Rotating convection at "extreme" parameters on a logarithmic Fourier lattice

Keaton Burns (MIT), Steve Tobias (Leeds),
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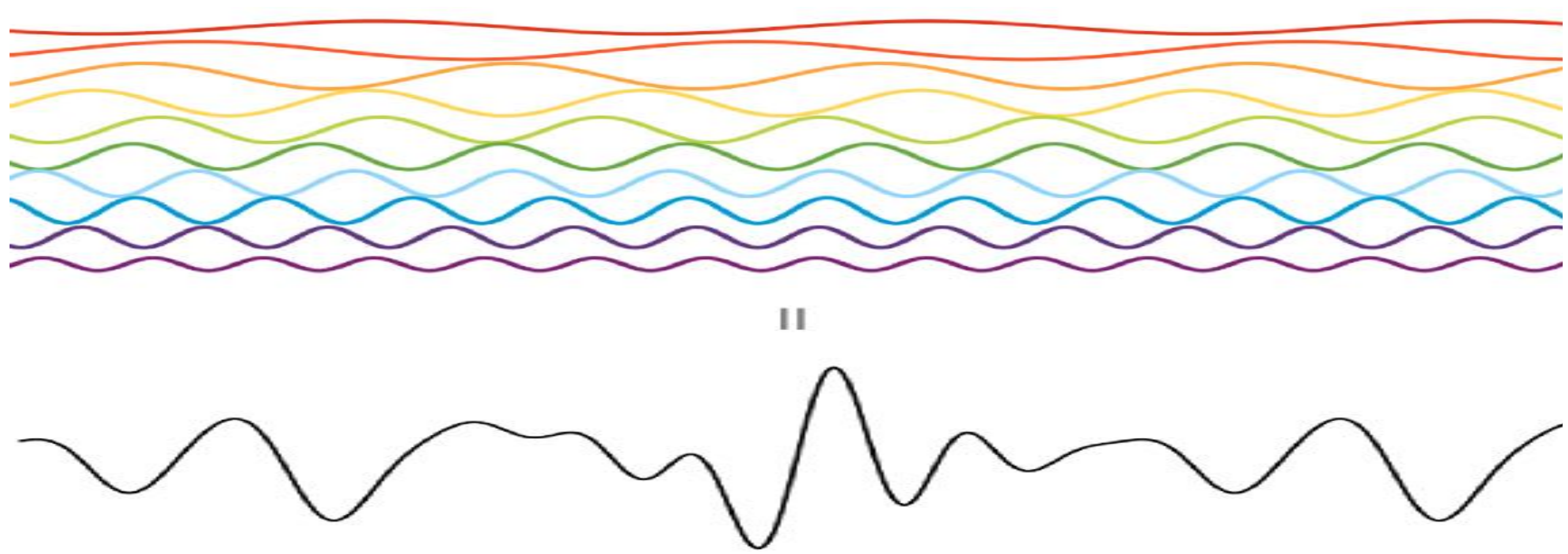
IPAM 2025

Fourier spectral methods

Fourier series: $\phi_n(x) = e^{inx}$

- **Exponential convergence** for smooth **periodic** functions
- **Fast transforms** for computing coefficients
- **Diagonal** differential operators:

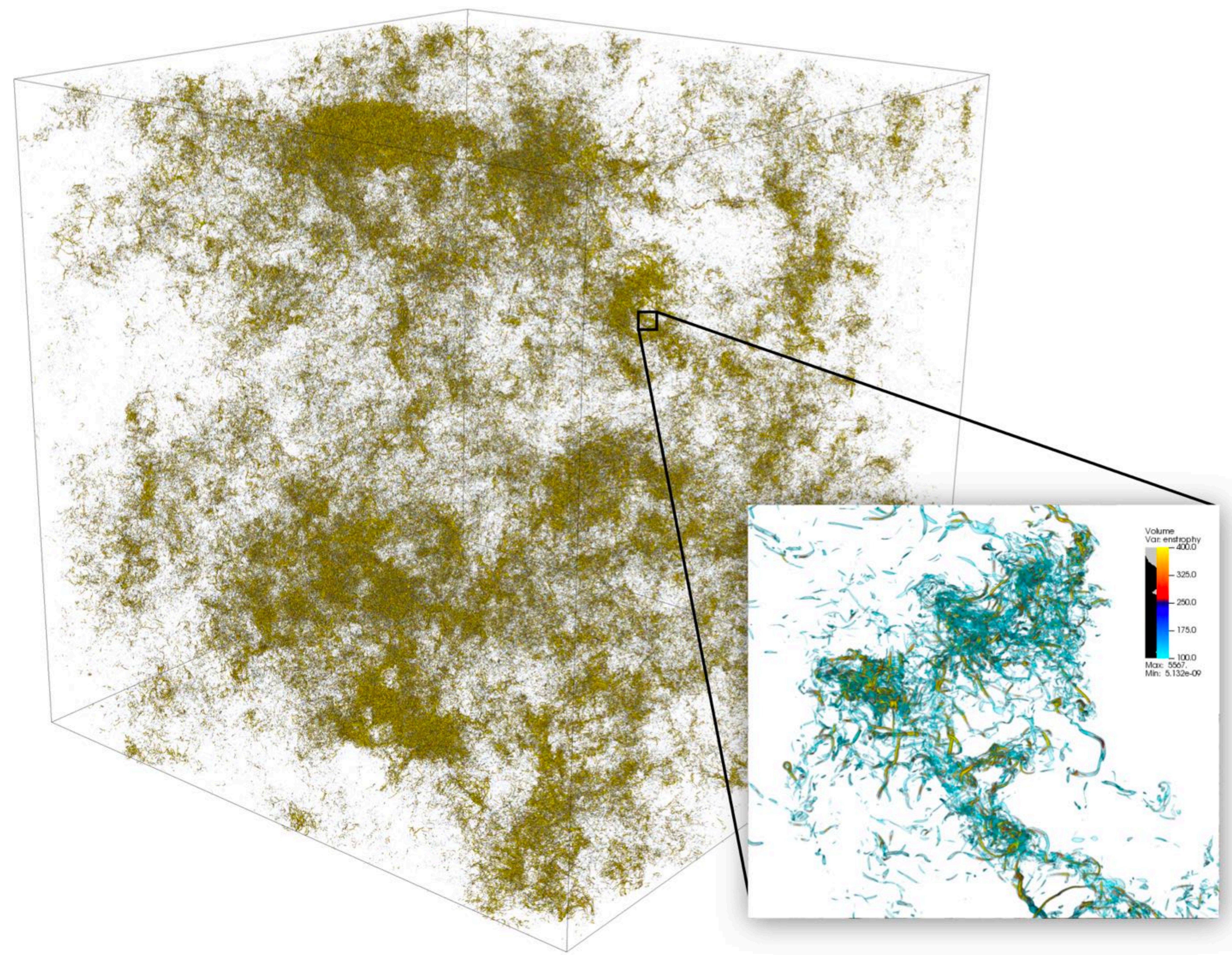
$$\langle \phi_m | \partial_x \phi_n \rangle = in\delta_{m,n}$$



World's largest turbulence simulations

Yeung & Ravikumar (2024)

- Fourier pseudospectral method
- $32{,}768^3$ grid points
- 32,768 GPUs (Frontier)
- $R_\lambda \approx 2550$



Cost of Direct Numerical Simulations (DNS)

Spatial dynamic range:

$$N_x = \frac{L_x}{\Delta x} \sim \mathrm{Re}^{3/4}$$

Temporal dynamic range:

$$N_t = \frac{T}{\Delta t} \sim \frac{L_x}{\Delta x}$$

Simulation degrees of freedom (DOF):

$$N = N_x^3 N_t \sim \mathrm{Re}^3$$

Decade of Moore's Law ~ Decade in Reynolds

DNS today: $\mathrm{Re} \sim 10^6$

DNS of Earth: ~50 years

DNS of Sun: ~90 years

“Log-Lattice” Fourier discretizations

Fourier discretization:

$$u(x) = \sum_k \hat{u}(k) e^{ikx}$$

Standard (linear) wavenumbers:

$$k \in \frac{2\pi}{L}\mathbb{Z} \qquad N_k \sim \frac{k}{k_0} \sim \frac{L}{\Delta x}$$

Logarithmic wavenumbers:

$$k \in \frac{2\pi}{L}\{\pm\lambda^n\}_{n\in\mathbb{N}}$$

Spacing factor λ

$$N_k \sim \log(k/k_0) \sim \log(L/\Delta x)$$

Properties:

$$\hat{u}(-k) = \hat{u}^*(k)$$

$$\widehat{\partial_x u}\,(k) = i\,k\,\hat{u}(k)$$

$$(u, v) = \sum_k \hat{u}(k)\,\hat{v}^*(k)$$

$$(\partial_x u, v) = -\,(u, \partial_x v)$$

Correct linear dynamics

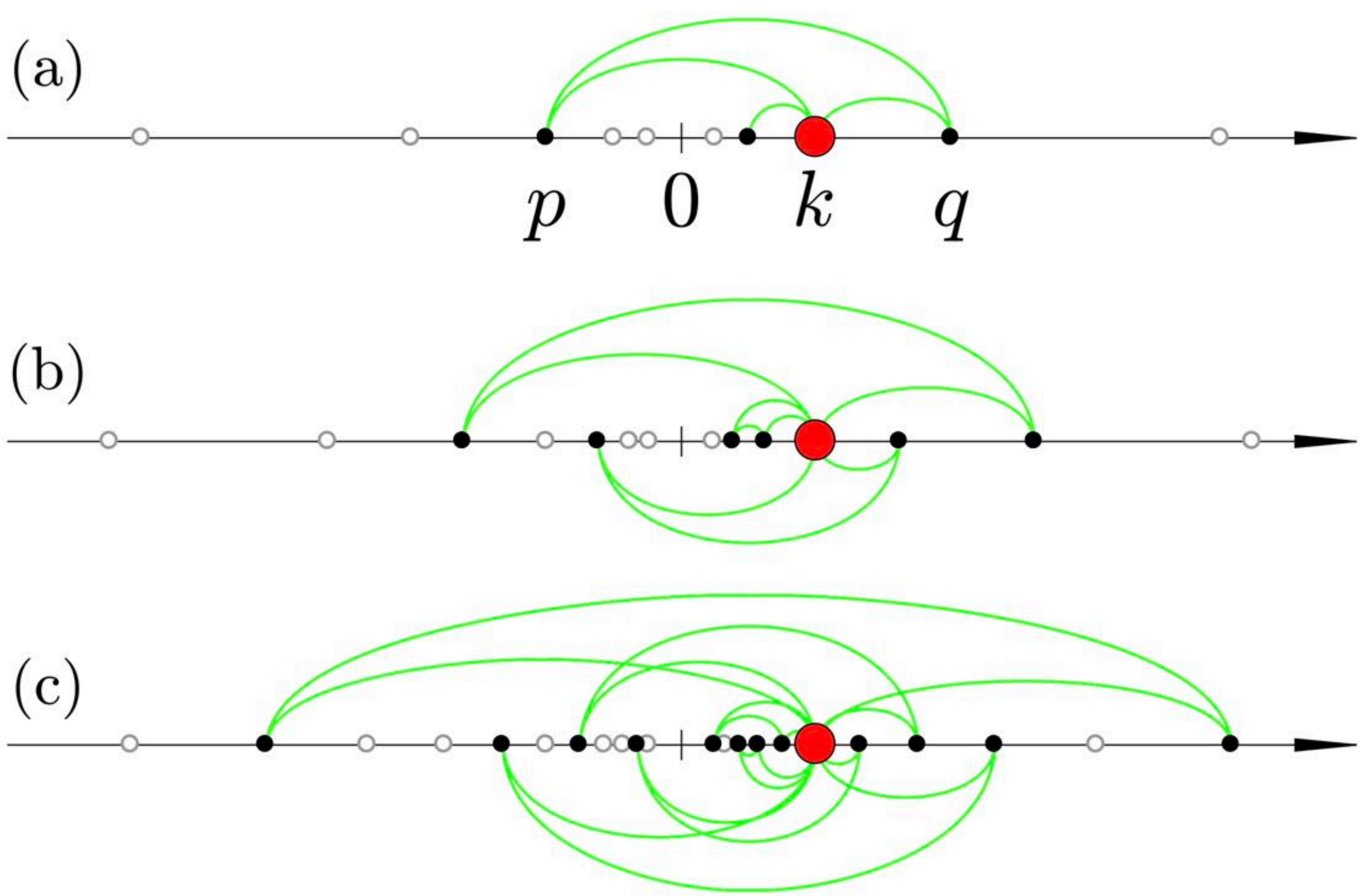
Log-Lattice triads

Theorem 1. *The following three cases describe all nondegenerate lattices with spacing factors $\lambda \geqslant 1.05$:*

(*a*) $\lambda = 2$, *and all triads at the unity are given in table 1(a);*

(*b*) $\lambda = \sigma$, *the plastic number (2), and all triads at the unity are given in table 1(c);*

(*c*) λ *satisfies* $1 = \lambda^b - \lambda^a$, *where* (a, b) *are mutually prime integers not larger than 62, excluding also the pairs* $(a, b) = (1, 3)$ *and* $(4, 5)$. *All triads at the unity are given in table 1(b).*



(a)

i	1	2	3
p_i	2	-1	1/2
q_i	-1	2	1/2

(b)

i	1	2	3	4	5	6
p_i	λ^b	$-\lambda^a$	λ^{b-a}	$-\lambda^{-a}$	λ^{-b}	λ^{a-b}
q_i	$-\lambda^a$	λ^b	$-\lambda^{-a}$	λ^{b-a}	λ^{a-b}	λ^{-b}

(c)

i	1	2	3	4	5	6	7	8	9	10	11	12
p_i	σ^3	$-\sigma$	σ^2	$-\sigma^{-1}$	σ^{-3}	σ^{-2}	σ^5	$-\sigma^4$	σ	$-\sigma^{-4}$	σ^{-5}	σ^{-1}
q_i	$-\sigma$	σ^3	$-\sigma^{-1}$	σ^2	σ^{-2}	σ^{-3}	$-\sigma^4$	σ^5	$-\sigma^{-4}$	σ	σ^{-1}	σ^{-5}

Campolina & Mailybaev (2021)

Log-Lattice triads

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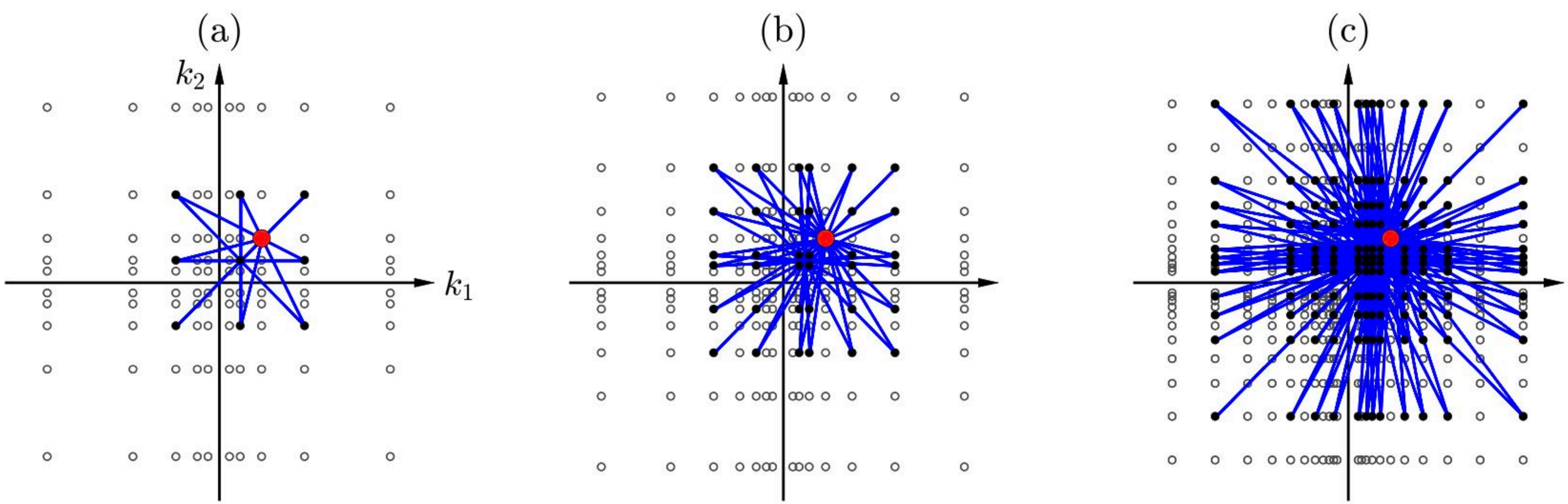


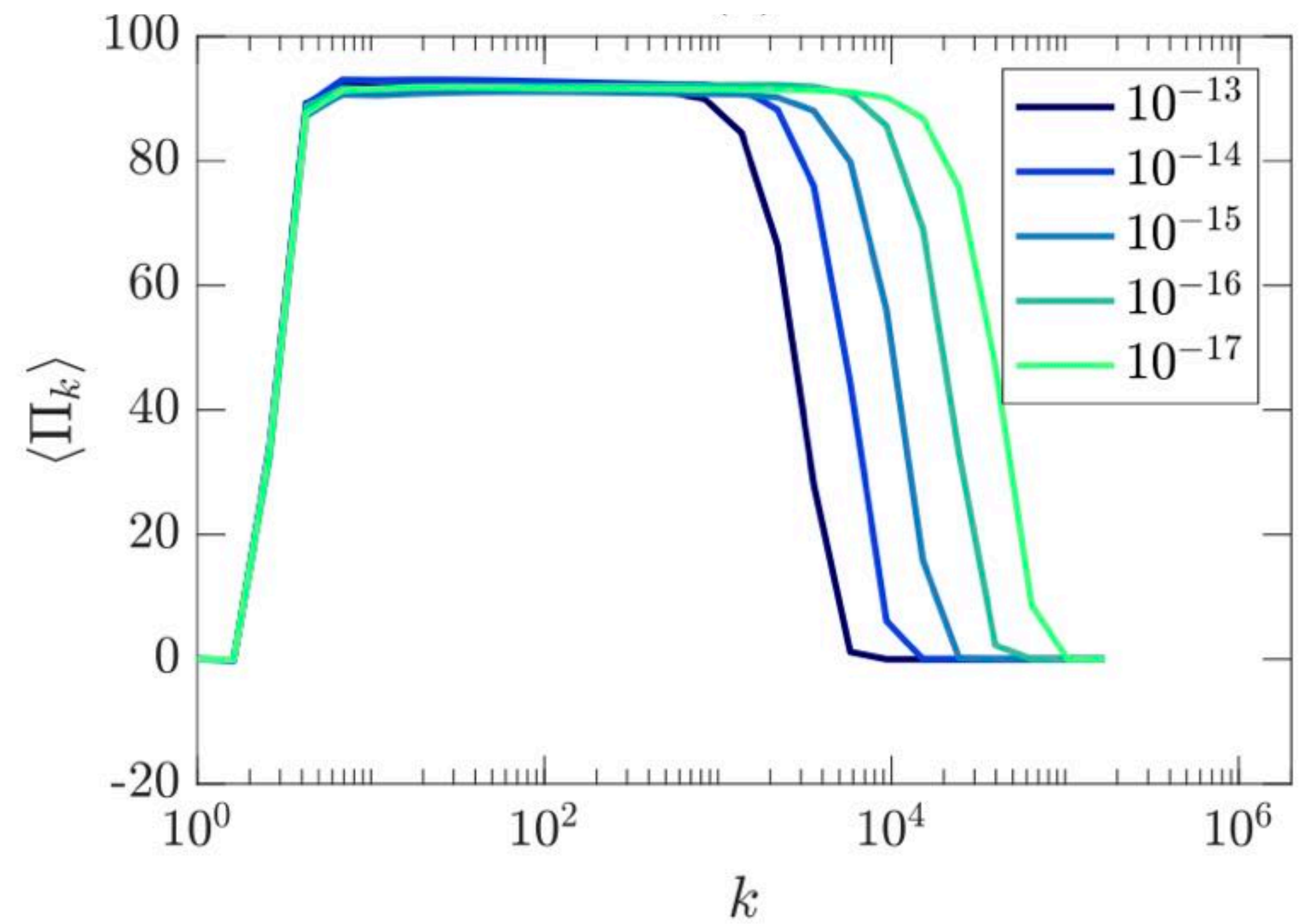
Campolina & Mailybaev (2021)

Log-Lattice isotropic turbulence

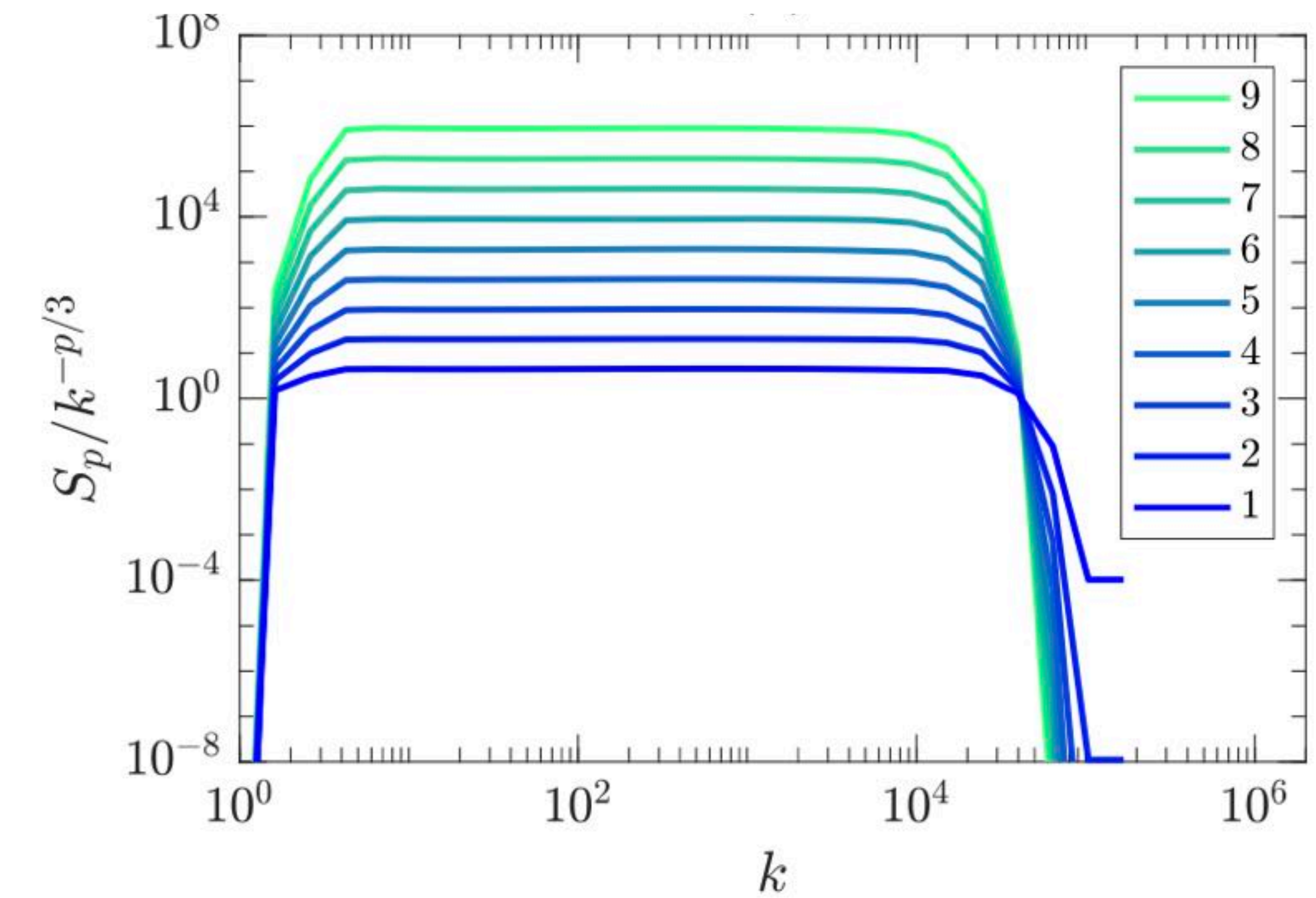
Campolina & Mailybaev (2021)

- 3D log-lattice
- Forward cascade, wrong slope
- Good structure function scaling
- Good velocity PDFs

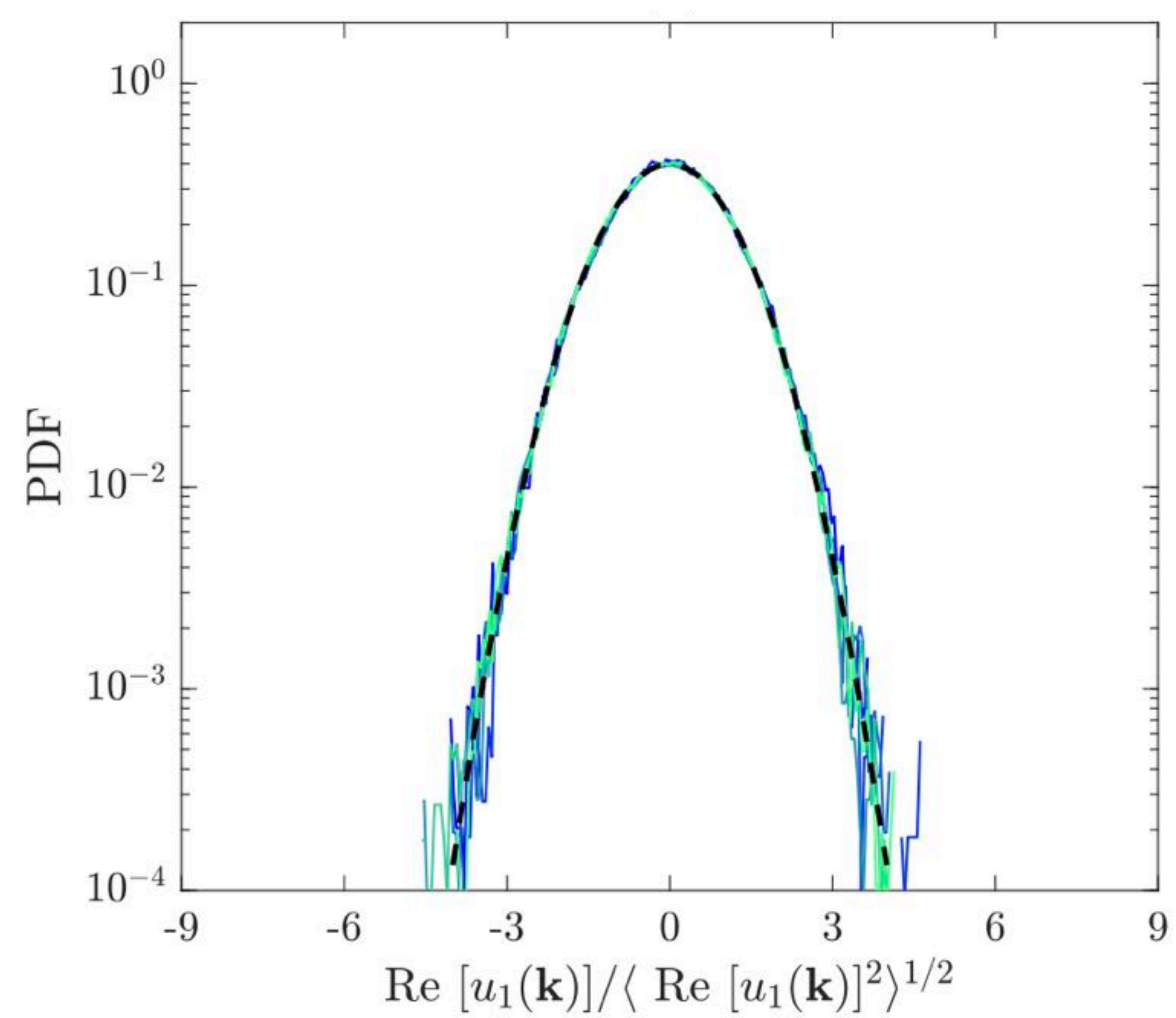
Mean energy flux



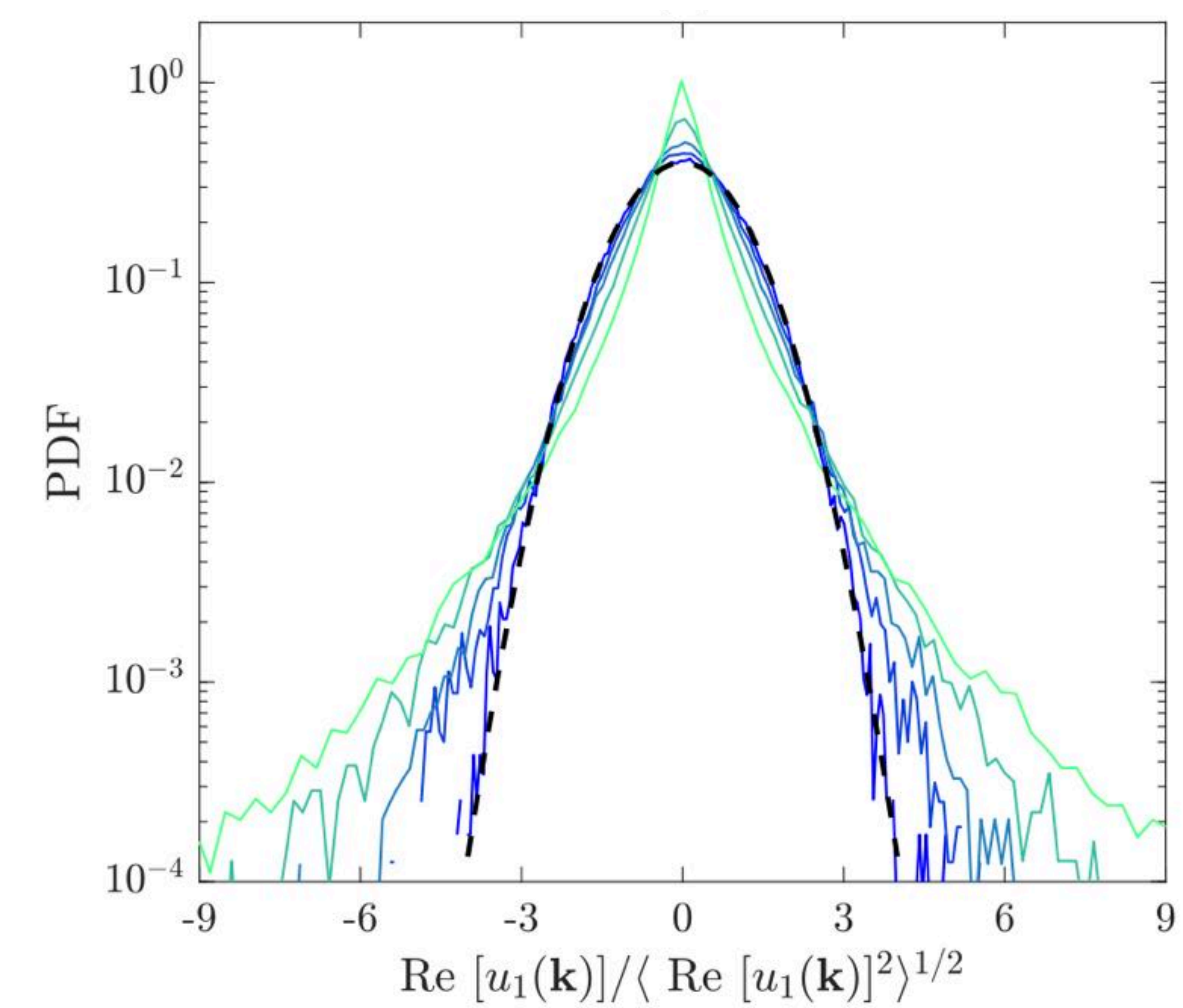
Comp. structure func.



Inertial range velocity PDF



Viscous range velocity PDF

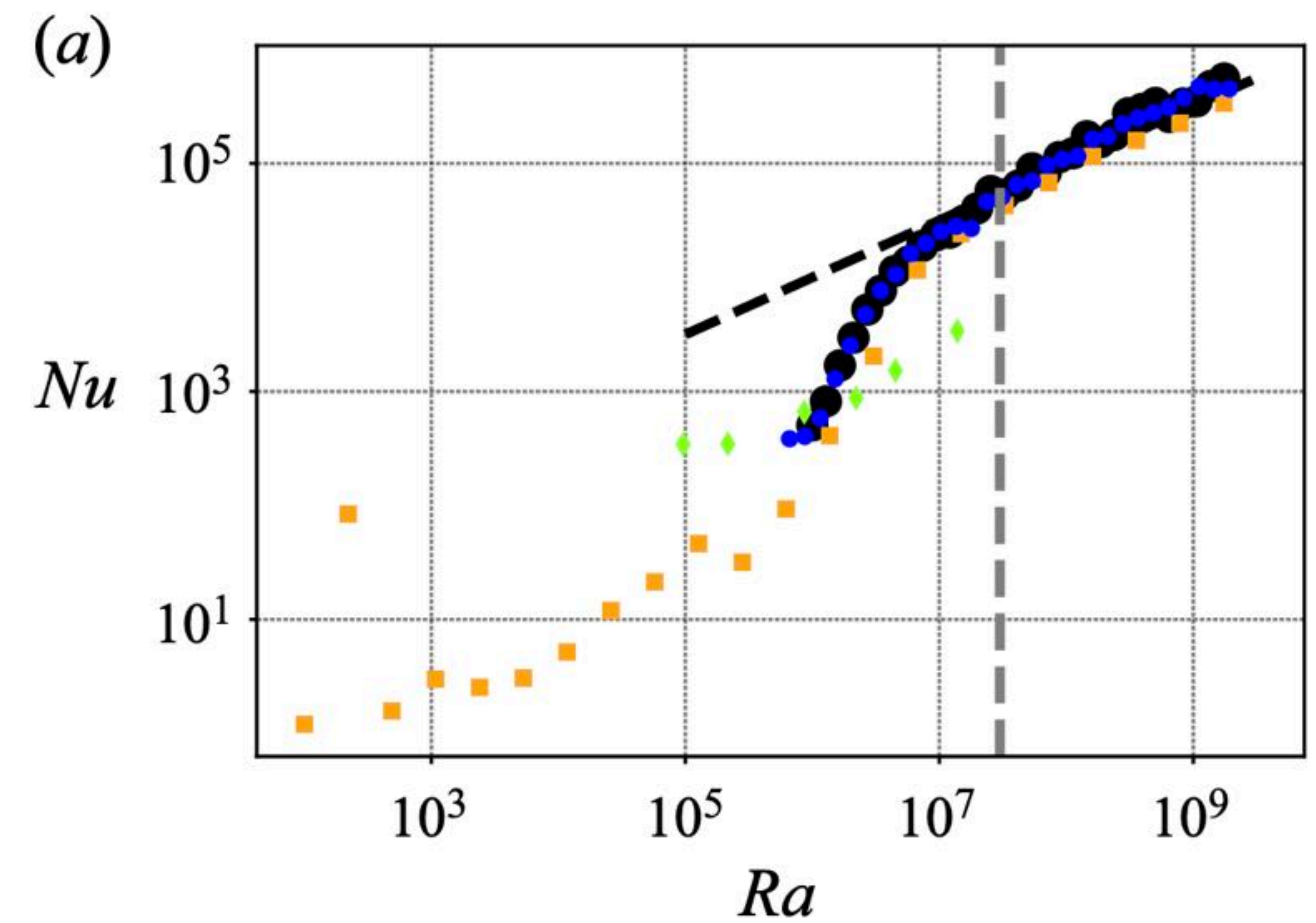


Log-Lattice homogeneous RBC

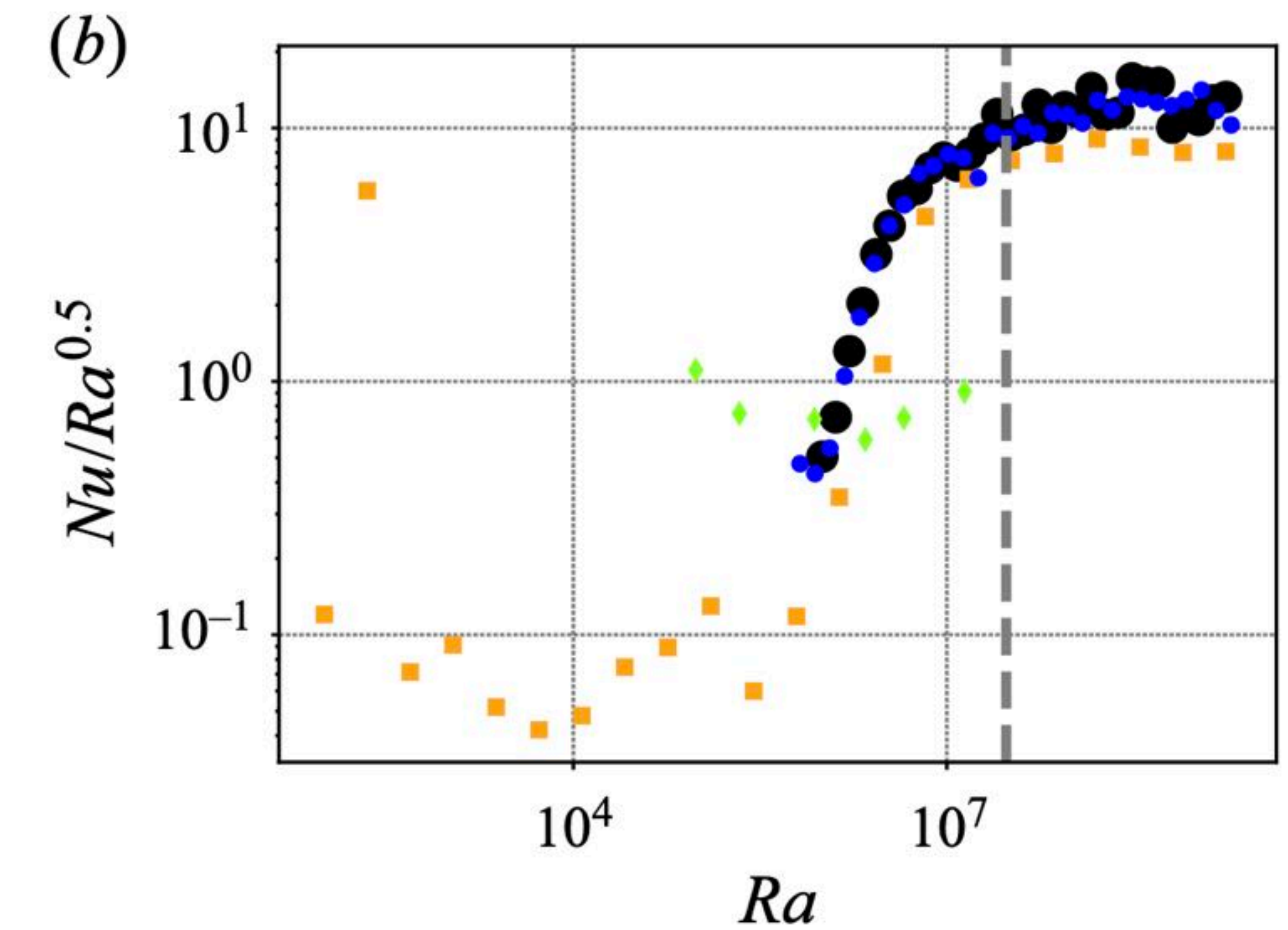
Barral & Dubrulle (2023)

- 3D log-lattice
- Need large-scale drag
- Friction modifies low Pr
- See hom. “ultimate regime” with GL scalings for Nu, Re

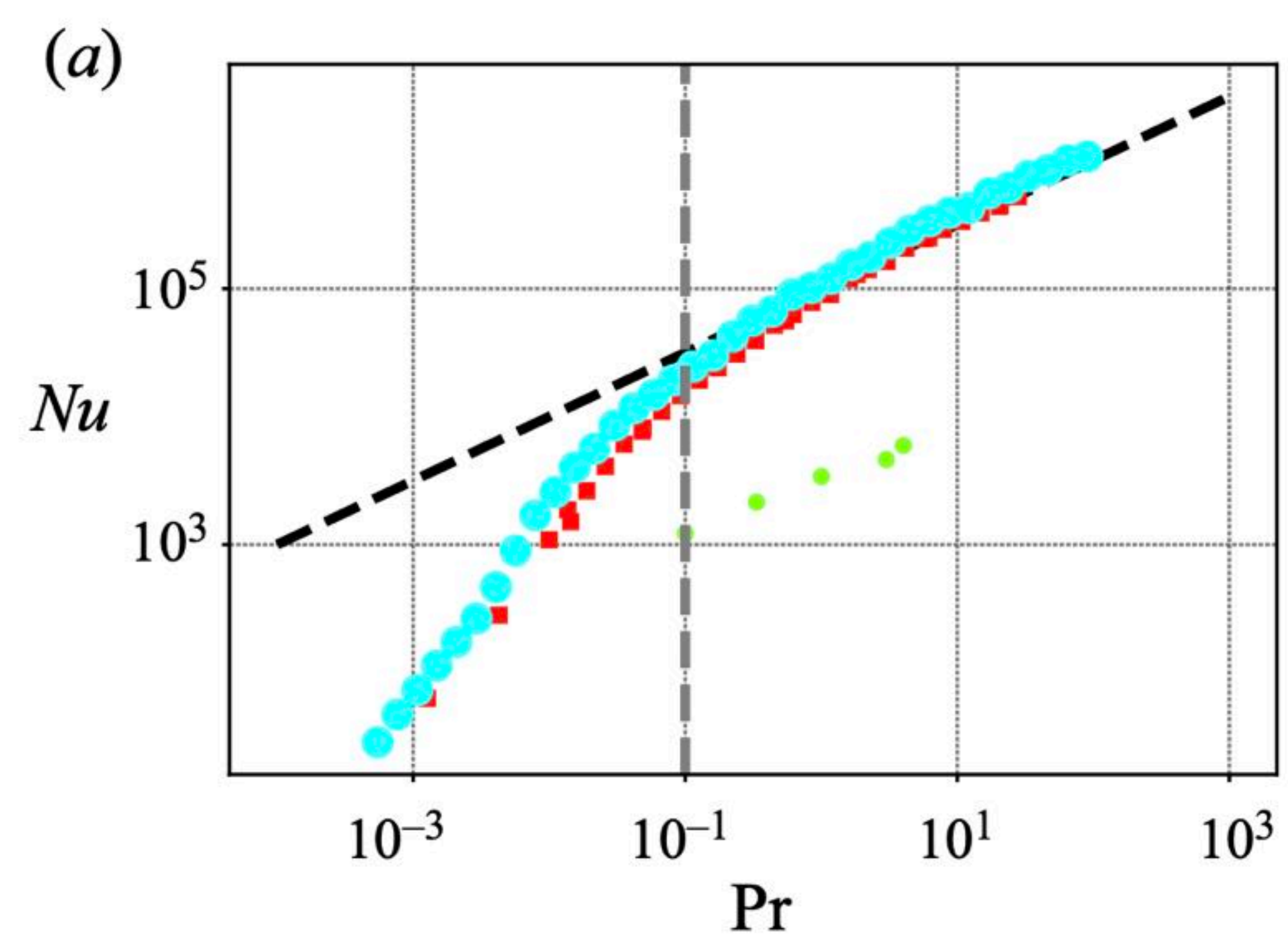
Nusselt at Pr=1



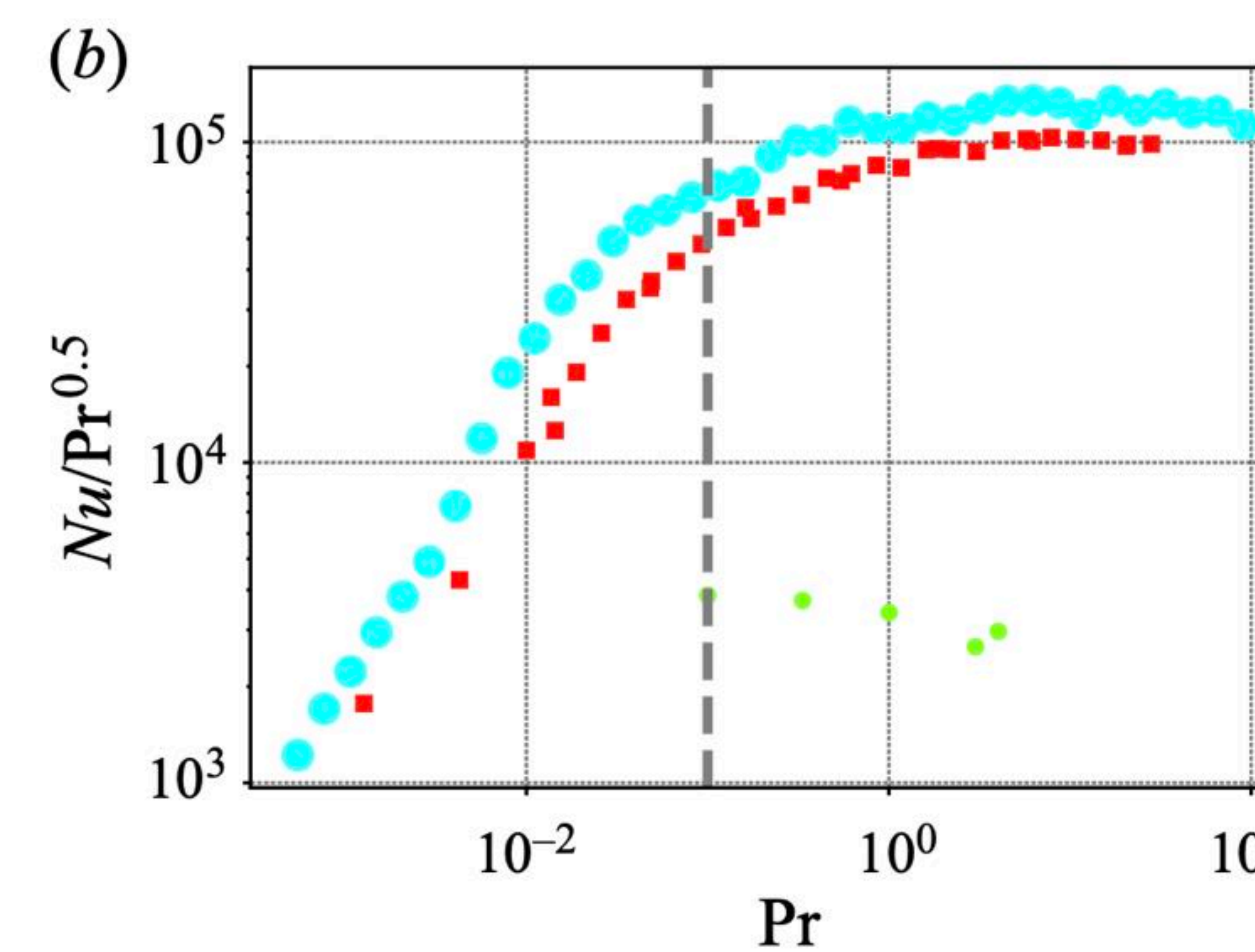
Nu / Ra$^{1/2}$



Nusselt at Ra=10^8



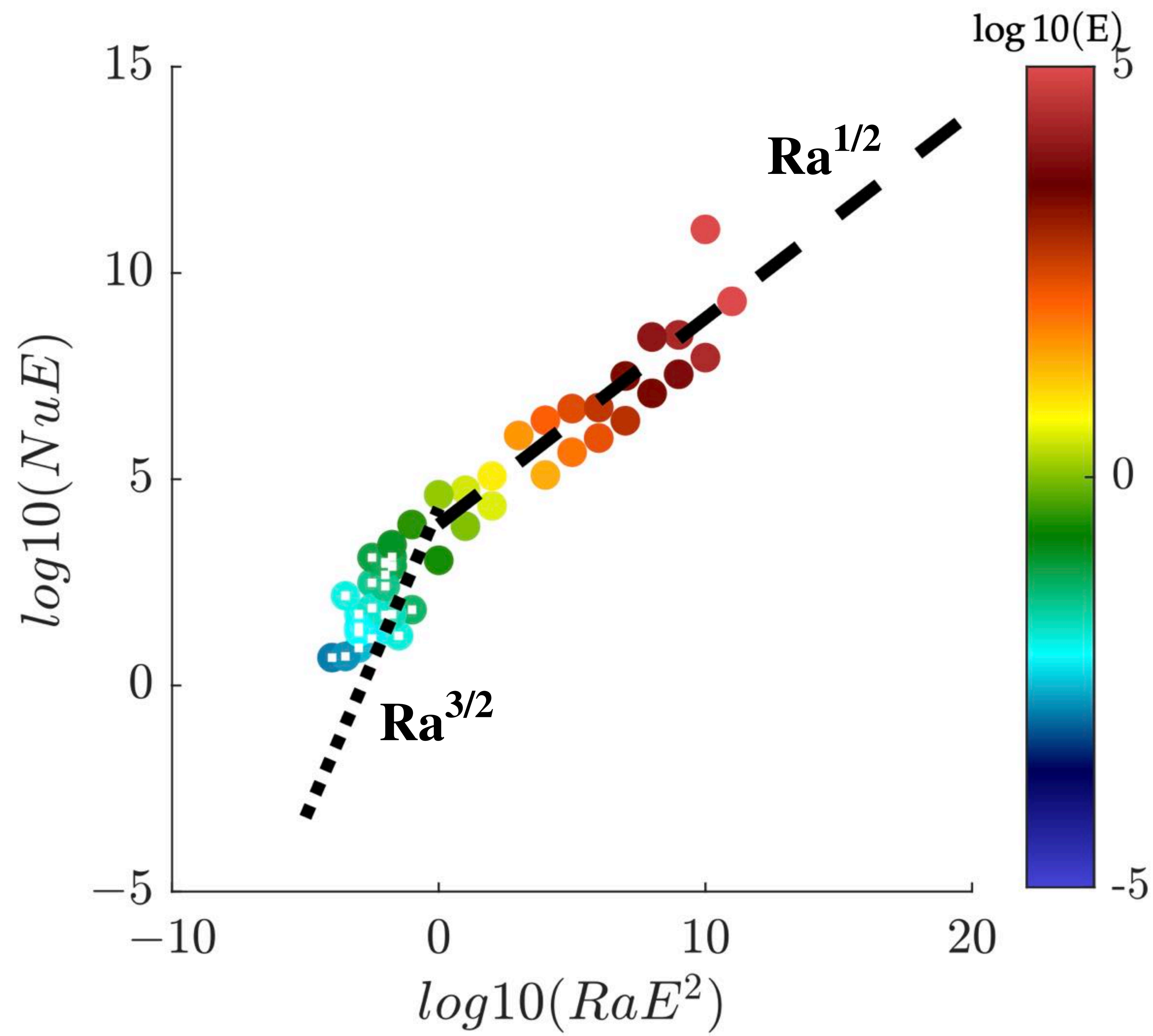
Nu / Pr$^{1/2}$



Log-Lattice homogeneous RRBC

Pikeroen et al. (2023)

- 3D log-lattice
- Need large-scale drag
- See geostrophic and ultimate regime scalings



Extreme RBC requires BL resolution globally

Transition to the Ultimate Regime in Two-Dimensional Rayleigh-Bénard Convection

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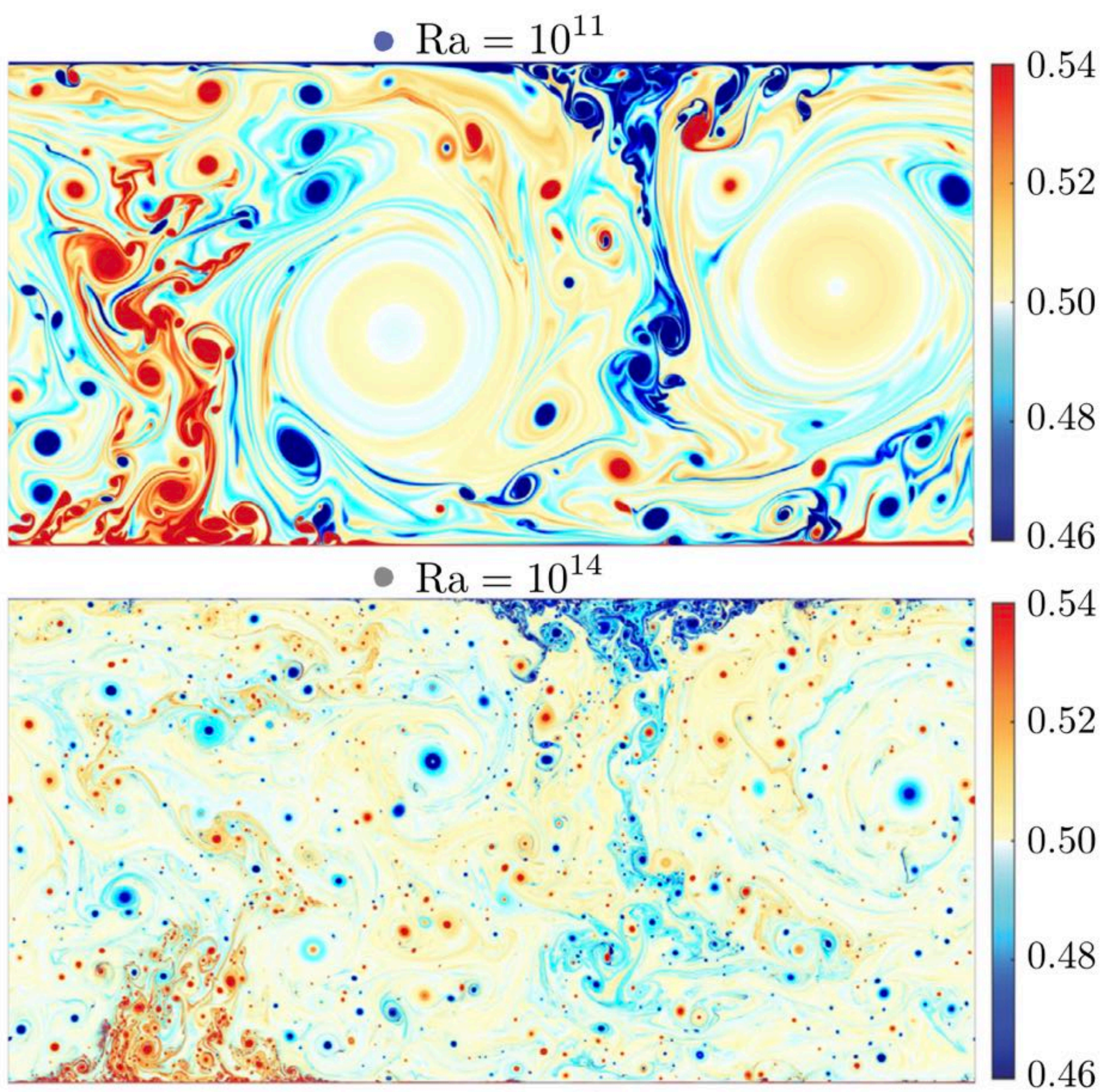
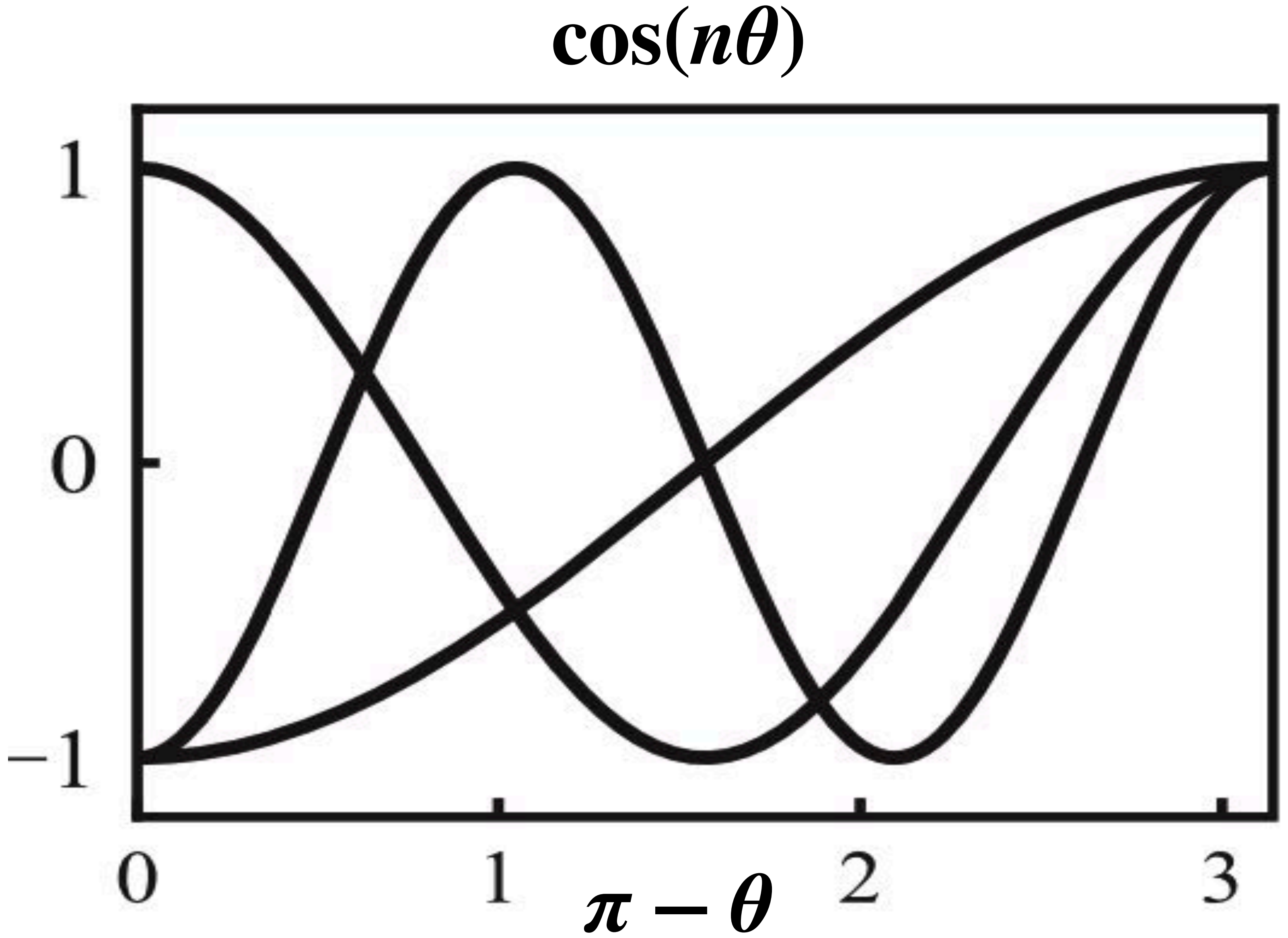
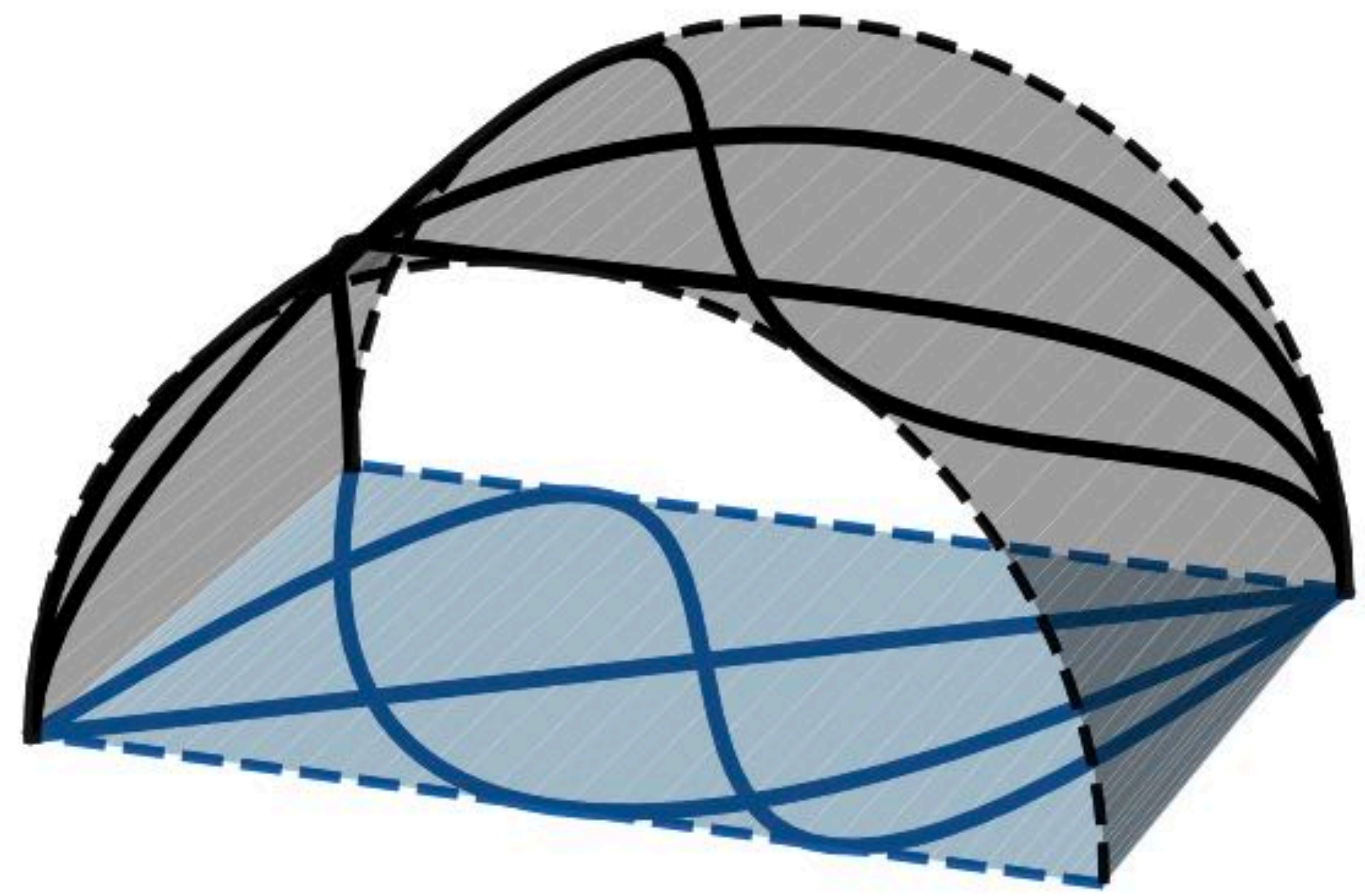


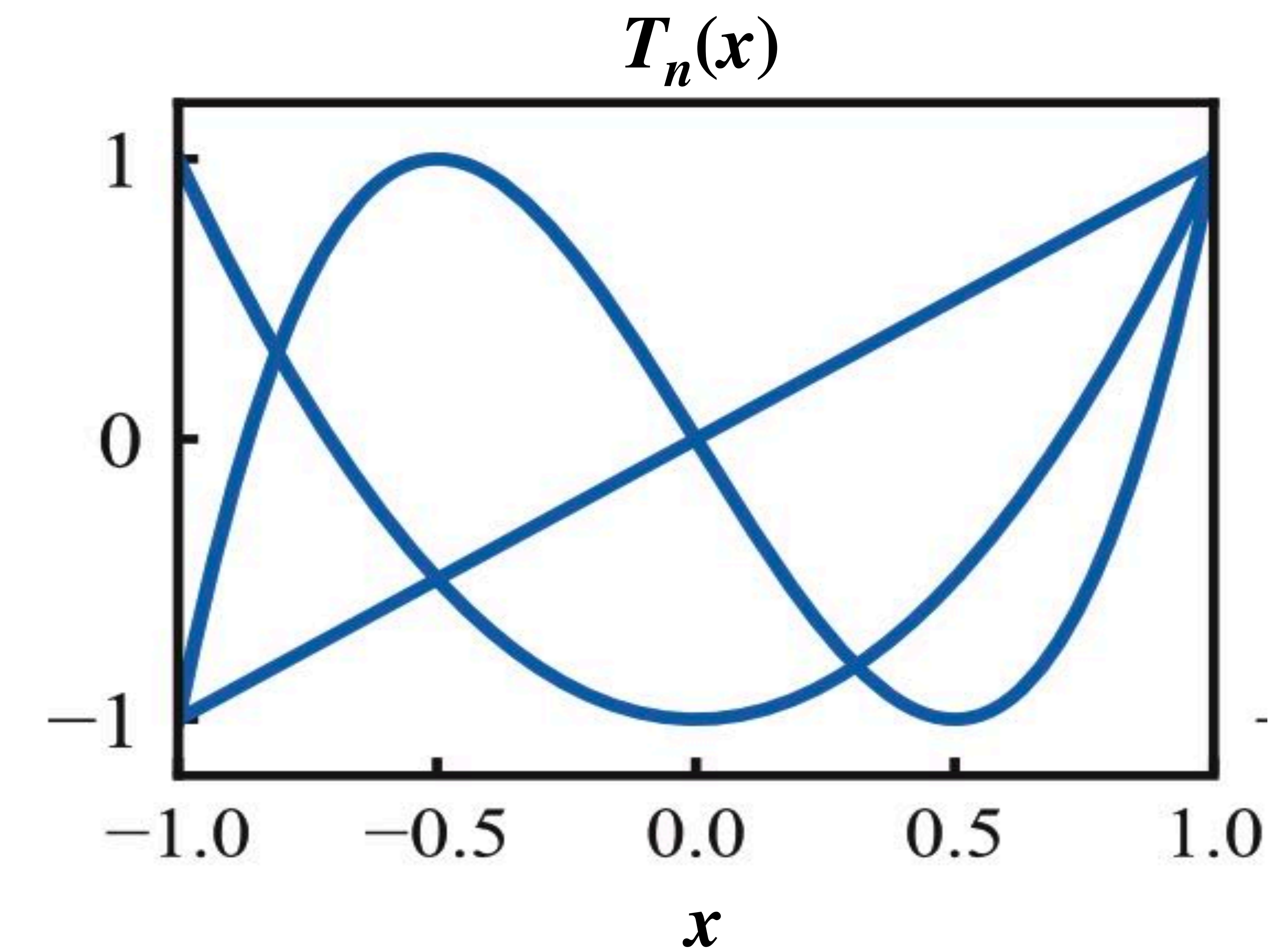
Ra	Pr	Γ	$N_x \times N_z$
10^8	1	2	512×256
2.15×10^8	1	2	768×384
4.64×10^8	1	2	768×384
10^9	1	2	1024×512
2.15×10^9	1	2	1280×640
4.64×10^9	1	2	1536×768
10^{10}	1	2	2048×1024
$2.15\times^{10}$	1	2	3072×1536
4.64×10^{10}	1	2	3072×1536
10^{11}	1	2	4096×2048
2.15×10^{11}	1	2	5120×2560
4.64×10^{11}	1	2	7680×3840
10^{12}	1	2	8192×4096
2.15×10^{12}	1	2	10240×5120
4.64×10^{12}	1	2	10240×5120
10^{13}	1	2	12288×6144
2.15×10^{13}	1	2	14336×7168
4.64×10^{13}	1	2	16384×8192
10^{14}	1	2	20480×10240

Sparse Chebyshev Methods

Chebyshev polynomials: Fourier in disguise

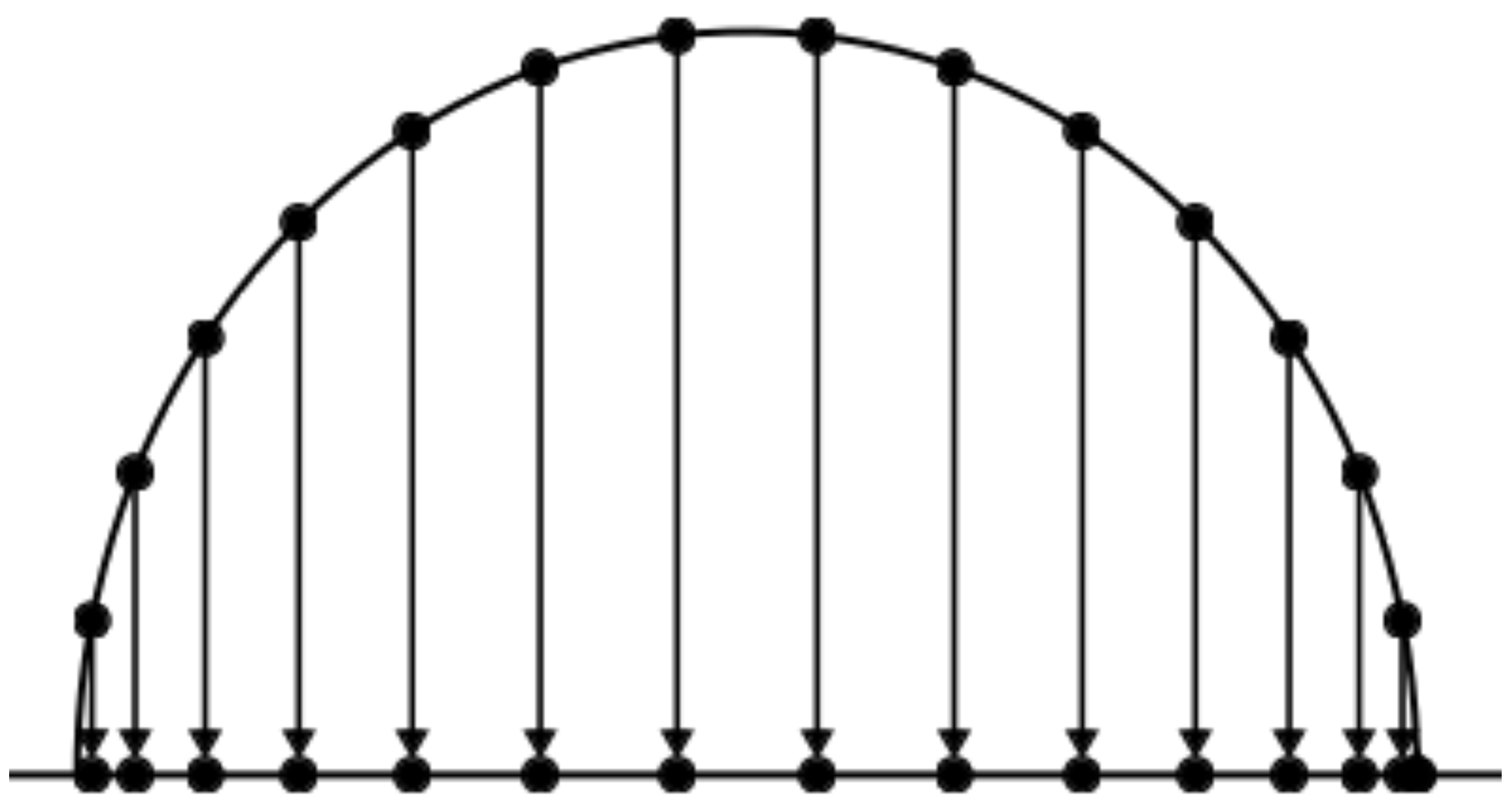




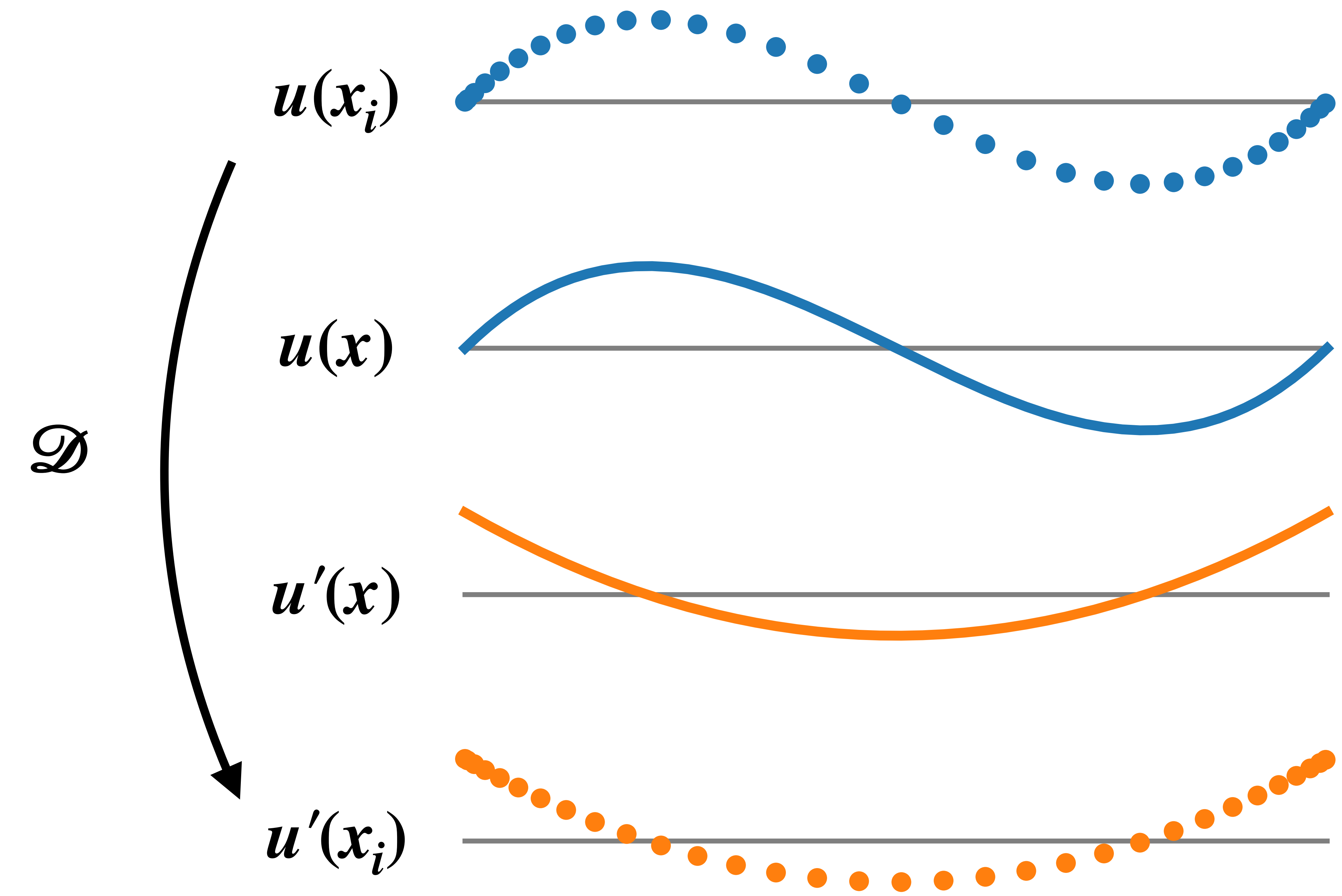


Chebyshev Collocation

Chebyshev nodes



Differentiation



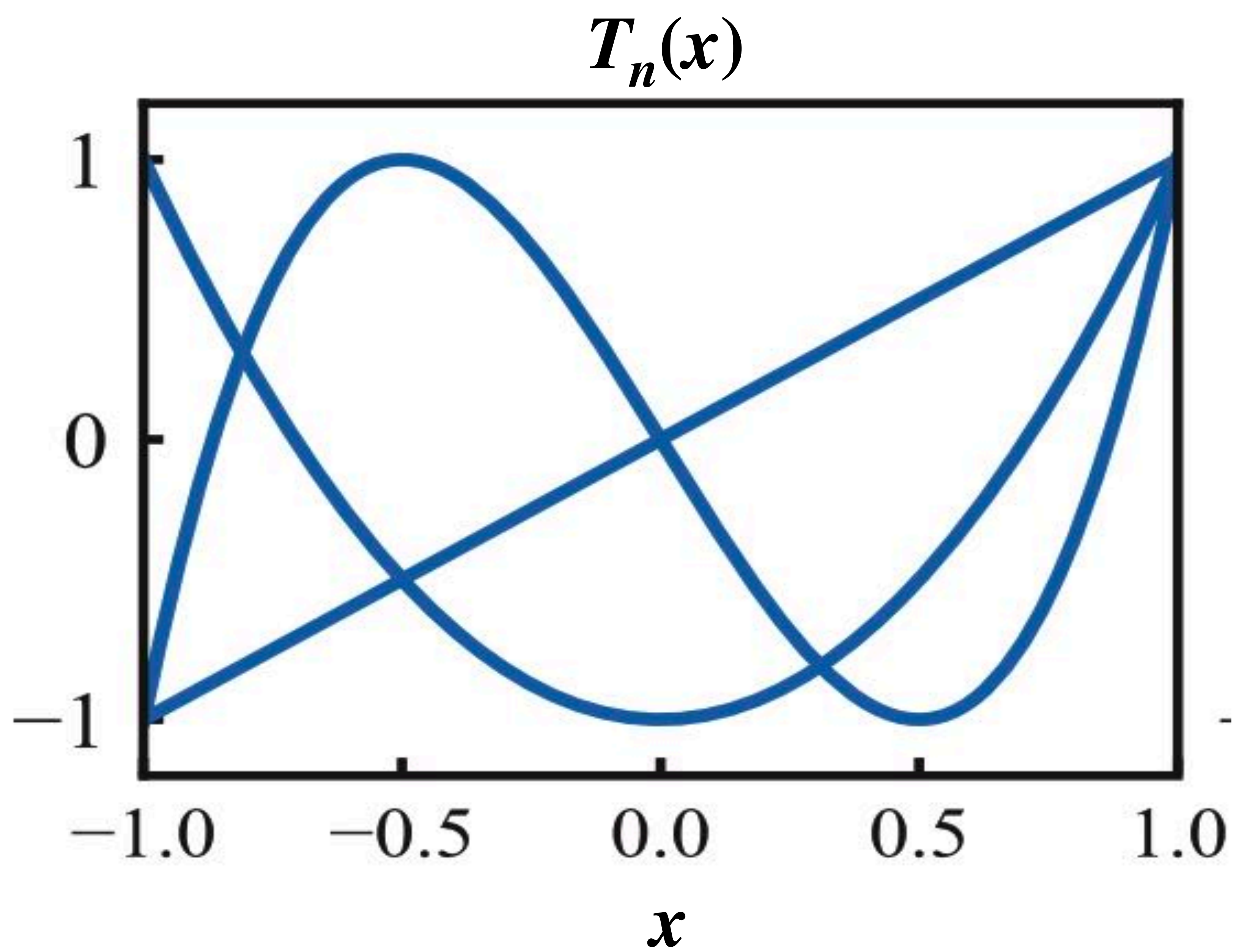
- Dense matrices
- Poor conditioning

Gottlieb, Orszag, Hussaini, Trefethen, Driscoll, Hale

Classical/Lanczos Tau Method

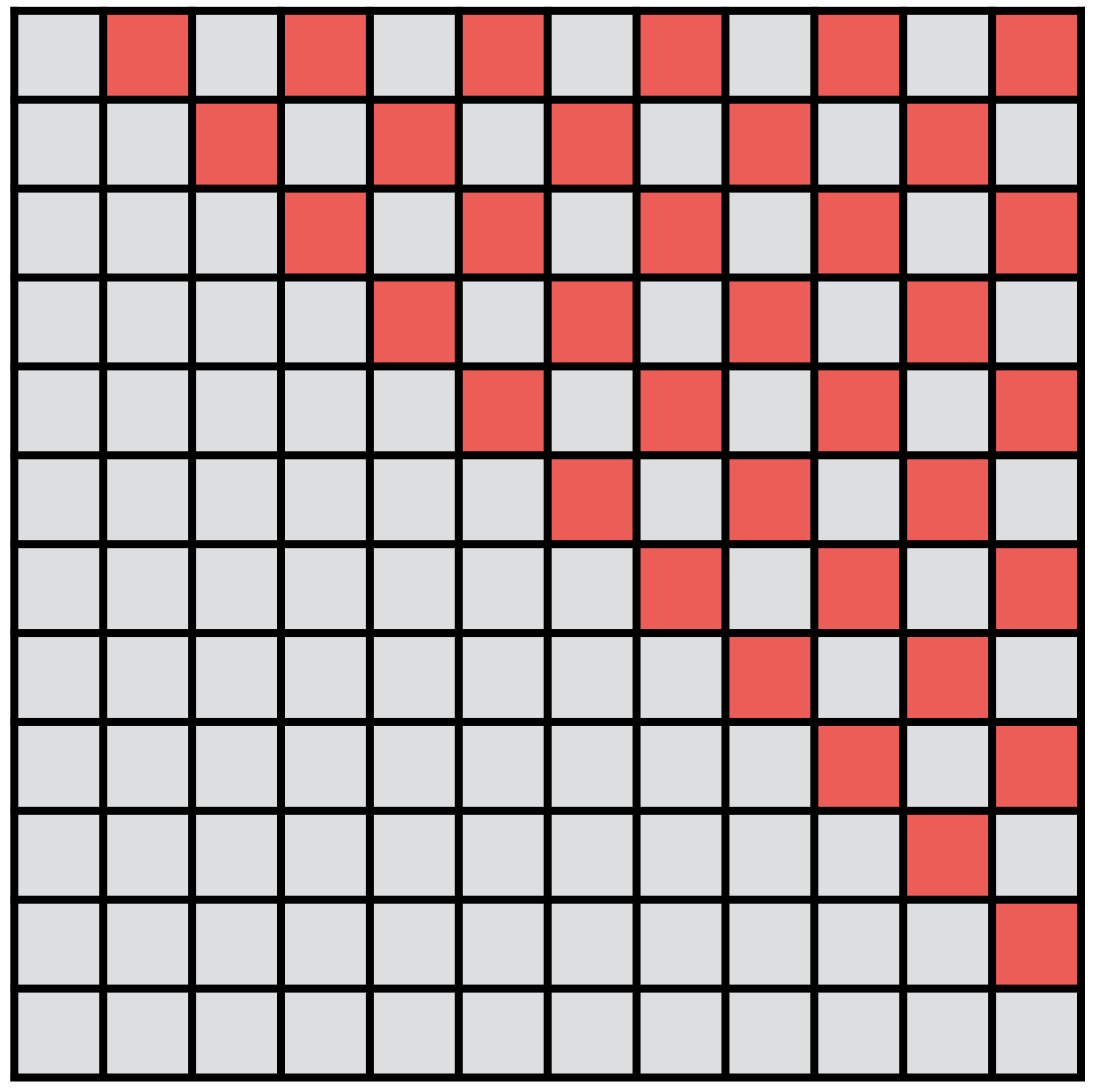
Chebyshev basis functions

$$u(x) = \sum_{n=0}^{N} u_n T_n(x)$$



Differentiation

$$\mathscr{D}_{m,n} = \langle T_m | \partial_x T_n \rangle$$



- Dense matrices
- Poor conditioning

Lanczos, Liu, Ortiz

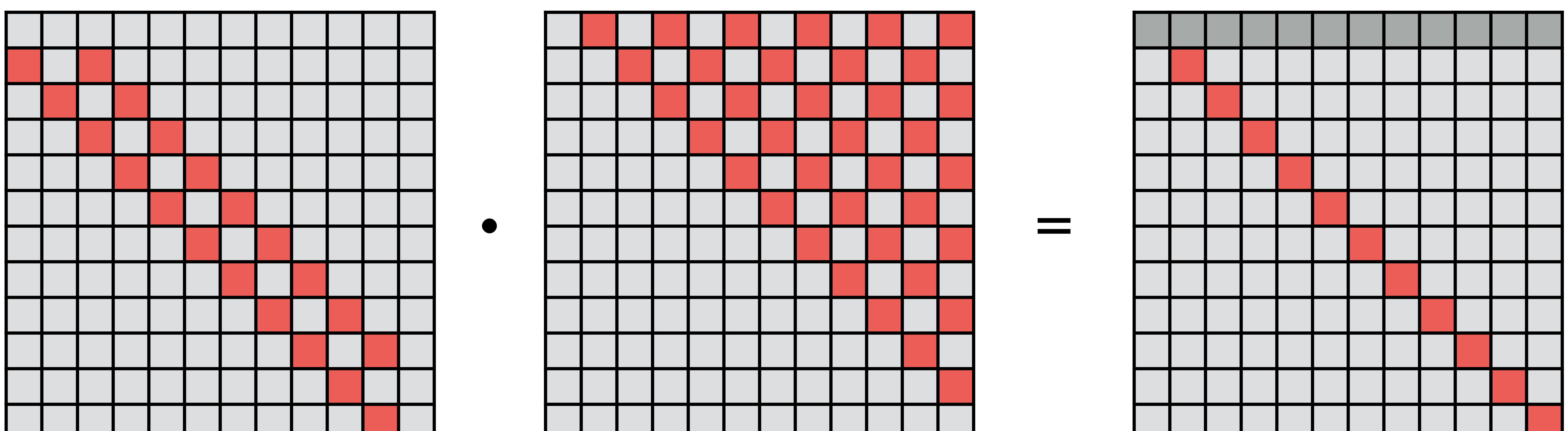
Quasi-Inverse Technique

$$u(x) = \sum_{n=0}^{N} u_n T_n(x), \qquad u'(x) = \sum_{n=0}^{N-1} u'_n T_n(x)$$

Derivative recursion: $$T_n(x) = \frac{T'_{n+1}(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1} \implies c_{n-1}u'_{n-1} - u'_{n+1} = 2nu_n$$

Pseudoinverse

$$\mathscr{B} \cdot \mathscr{D} = I^{(1)}$$



Gottlieb, Orszag, Greengard, Dang-Vu, Julien & Watson + friends

Ultraspherical Method

Chebyshev "trial" functions:

$$u(x) = \sum_{n=0}^{N} u_n T_n(x)$$

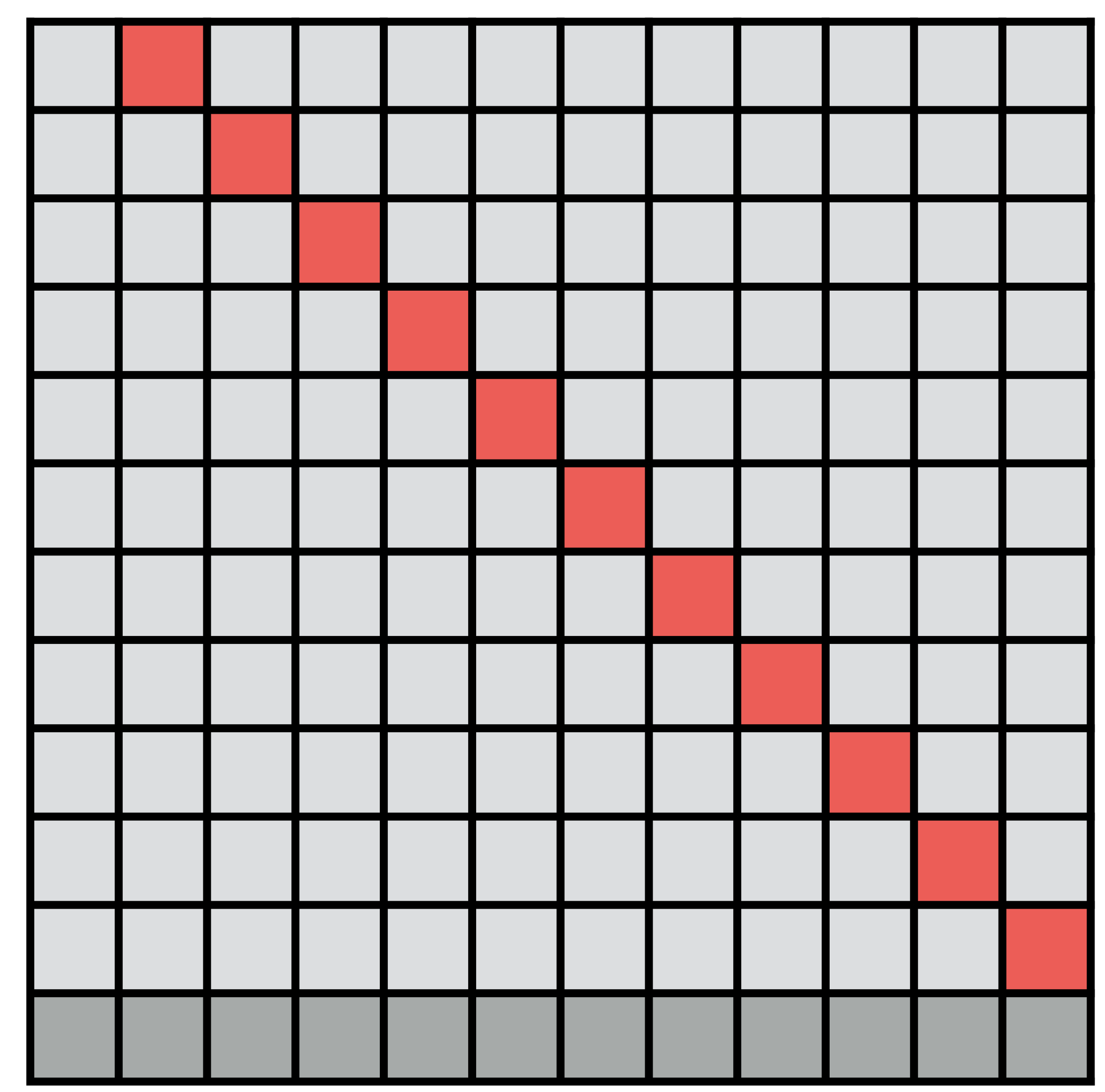
Ultraspherical "test" functions:

$$C_n^{(k)}(x) \;\propto\; \partial_x^k T_{n+k}(x)$$

$$w(x) = (1 - x^2)^{k-1/2}$$

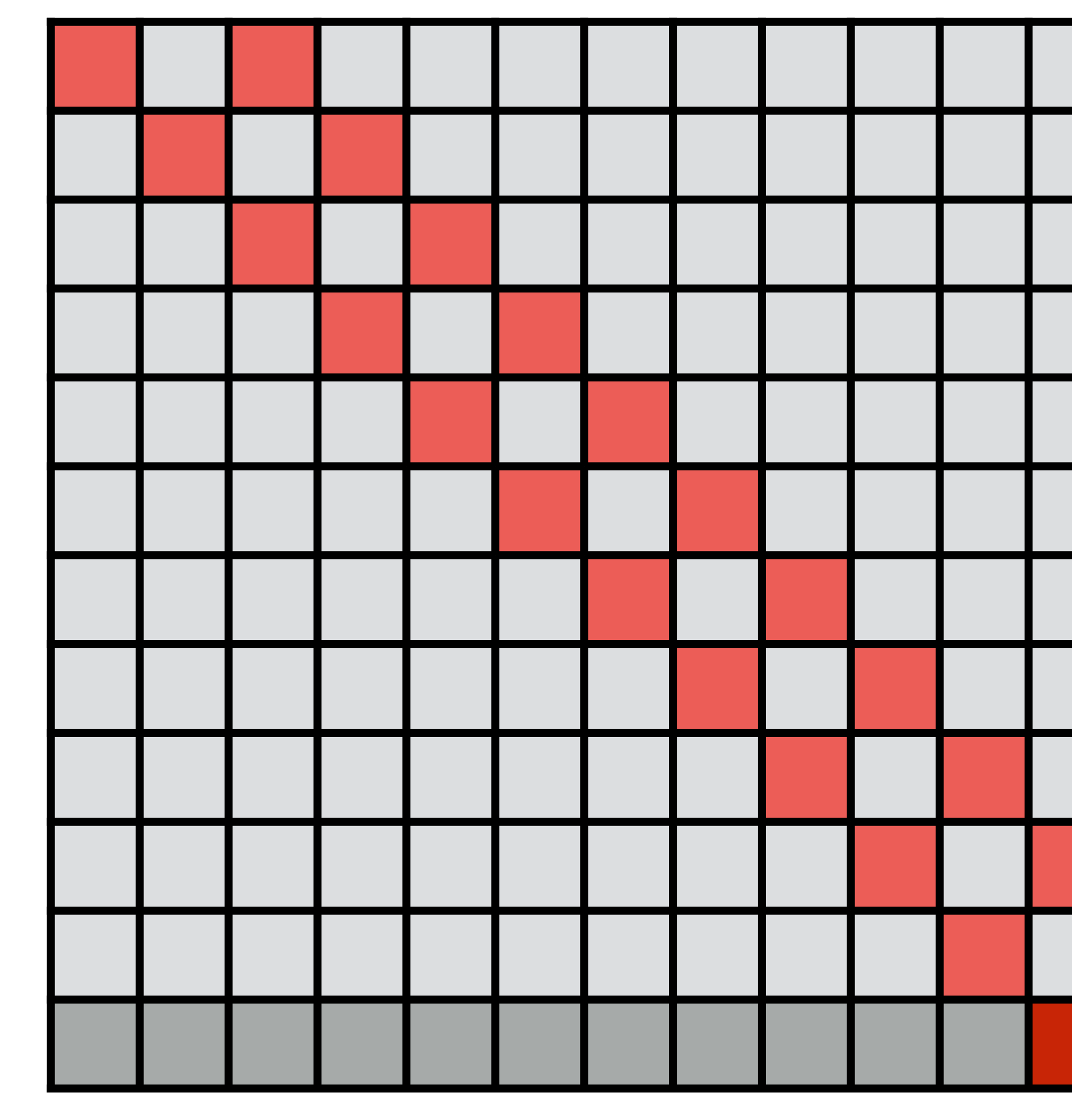
Differentiation

$$\mathscr{D}_{m,n} = \langle C_m^{(1)} \,|\, \partial_x T_n \rangle$$



Conversion

$$\tilde{I}_{m,n} = \langle C_m^{(1)} \,|\, T_n \rangle$$



$$\boxed{\tilde{I} \sim \mathscr{B}}$$

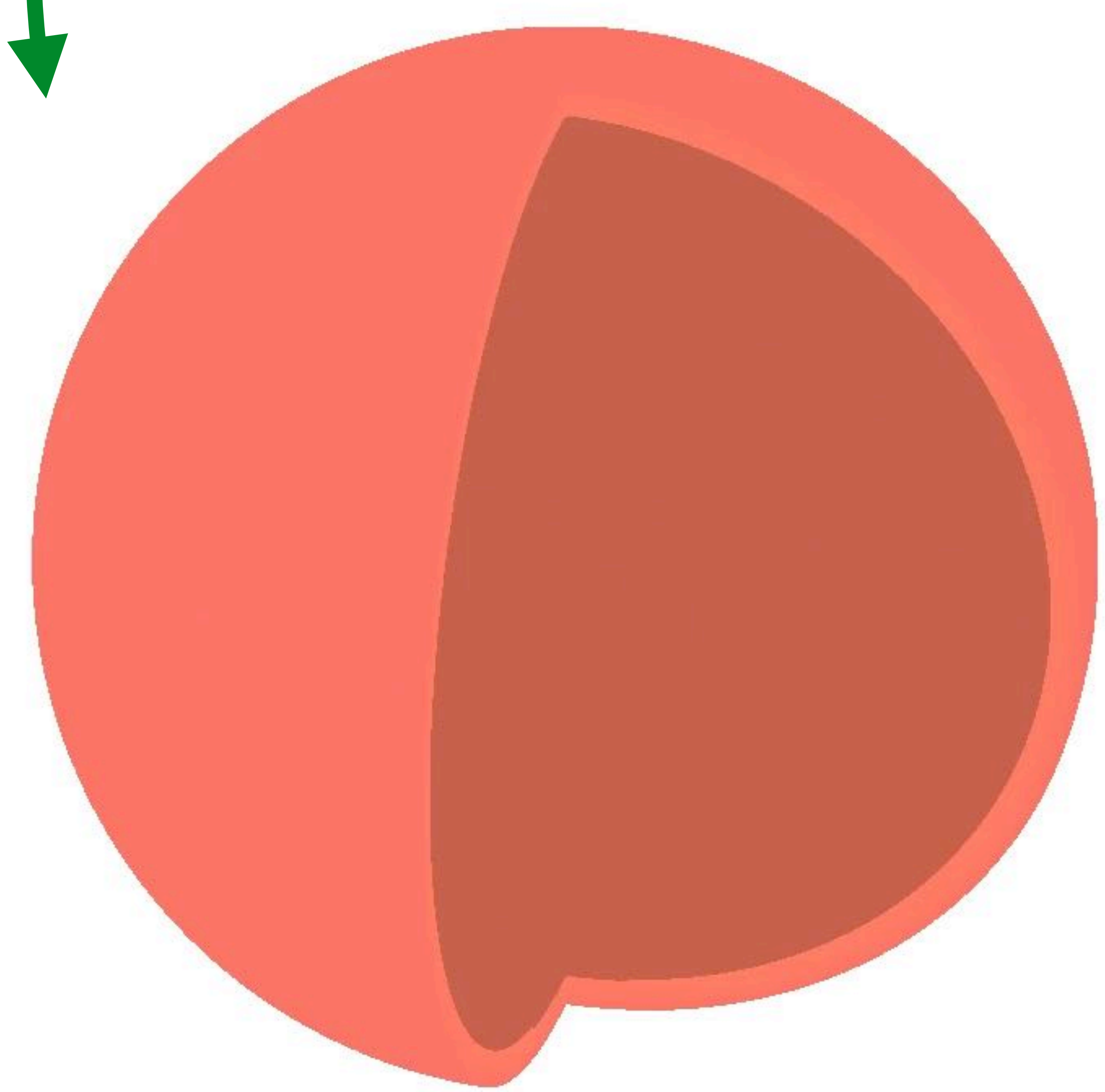
Orszag, Coutsias, Doha, ***Olver, Townsend, Vasil***

Dedalus Project

```
problem.add_equation("div(u) = 0")
problem.add_equation("dt(u) - ν*Lap(u) + grad(p) + b*g = - u@grad(u)")
problem.add_equation("dt(b) - κ*Lap(b) = - u@grad(b)")
```

Rapid solver development

Spiral-defect chaos

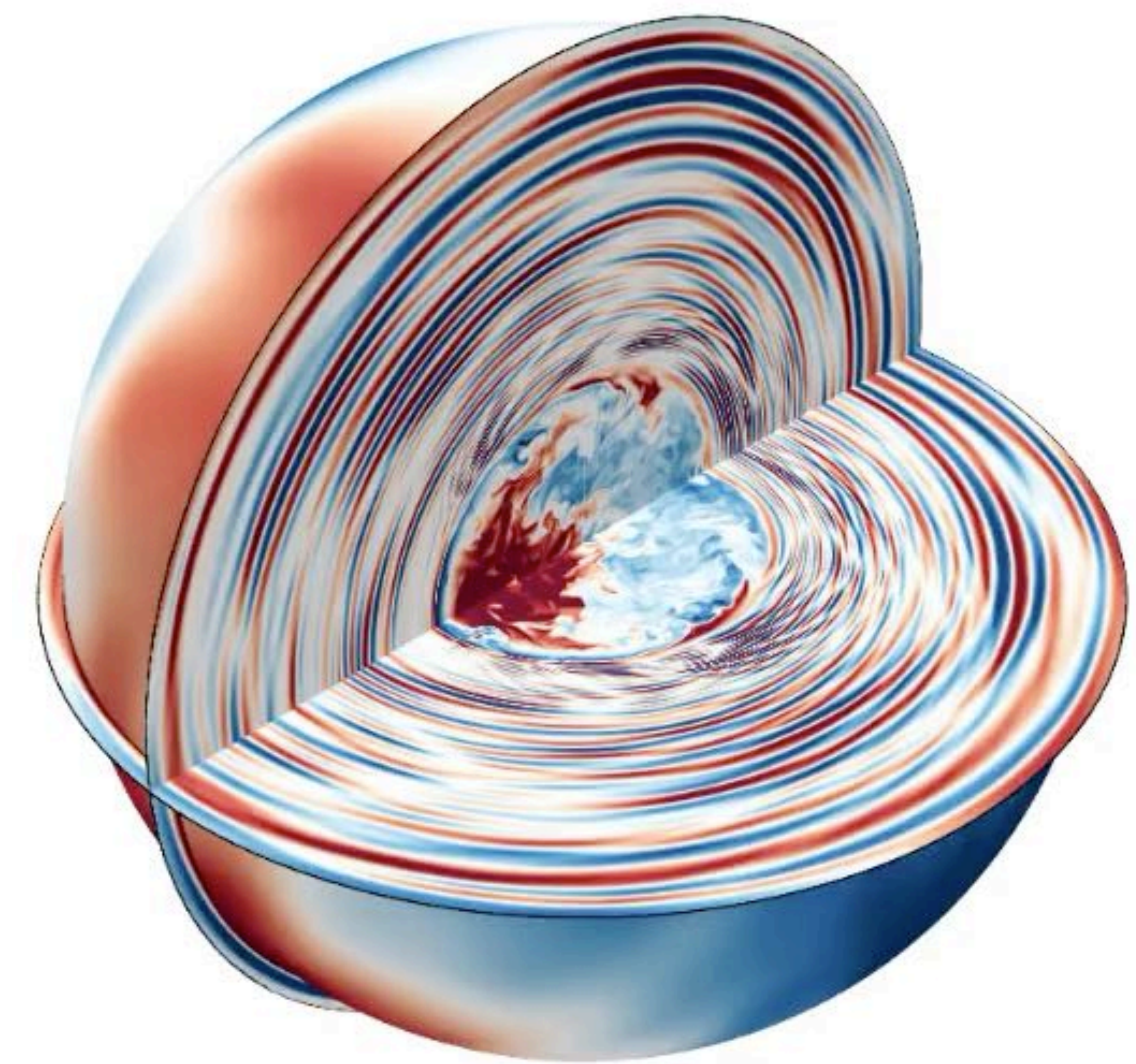


Flexible equations

NLS quantum graphs

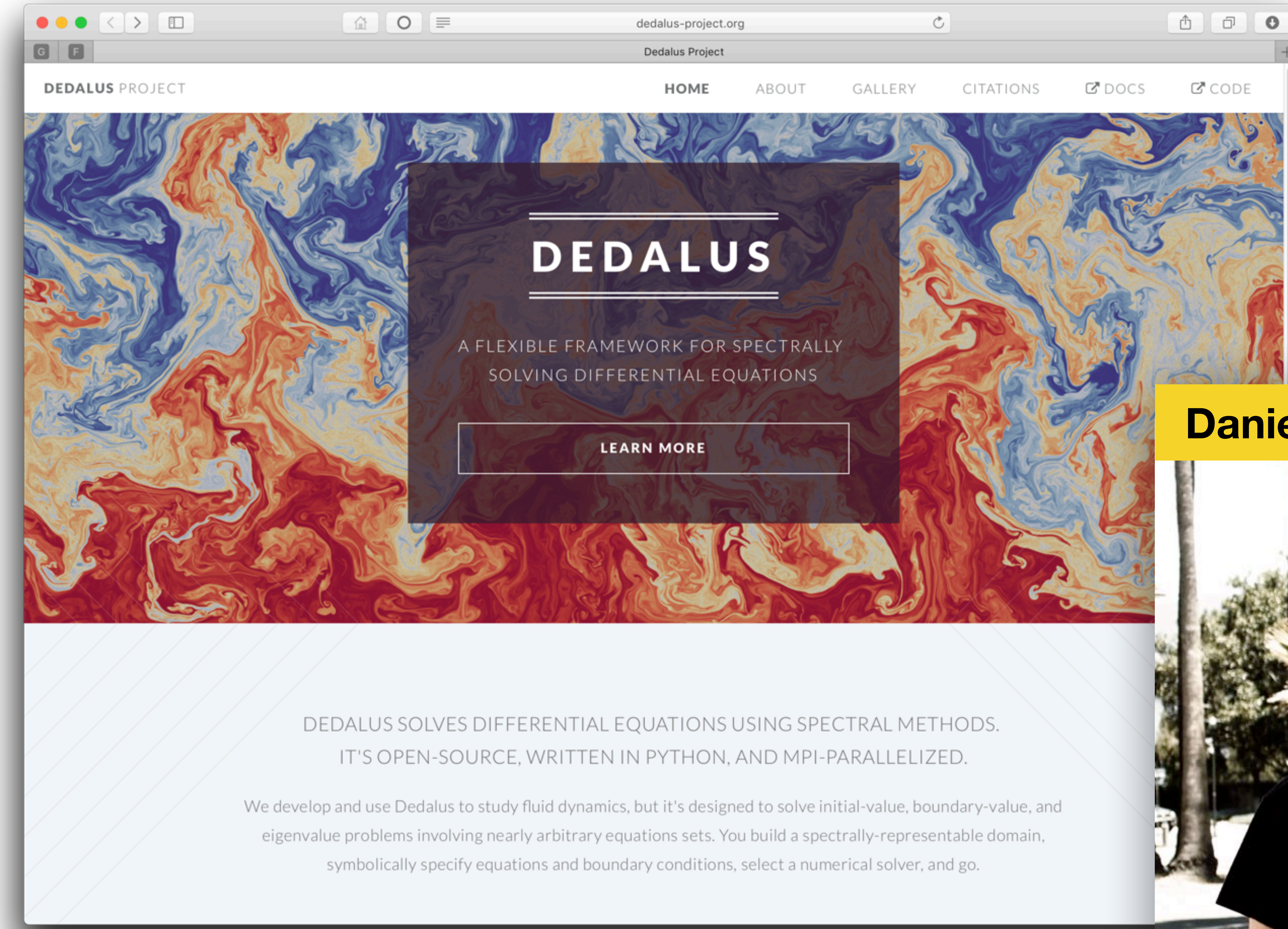
High performance

Turbulent wave excitation



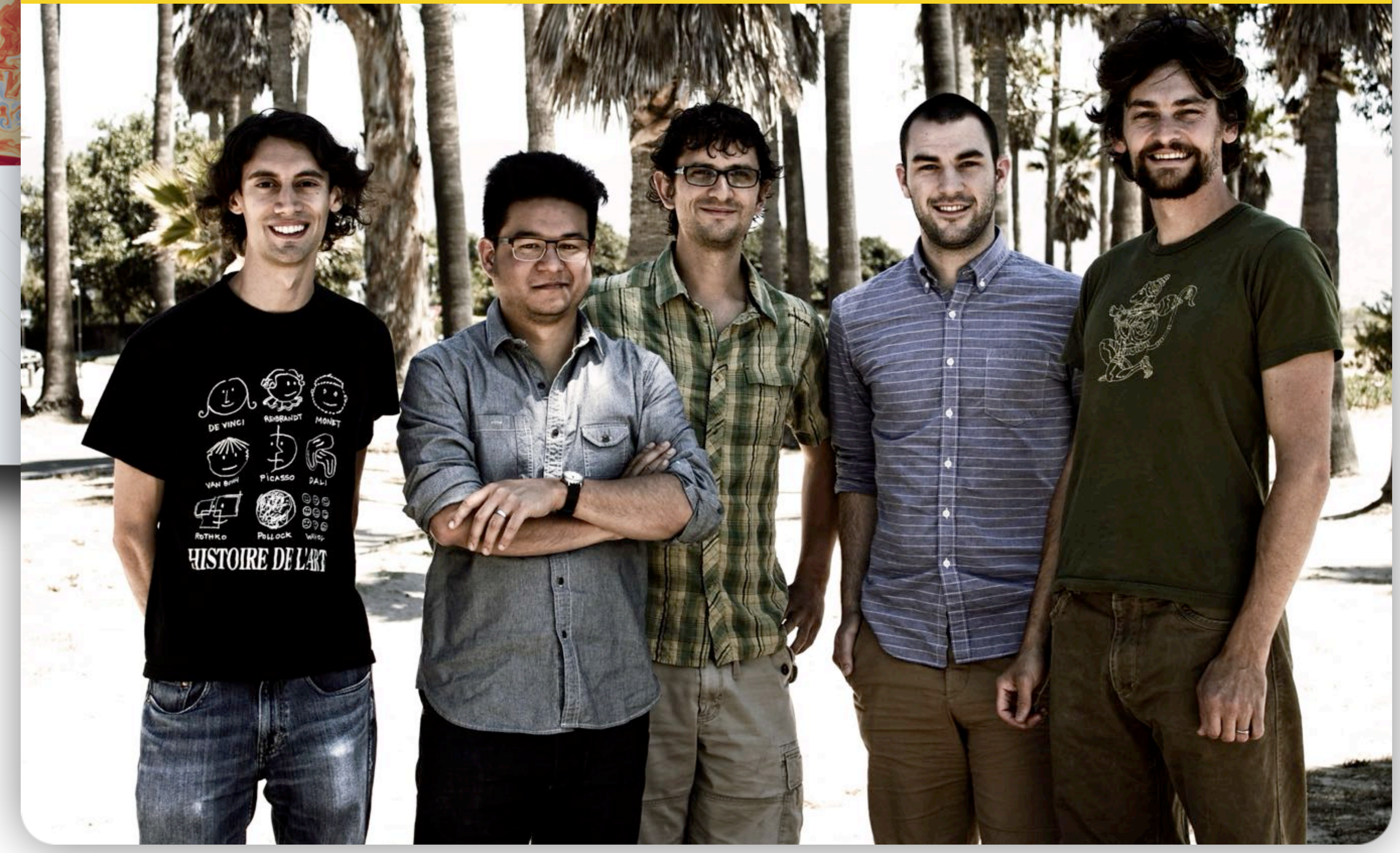
w/ Evan Anders

www.dedalus-project.org



Core developers

Daniel Lecoanet, Jeff Oishi, Geoff Vasil, Keaton Burns, Ben Brown



Log-Lattice Rayleigh Benard

Log-Fourier + Chebyshev

LFL resolution:

$$N_k \sim \log(L/\Delta x) \sim \log \mathrm{Re}$$

Chebyshev resolution:

$$N_z = \frac{L_x}{\Delta x} \sim \mathrm{Re}^{3/4}$$

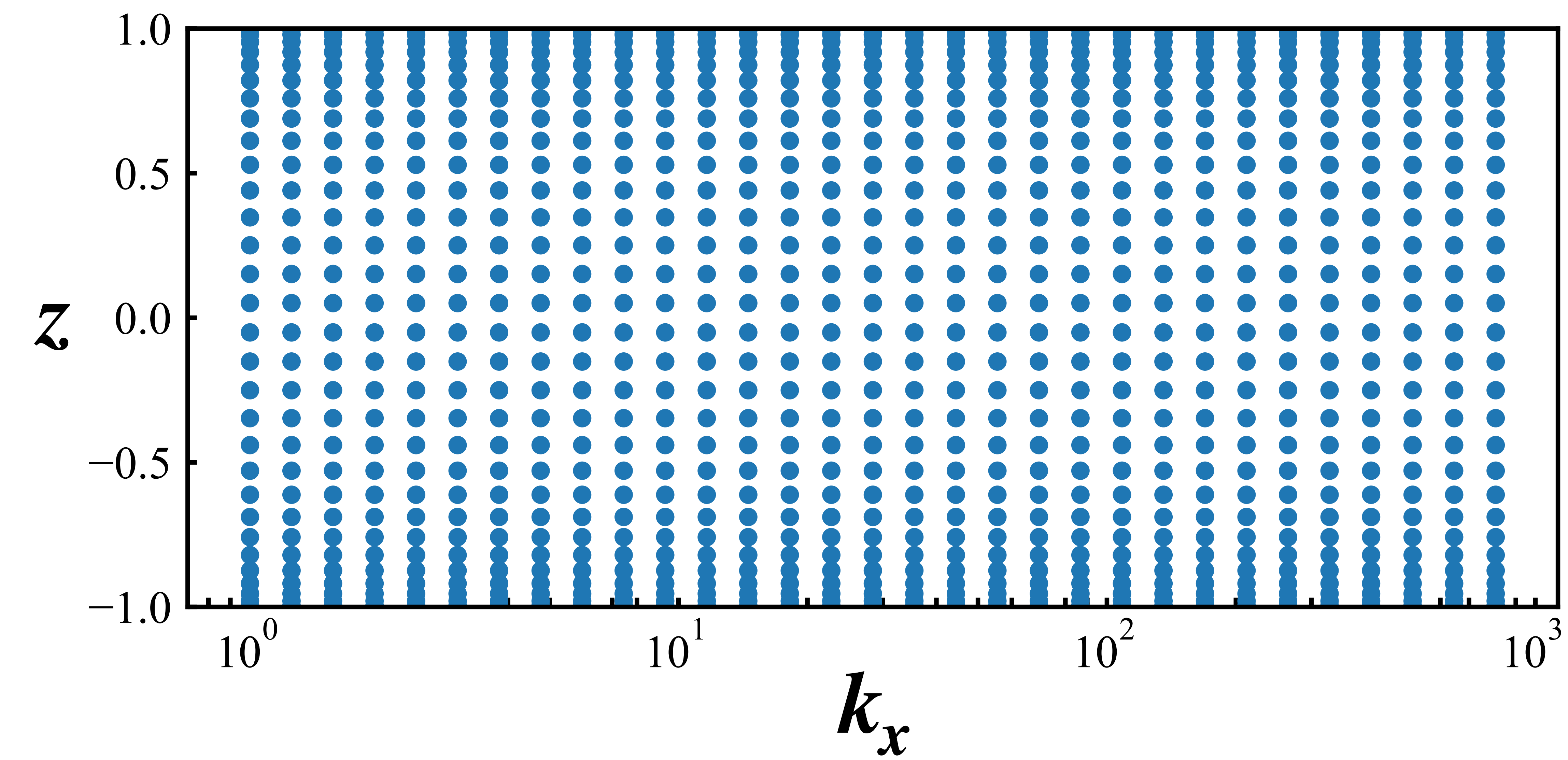
Temporal resolution:

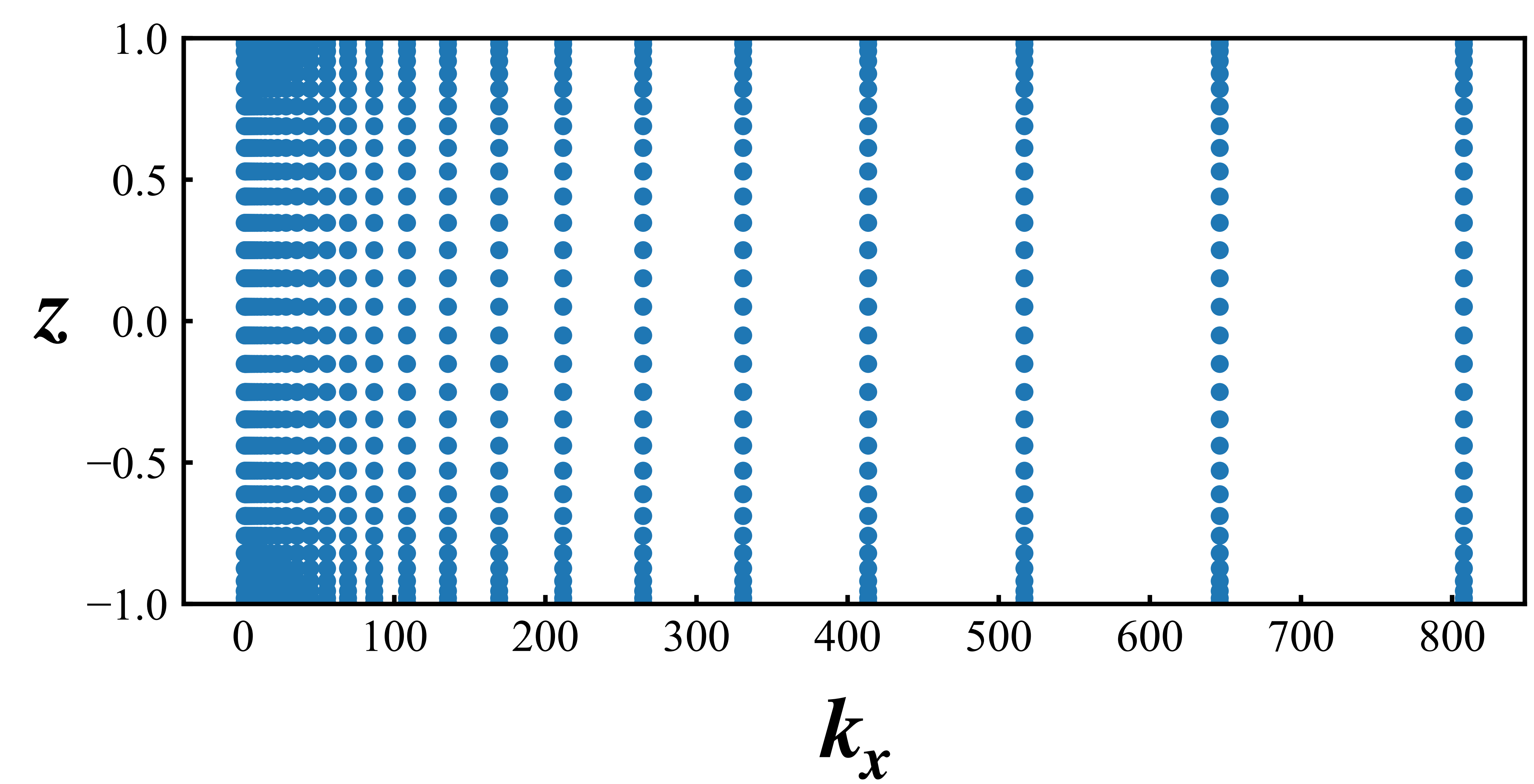
$$N_t = \frac{T}{\Delta t} \sim \frac{L_x}{\Delta x} \sim \mathrm{Re}^{3/4}$$

Total model DOF:

$$N_{\mathrm{hom}} = N_k^d N_t \sim (\log \mathrm{Re})^d \, \mathrm{Re}^{3/4}$$

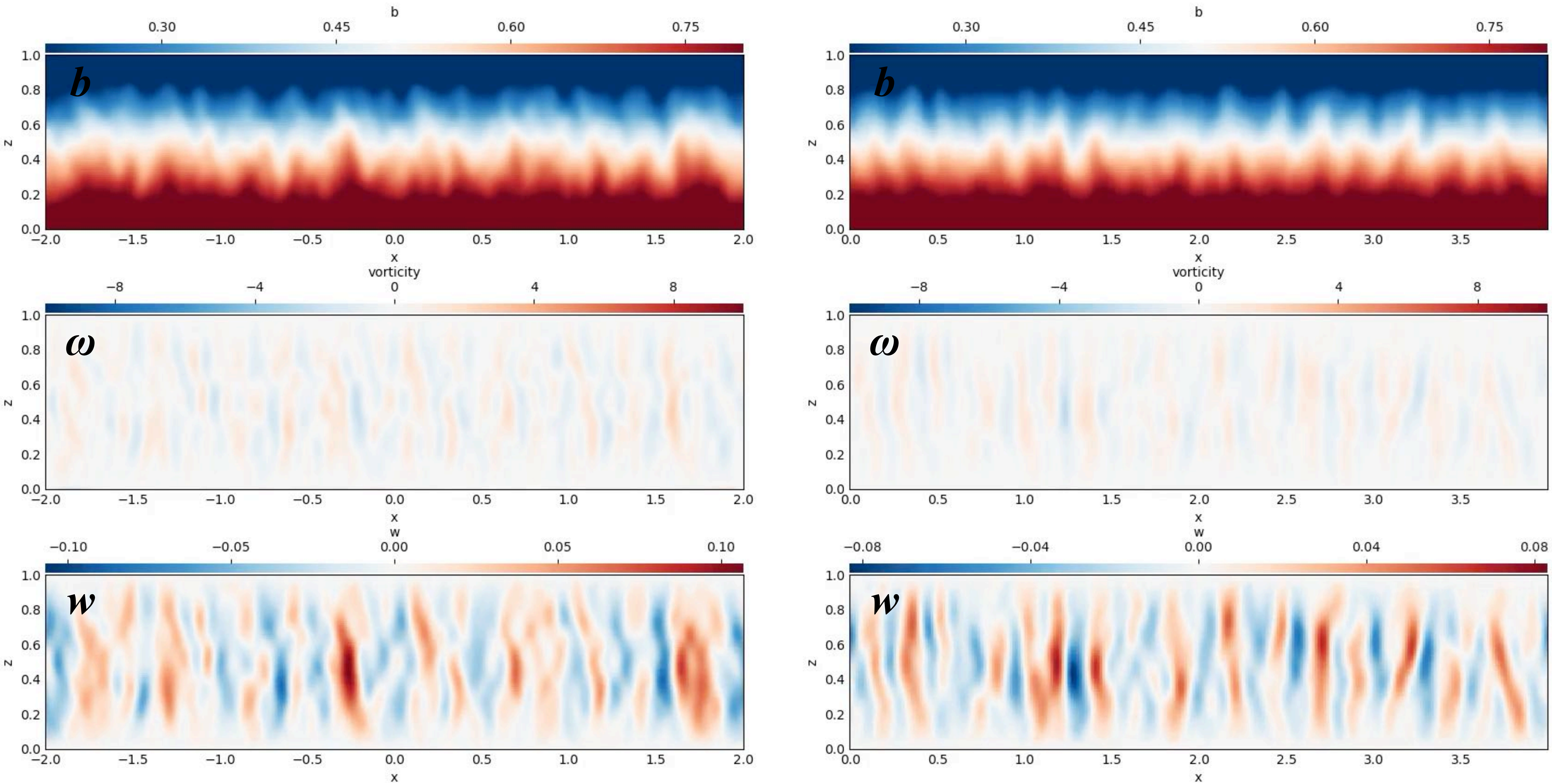
$$N_{\mathrm{ch}} = N_k^{d-1} N_z N_t \sim (\log \mathrm{Re})^{d-1} \, \mathrm{Re}^{3/2}$$





Log-Fourier Rayleigh-Benard

Ra = 10^8

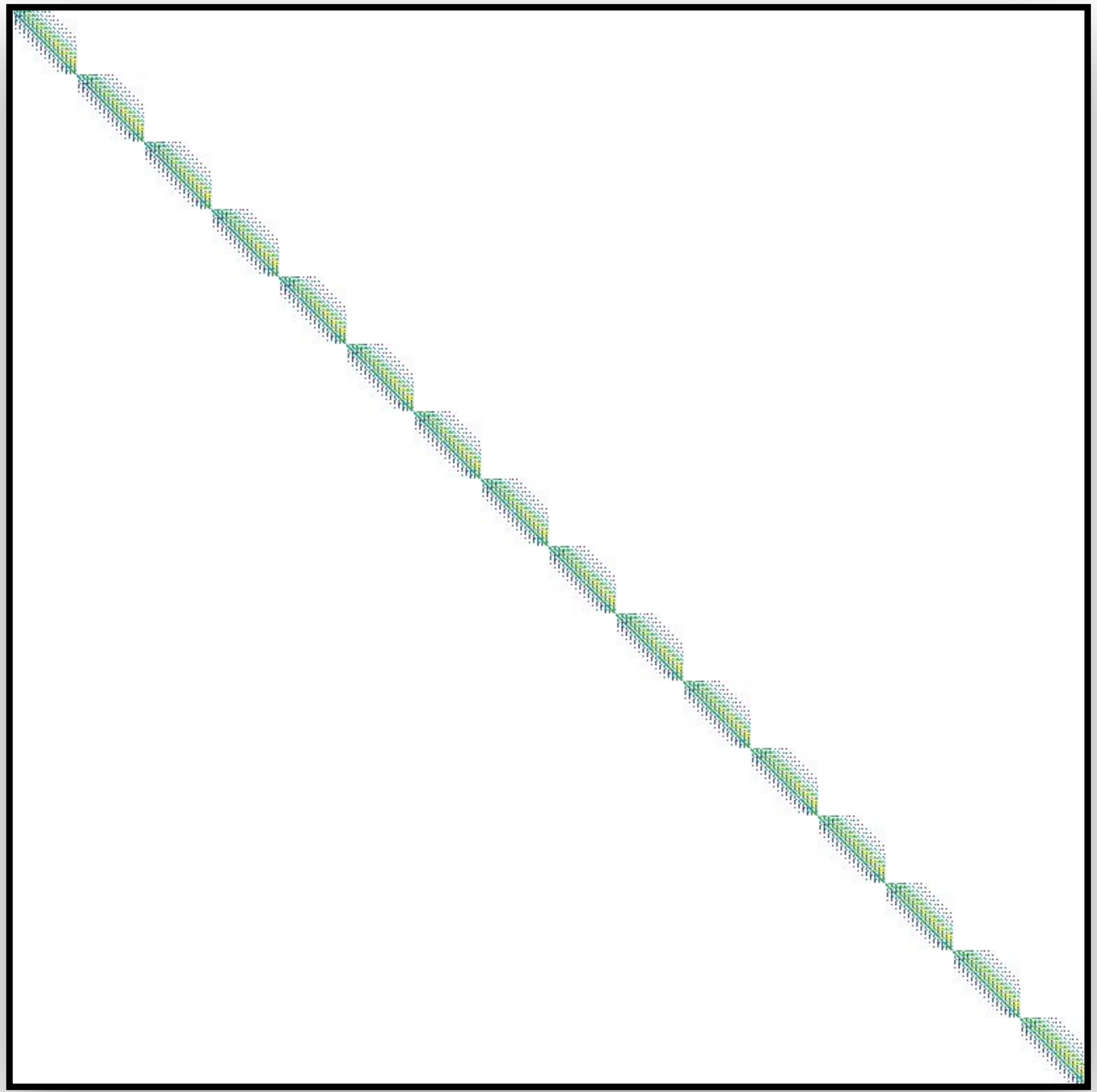


Log-Fourier Rayleigh-Benard

$\lambda = 1.16$

Ra	DNS resolution (Zhu et al. 2018)	Log lattice resolution
10^{10}	2,048 x 1,024	112 x 1,024
10^{11}	4,096 x 2,048	128 x 2,048
10^{12}	8,192 x 4,096	128 x 4,096
10^{13}	12,288 x 6,144	136 x 4,096
10^{14}	20,480 x 10,240	144 x 8,192
10^{15}		152 x 16,384
10^{16}		156 x 24,576
10^{17}		160 x 32,768

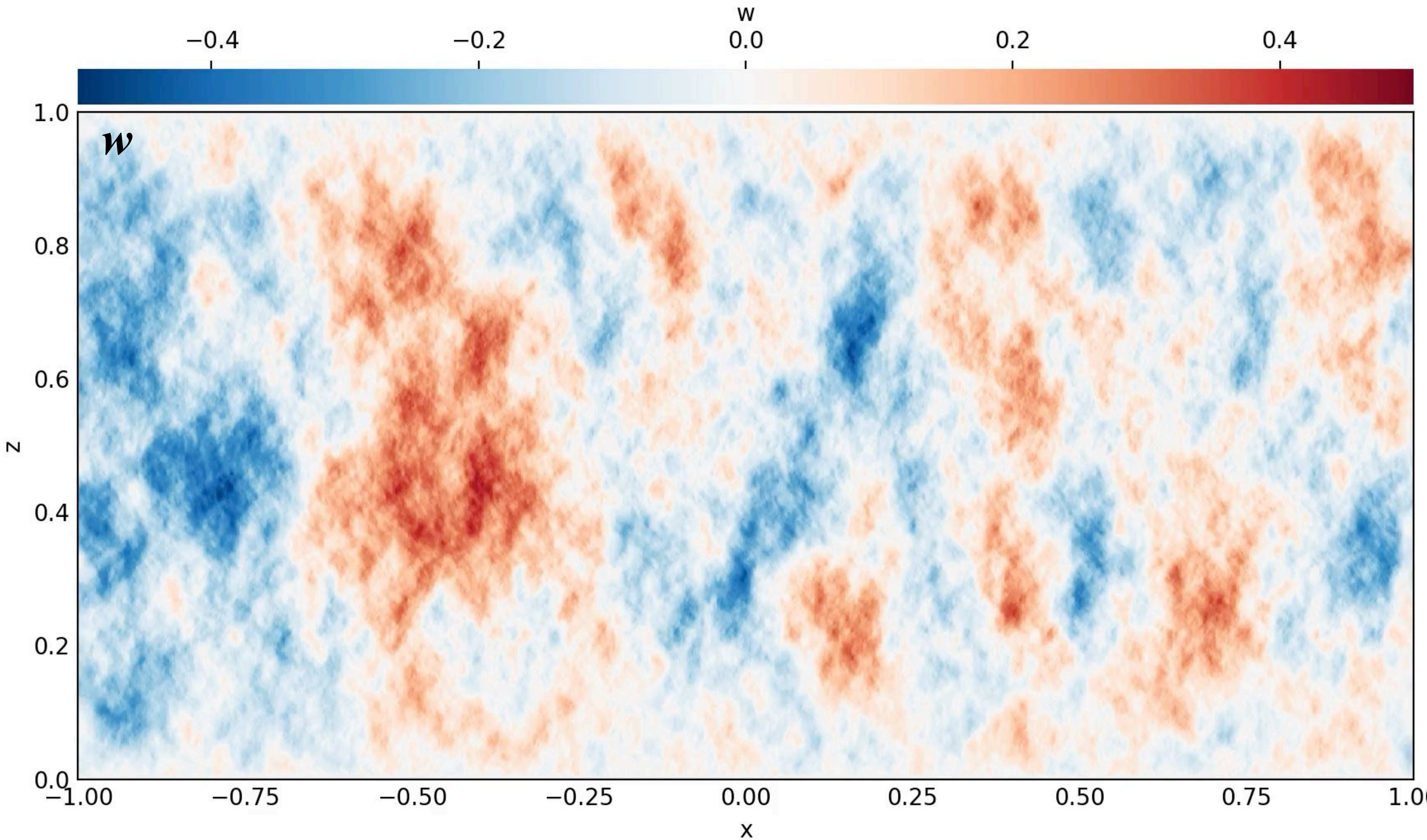
Sparse Chebyshev solver (Dedalus)



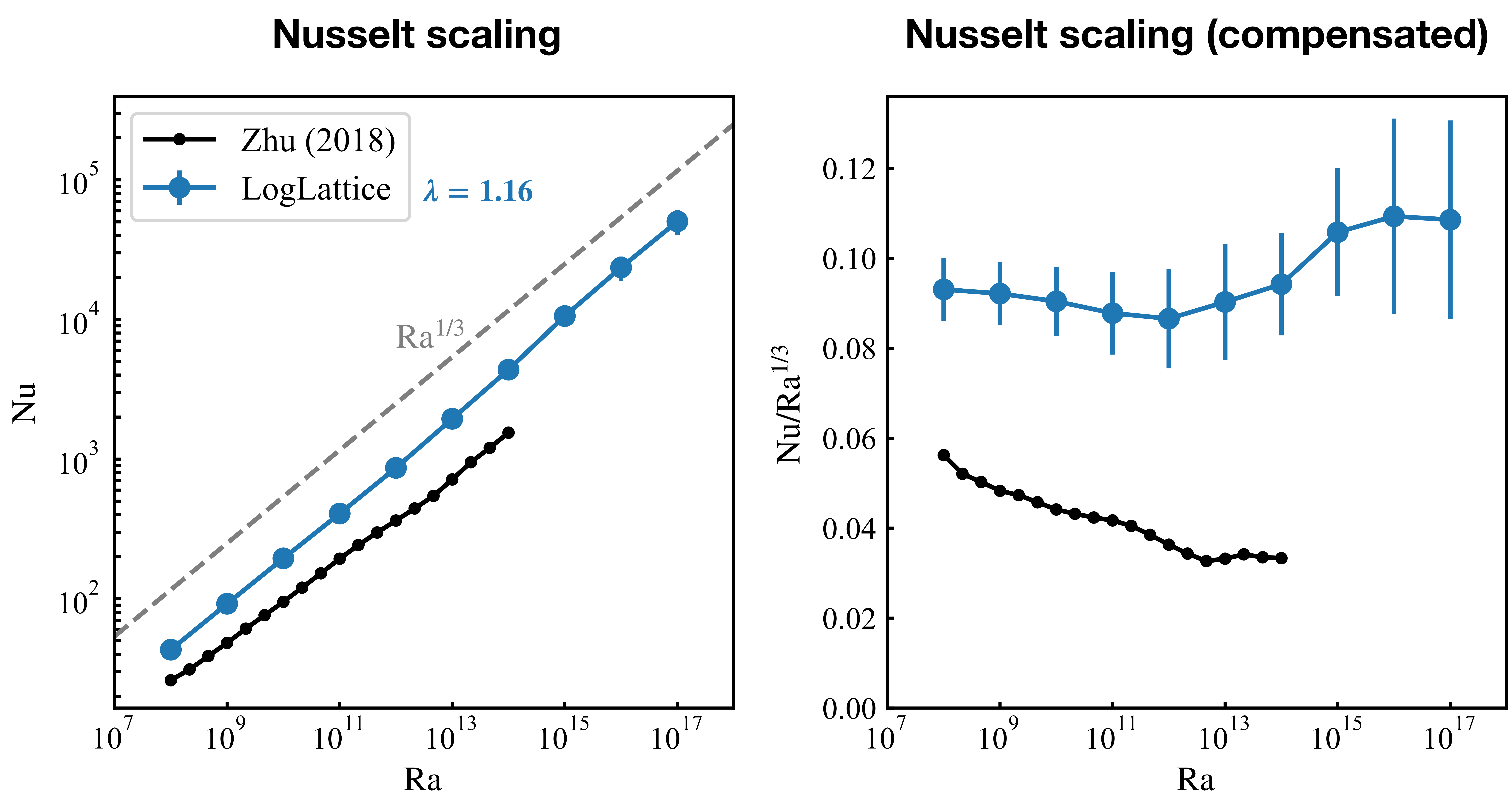
Burns et al., PRR (2020)

Log-Fourier Rayleigh-Benard

Ra = 10^{14} $\lambda = 1.16$



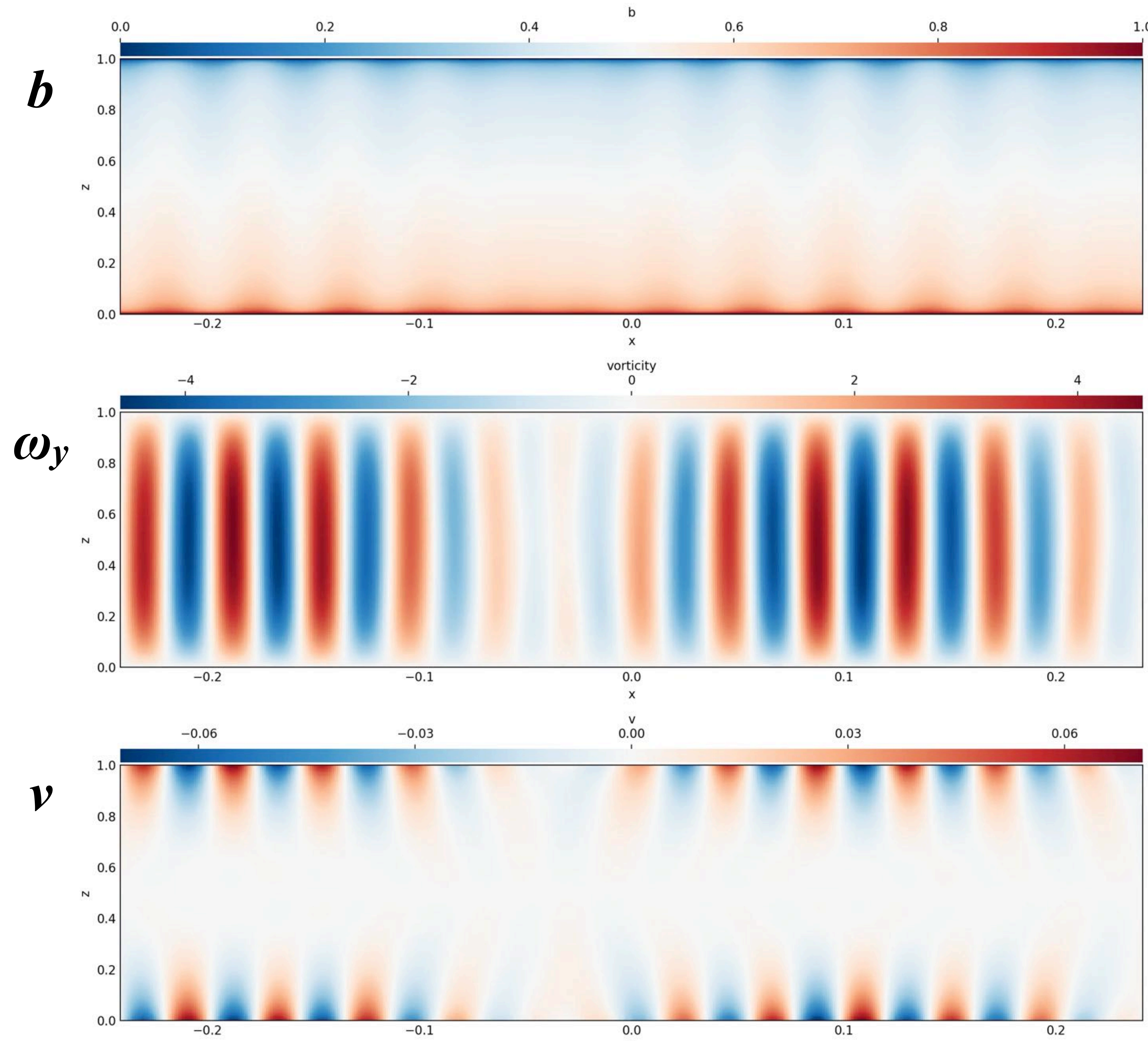
Log-Fourier Rayleigh-Benard



Log-Lattice Rotating Convection

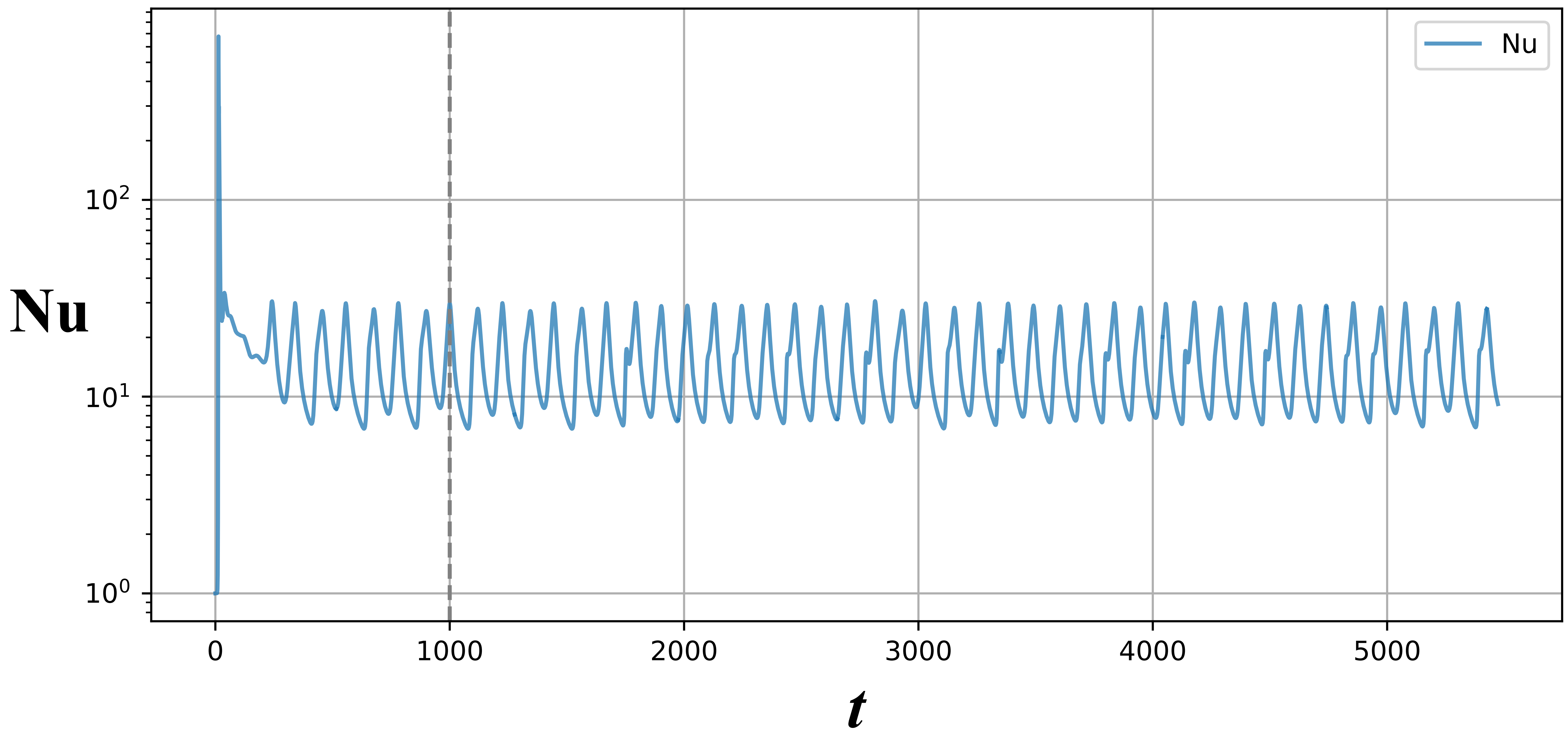
Log-Fourier Rotating Rayleigh-Benard

$$\widetilde{\mathbf{Ra}} = 40$$
$$\mathbf{Ek} = 10^{-6}$$
$$L_x = 10\ell_c$$
$$\lambda = 1.16$$



Nusselt number in Log-Lattice RRBC

$\widetilde{\mathbf{Ra}} = 40$
$\mathbf{Ek} = 10^{-6}$
$L_x = 10\ell_c$
$\lambda = 1.16$

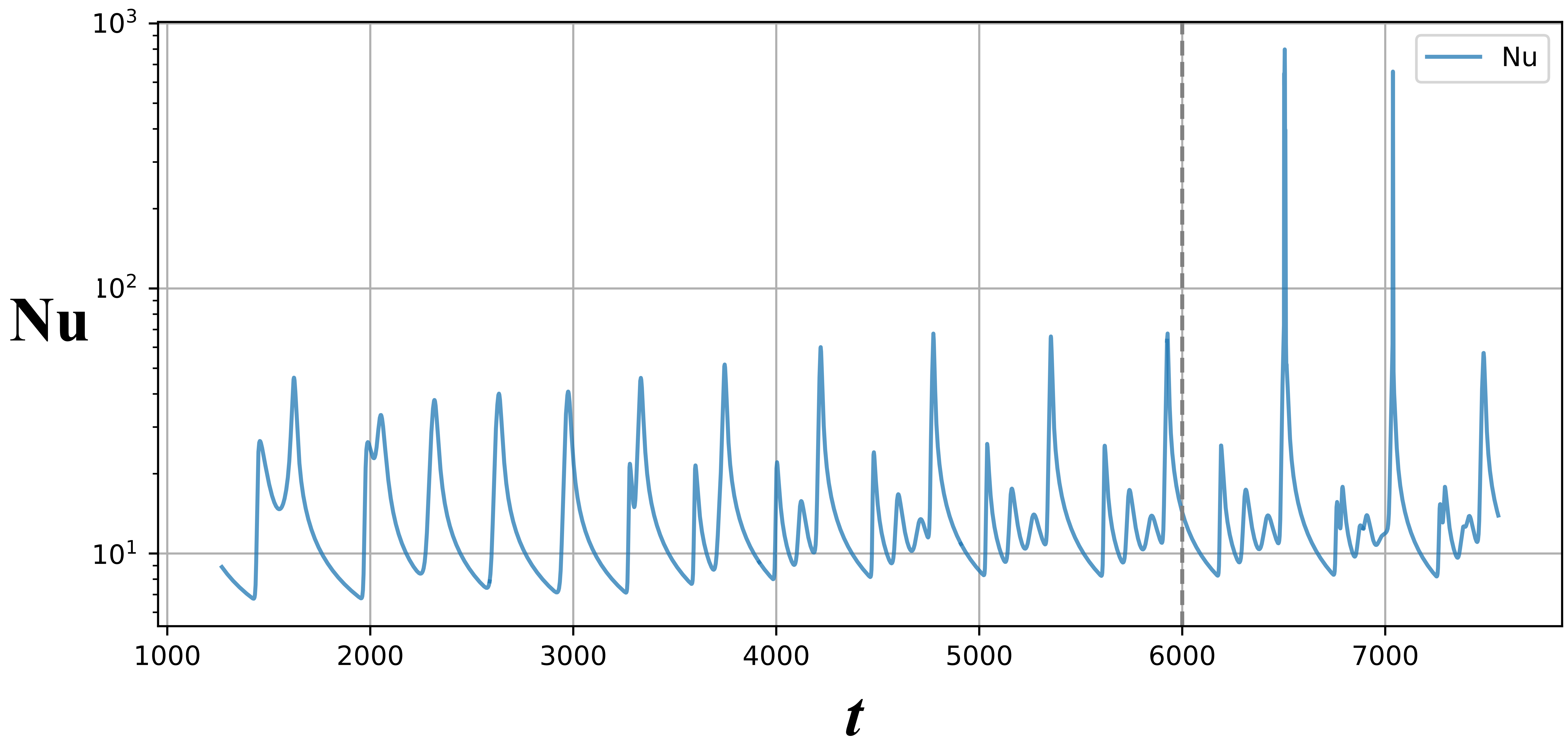


Measured: $\langle \mathrm{Nu} \rangle_t = 16.0 \pm 6.2$

Reference: $\mathrm{Nu}_{\mathrm{AvK}} = 15.8 \pm 1.0$

Strong intermittency at lower Ekman

$\widetilde{\mathrm{Ra}} = 40$
$\mathrm{Ek} = 10^{-7}$
$L_x = 10\ell_c$
$\lambda = 1.16$



Measured: $\langle \mathrm{Nu} \rangle_t = 13.1 \pm 8.3$

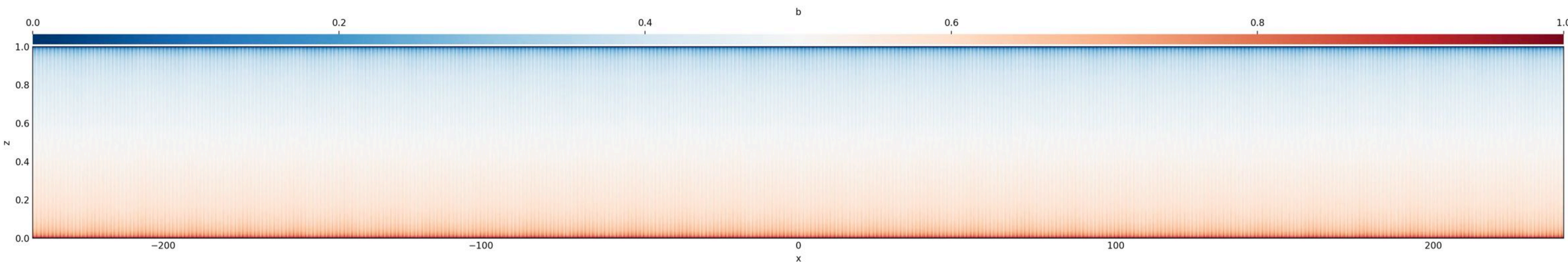
Reference: $\mathrm{Nu}_{\mathrm{AvK}} = 13.2 \pm 0.7$

Extreme domain sizes

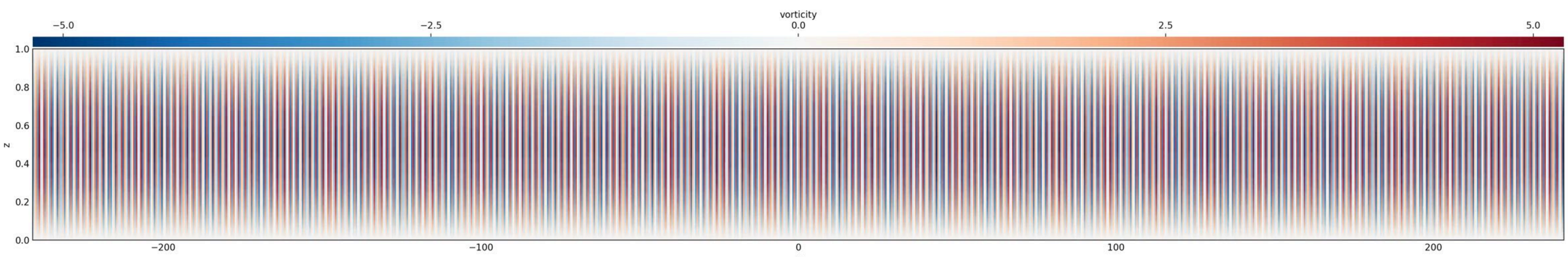
$\widetilde{\mathrm{Ra}} = 40 \qquad L_x = 10{,}000\,\ell_c$

$\mathrm{Ek} = 10^{-6} \qquad \lambda = 1.16$

b



ω_y



Measured: $\langle \mathrm{Nu} \rangle_t = 14.9 \pm 6.7$

Next steps

Lower Ekman

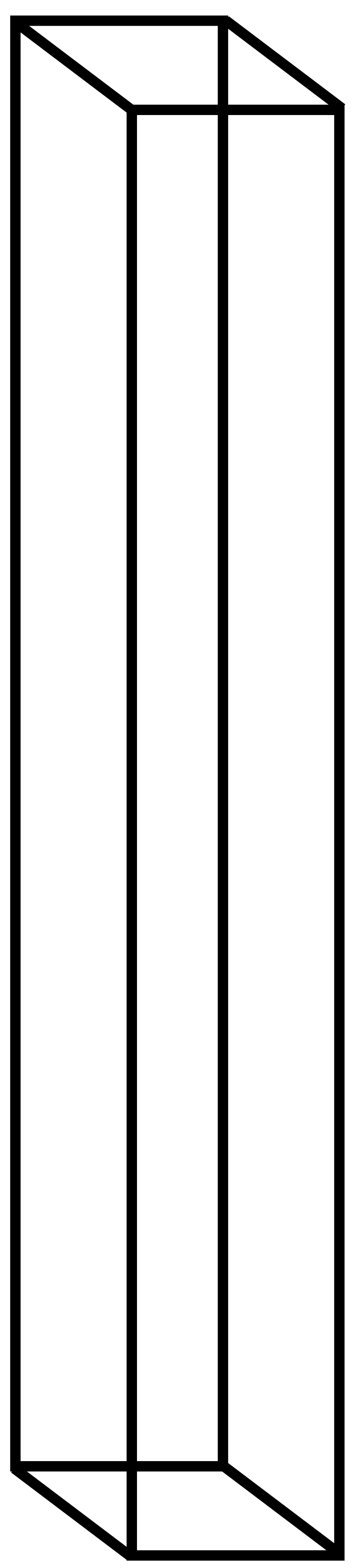
- Artifacts from 2.5D?
- Adaptive timestepping
- Examine large-scales in GT

Extend to 3D

- Even more relative speedup vs DNS
- Difficult parallelization

Optimize lattice

- Vary λ, a, b
- Different lattices
- Flow dependent



$Re^{3/2}$ vs Re^3

Summary

- **Logarithmic Fourier lattices**
  - Alternative to DNS for highly multiscale problems
  - Promising results in homogeneous flows
  - Can mix with standard bases for bounded domains
- **Results for RBC**
  - Can scale to Ra = 10^{17} w/ sparse Chebyshev
  - General agreement with Malkus scaling
- **Results for RRBC**
  - Highly intermittent flows in 2.5D
  - Average statistics match DNS at modest Ek
  - Scales to large domains $L_x \gg \ell_c$
- **Ongoing work:**
  - Optimize λ, triad couplings, etc.
  - Extend to 3D, modifications for coherent structures