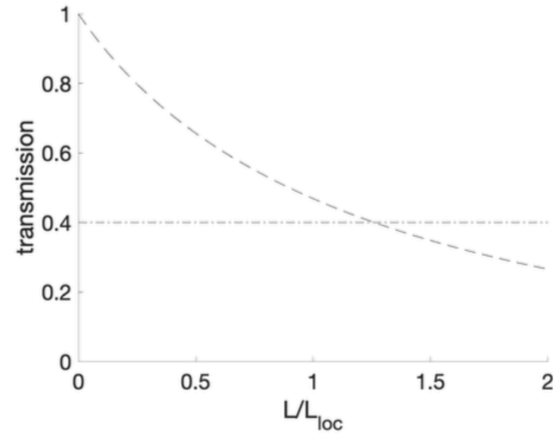
Thus, we need strong enough scattering at the barrier to get the useful interference due to the symmetry

• Conclusion extends to strong media ($L > L_{\text{loc}}$) where the expression of the mean intensity is more complicated

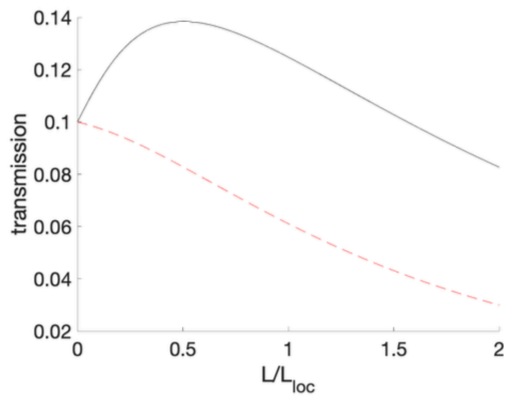
Effect of barrier opacity - strong random media

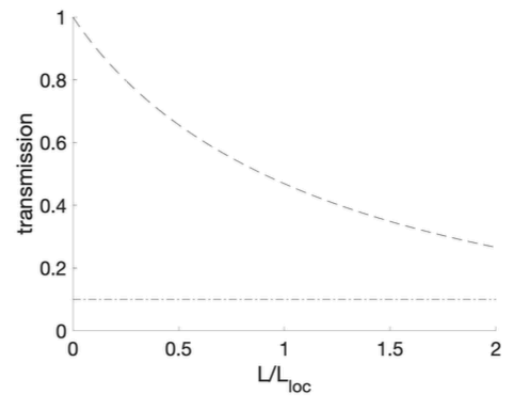
$|T_1(\omega)|^2 = 0.4$





$|T_1(\omega)|^2 = 0.1$





Left plots: Mean transmitted intensity with symmetry (black) and with independent media (red)

Right plots: Mean transmitted intensity for one random section (dashed) and for barrier (dot-dash)

Conclusion

• We analyzed a striking phenomenon of enhanced transmission through a random system, that does not require any measurements and wavefront shaping

• The enhancement is due entirely to constructive interference between symmetric scattering processes

• Symmetric random systems do not arise naturally in applications, but they can be manufactured. In addition, they are quite relevant when analyzing wave propagation in a half space with reflecting boundary (method of images Gomez et al)