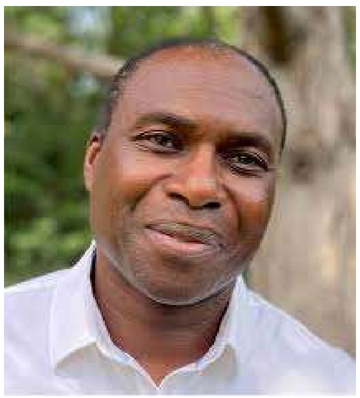
Simulating rotating convection at very small but finite Ekman numbers



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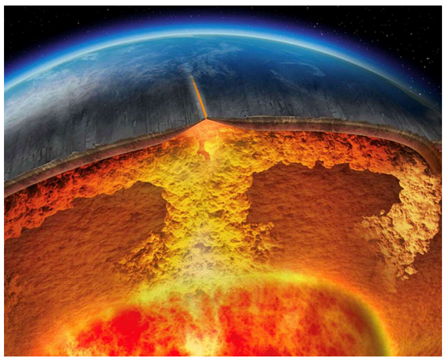
Overview

1) Introduction
2) Non-hydrostatic quasi-geostrophic equations (NHQG)
3) Rescaled incompressible Navier-Stokes equations (RiNSE)
4) Numerical results
5) Conclusions and Outlook

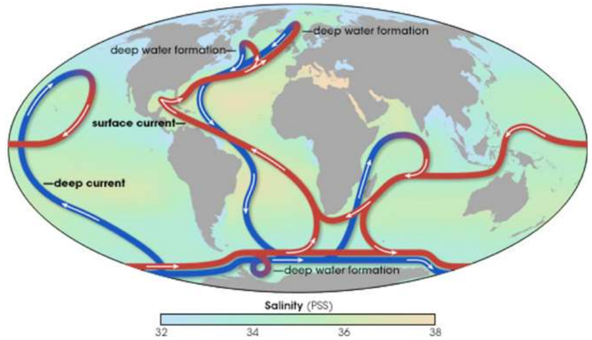
1) Introduction

Convection drives fluid flows on Earth & beyond

Planetary interiors

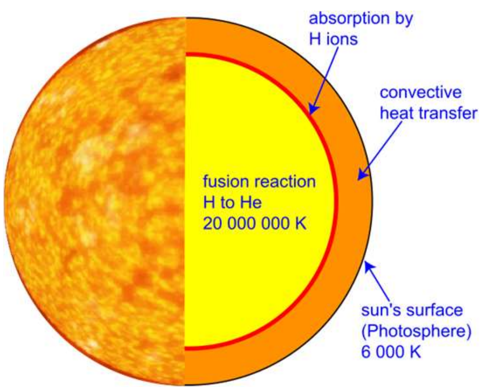


Oceans on Earth …

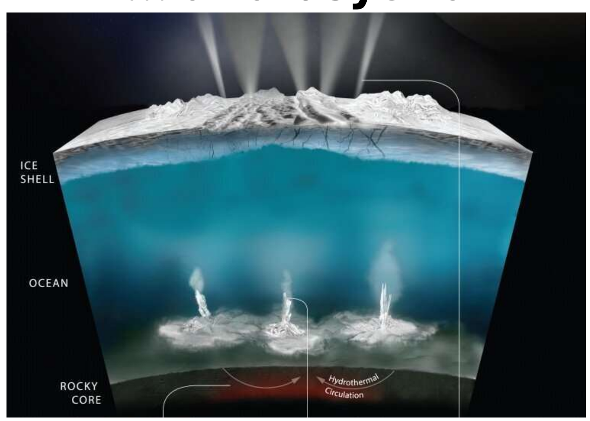


Planetary atmospheres

Stellar interiors

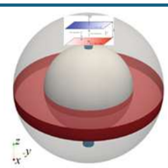


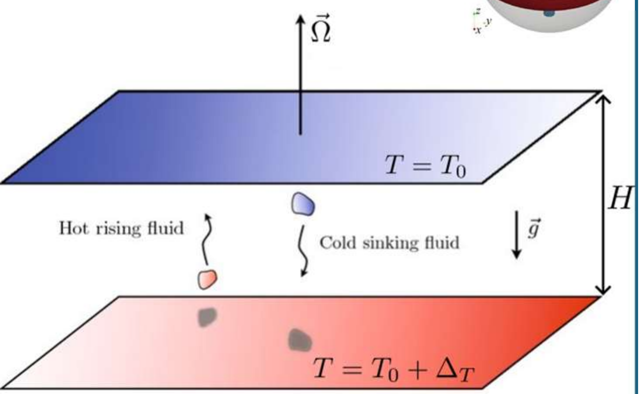
… and beyond



Paradigm: Rotating Rayleigh-Bénard convection

Polar Setup:





Control parameters:

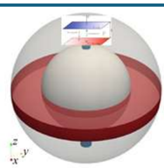
$$Ek = \frac{\nu}{2\Omega H^2}$$

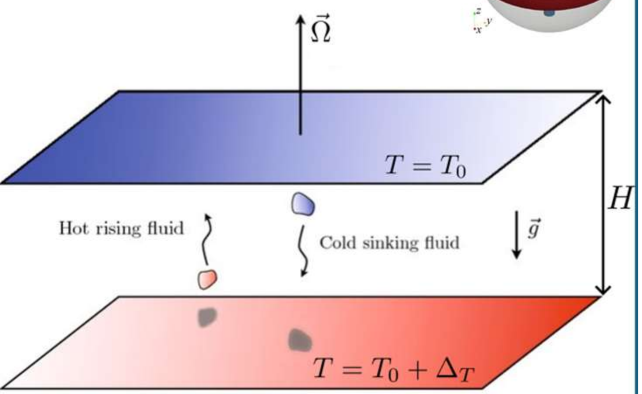
$$Ra = \frac{\alpha g \Delta_T H^3}{\nu\kappa}$$

$$Pr = \nu/\kappa$$

Paradigm: Rotating Rayleigh-Bénard convection

Polar Setup:





Nondimensional Boussinesq-Navier-Stokes equations:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{Ek} \hat{\mathbf{z}} \times \mathbf{u} = - \nabla p + \frac{Ra}{Pr} \theta \hat{\mathbf{z}} + \nabla^2 \mathbf{u}$$

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \hat{\mathbf{z}} \cdot \mathbf{u} + \frac{1}{Pr} \nabla^2 \theta$$

$$\nabla \cdot \mathbf{u} = 0$$

Control parameters:

$$Ek = \frac{\nu}{2\Omega H^2}$$

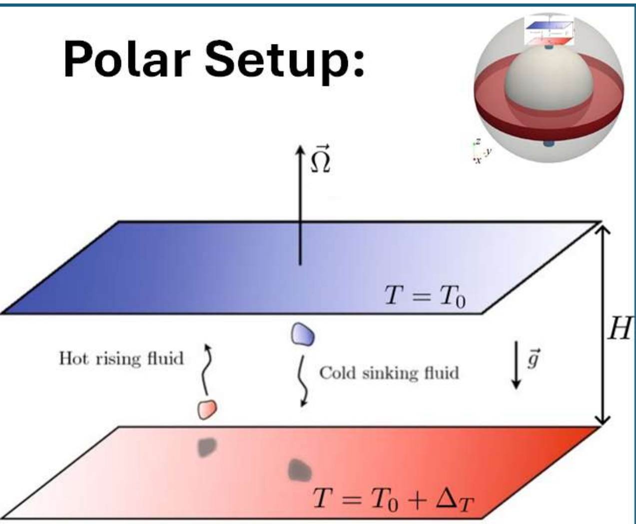
$$Ra = \frac{\alpha g \Delta_T H^3}{\nu \kappa}$$

$$Pr = \nu / \kappa$$

- Constant temperature at top and bottom:
  - $\theta|_{z=0,1} = 0$
- Velocity $\mathbf{u} = (u, v, w)$ is subject to:
  - **Stress-free BC** $\partial_z u|_{z=0,1} = \partial_z v|_{z=0,1} = 0$
  - Impenetrability $w|_{z=0} = w|_{z=1} = 0$
  - Periodicity in the horizontal direction

Paradigm: Rotating Rayleigh-Bénard convection

Polar Setup:



Nondimensional Boussinesq-Navier-Stokes equations:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{Ek}\hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \frac{Ra}{Pr}\theta\hat{\mathbf{z}} + \nabla^2 \mathbf{u}$$

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \hat{\mathbf{z}} \cdot \mathbf{u} + \frac{1}{Pr}\nabla^2 \theta$$

$$\nabla \cdot \mathbf{u} = 0$$

Control parameters:

$$Ek = \frac{\nu}{2\Omega H^2}$$

$$Ra = \frac{\alpha g \Delta_T H^3}{\nu\kappa}$$

$$Pr = \nu/\kappa$$

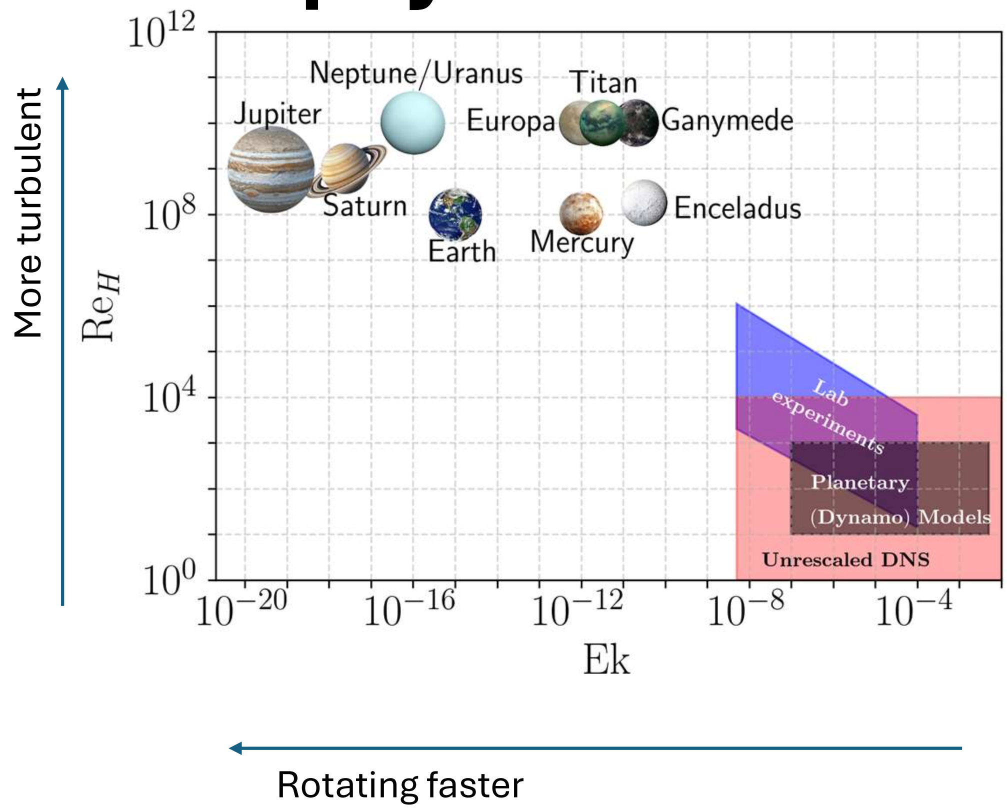
Output parameters include

$$Nu = 1 + Pr\langle w\theta\rangle_D$$

$$U_{rms} = \sqrt{\langle|\mathbf{u}|^2\rangle_D}$$

$$Re_H = \frac{U_{rms}H}{\nu}.$$

The astrophysical limit: $Ek \to 0\,, Re_H \to \infty$



State-of-the-art numerical and experimental techniques cannot reach realistic Ek

<u>Lowest Ek achieved:</u>

DNS (plane layer geometry):

$$\mathrm{Ek} \geq 5 \times 10^{-9}$$

[Song, Shishkina, Zhu, *JFM 2024 a, b*]

DNS (spherical geometry):

$$\mathrm{Ek} \geq 10^{-7}$$

[Gastine and Aurnou, *JFM* 2023]

Laboratory experiments (Eindhoven):

$$\mathrm{Ek} \geq 5 \times 10^{-9}$$

[Cheng+al. *GAFD* 2018, Kunnen *JT* 2020]

Key Challenges

- Fast inertial waves with $O(Ek^{-1})$ frequencies → prohibitive time stepping requirements
- Columnar structures with aspect ratio $\sim Ek^{-1/3} \gg 1$ (Taylor-Proudman) + Ekman layers of depth $O(Ek^{1/2})$ are difficult to resolve
- Scale disparity causes ill-conditioned matrices in discretized problem → numerical errors

How to tackle these?

Linear stability theory:

[Chandrasekhar (1953)]

Linearize about conduction state $\mathbf{u} = \mathbf{0}, \theta = 0$ to obtain the system

$$\partial_t \mathbf{u} + \frac{1}{Ek}\hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \frac{Ra}{Pr}\theta\hat{\mathbf{z}} + \nabla^2 \mathbf{u}$$

$$\partial_t \theta = \hat{\mathbf{z}} \cdot \mathbf{u} + \frac{1}{Pr}\nabla^2 \theta$$

$$\nabla \cdot \mathbf{u} = 0$$

which admits solutions of the form $(\mathbf{u}, \theta) = (\mathbf{U}(z), \Theta(z)) \exp(ik_x x + ik_y y + \sigma t)$

For $Pr > 0.68$: **Onset:** $Ra_c \overset{Ek \to 0}{\sim} 3\left(\frac{\pi^2}{2}\right)^{2/3} Ek^{-4/3} \approx 8.7\underline{Ek^{-4/3}}$

→ **Rotation suppresses convection**

Critical horiz. wavenumber: $k_c \overset{Ek \to 0}{\sim} (2\pi^4)^{1/3}\underline{Ek^{-1/3}H^{-1}}$

→ **Thin columns**

2) Nonhydrostatic quasi-geostrophic equations

Sprague et al., J. Fluid Mech. 2006

Change of units

- Let $\epsilon \equiv Ek^{1/3}$ and define $\widetilde{Ra} = \epsilon^4 Ra$ (cf. linear stability analysis).
- Measure **lengths in units of** $\ell = \epsilon H$, **time in units of** $\tau_\nu = \ell^2/\nu = \epsilon^2 H^2/\nu$, with **velocity scale** $U = \dfrac{\ell}{\tau_\nu} = \epsilon^{-1}\nu/H$, such that

$$\nabla \to \frac{1}{\epsilon}\nabla, \qquad D_t \to \frac{1}{\epsilon^2} D_t, \qquad \boldsymbol{u} \to \frac{1}{\epsilon}\boldsymbol{u}, \qquad p \to \frac{1}{\epsilon^3} p$$

leading to

$$\boxed{\begin{aligned} D_t \mathbf{u} + \frac{1}{\epsilon}\hat{\mathbf{z}} \times \mathbf{u} &= -\frac{1}{\epsilon}\nabla p + \frac{1}{\epsilon}\frac{\widetilde{Ra}}{Pr}\theta\hat{\mathbf{z}} + \nabla^2 \mathbf{u} \\ D_t\theta &= w + \frac{1}{Pr}\nabla^2\theta \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}}$$

Multi-scale asymptotic expansion

Consider $\epsilon = Ek^{1/3} \to 0$, assume $\widetilde{Ra} = RaEk^{4/3} = O(1)$

1) Introduce *slow variables* $Z = \epsilon z, T = \epsilon^2 t$ in addition to fast variables x, y, z, t

substituting $\partial_z \to \partial_z + \epsilon \partial_Z, \quad \partial_t \to \partial_t + \epsilon^2 \partial_T$

2) Decompose $\boldsymbol{v} = (\boldsymbol{u}, p, \theta)$ into $\boldsymbol{v} = \overline{\boldsymbol{v}}(Z, T) + \boldsymbol{v}'(\boldsymbol{x}, Z, t, T)$, with

$$\overline{f}(Z,T) \equiv \lim_{L,\tau\to\infty} \frac{1}{\tau \mathrm{Vol}(D)} \int_D \int_0^\tau f(\mathbf{x}, Z, t, T) d\mathbf{x} dt.$$

3) Expand $\boldsymbol{w} = (\overline{\boldsymbol{v}}, \boldsymbol{v}')$ as $\boldsymbol{w} = \boldsymbol{w}_0 + \epsilon \boldsymbol{w}_1 + \epsilon^2 \boldsymbol{w}_2 + O(\epsilon^3)$, solve order by order

4) **Leading-order geostrophic balance:** $\boxed{\hat{\mathbf{z}} \times \mathbf{u}_0' = -\nabla p_0'}$

Taylor-Proudman constraint: $\partial_z \boldsymbol{u}_0' = \partial_z p_0' = \partial_z \theta_1' = 0$

5) At next order: avoid secular growth of $\boldsymbol{w}_2$ with z by **solvability condition**

→ **closed set of equations**

Non-hydrostatic quasi-geostrophic equations

$$D_t^{\perp} w + \frac{\partial}{\partial Z}\left((\nabla_{\perp}^2)^{-1}\omega\right) = \frac{\widetilde{Ra}}{Pr}\theta' + \nabla_{\perp}^2 w \qquad (1)$$

$$D_t^{\perp}\omega - \frac{\partial w}{\partial Z} = \nabla_{\perp}^2 \omega \qquad (2)$$

$$D_t^{\perp}\theta' + w\partial_Z \overline{\theta} = \frac{1}{Pr}\nabla_{\perp}^2 \theta' \qquad (3)$$

$$\partial_T \overline{\theta} + \partial_Z \overline{w\theta'} = \frac{1}{Pr}\partial_Z^2 \overline{\theta} \qquad (4)$$

$$\partial_x u + \partial_y v = 0. \qquad (5)$$

Where $D_t^{\perp} = \partial_t + u\partial_x + v\partial_y$, $\nabla_{\perp}^2 = \partial_x^2 + \partial_y^2$,
and the vorticity $\omega = \partial_x v - \partial_y u$.

__Key features:__

Small parameter ϵ & **fast scale** z are **eliminated**.
→ **Filtering fast waves** with $O(Ek^{-1})$ frequencies and Ekman layers

Eqs. (1-5), a.k.a. **3D Hasegawa-Mima Eqs.**, related equations were studied by **Edriss Titi** + al.

- *J. Evol. Equ.* 21 (2021), 2923–2954.
- *J. Diff. Equ.* 269.10 (2020): 8736-8769.
- *J. Math. Phys.* 59.7 (2018)
- *Comm. Math. Phys.* 319 (2013): 195-229.

Global existence & uniqueness, cont. dependence on initial data of solutions to (1-5) with small vertical dissipation in (2).

Equivariance under **reflections**

$$\begin{cases} (x,y) & \to (-x,y) \\ (u,v,w,\omega) & \to (-u,v,-w,-\omega) \\ (\theta',\overline{\theta}) & \to (-\theta',\overline{\theta}) \end{cases}$$

and similar for $(x,y) \to (x,-y)$ indicates **absence of handedness**.

3) Rescaled incompressible Navier-Stokes equations

Julien, **AvK** et al., arXiv:2410.02702 (2025)
AvK, Julien et al., arXiv:2409.08536 (2025)

Rescaled Navier-Stokes equations (RiNSE)

$$\boxed{\begin{aligned} D_t\mathbf{u} + \frac{1}{\epsilon}\hat{\mathbf{z}}\times\mathbf{u} &= -\frac{1}{\epsilon}\nabla p + \frac{1}{\epsilon}\frac{\widetilde{Ra}}{Pr}\theta\hat{\mathbf{z}} + \nabla^2\mathbf{u} \\ D_t\theta &= w + \frac{1}{Pr}\nabla^2\theta \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}}$$

Starting point:
Boussinesq-Navier-Stokes equations
($\mathrm{D_t} \equiv \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$)

Idea: rescale **full** Boussinesq-Navier-Stokes equations based on NHQG scaling laws valid at $\mathrm{Ek} \to 0$

Two key steps:

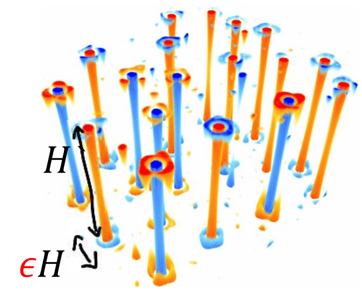
1) Adopt **anisotropic** length scales, such that

$$\nabla^2 \to \nabla_\epsilon^2 \equiv \partial_x^2 + \partial_y^2 + \epsilon^2\partial_z^2$$

and material derivatives become

$$D_t \to D_t^\epsilon \equiv \partial_t + u\partial_x + v\partial_y + \epsilon w\partial_z = \partial_t + \boldsymbol{u}_\perp\cdot\boldsymbol{\nabla}_\perp + \epsilon w\partial_z$$

2) Decompose temperature as $\theta = \Theta(z,t) + \epsilon\vartheta(x,y,z,t)$.



Grooms et al. PRL (2010)

Rescaled Navier-Stokes equations (RiNSE)

Equations governing horizontal fluctuations $(k_x, k_y) \neq (0,0)$

$$\boxed{\begin{aligned} D_t^{\epsilon}\mathbf{u} + \epsilon^{-1}\hat{\mathbf{z}} \times \mathbf{u} &= -\,\epsilon^{-1}\nabla_{\epsilon}p + \frac{\widetilde{Ra}}{Pr}\vartheta\hat{\mathbf{z}} + \nabla_{\epsilon}^2\mathbf{u} \\ D_t^{\epsilon}\vartheta + w(\partial_z\Theta - 1) &= \frac{1}{Pr}\nabla_{\epsilon}^2\vartheta \end{aligned}}$$

Where $\epsilon = \mathrm{Ek}^{1/3}$ and the *reduced* Rayleigh number is $\widetilde{\mathrm{Ra}} = \mathrm{Ra}\ \mathrm{Ek}^{4/3} = \mathrm{Ra}\ \epsilon^4$.

Horizontally averaged temperature is governed by

$$\epsilon^{-2}\partial_t\Theta + \partial_z\overline{w\vartheta}^{xy} = \frac{1}{Pr}\partial_{zz}\Theta$$

Becomes negligible as horizontal domain size is increased

4) Numerical results

Julien, **AvK** et al., arXiv:2410.02702 (2025)
AvK, Julien et al., arXiv:2409.08536 (2025)

Numerical method

- Use spectral PDE solver **Coral** [Miquel JOSS 2021]
- Expand fields in Fourier basis in x, y and in Chebyshev polynomials in z

$$(\mathcal{M}\partial_t - \mathcal{L}_I)\,\mathbf{v} = \mathcal{N}(\mathbf{v}, \mathbf{v})$$

$$\Big\downarrow \text{discretization}$$

$$(\mathbf{M}\partial_t - \mathbf{L}_I)\mathbf{v} = \mathbf{b}$$

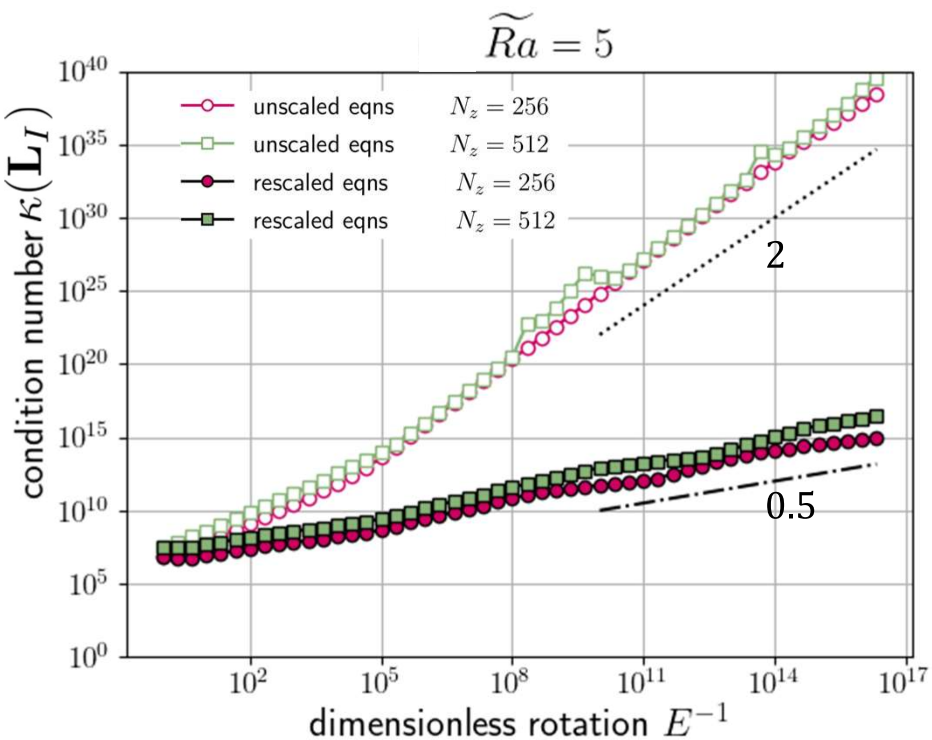
- Assess **numerical accuracy** at low Ek in two complementary ways:
  1. Compute condition number of matrices on LHS
  2. Consider generalized eigenvalue problem $(\mathbf{M}s - \mathbf{L}_I)\mathbf{v} = 0$

Conditioning properties at low Ek

Condition number of discretized linear operator:

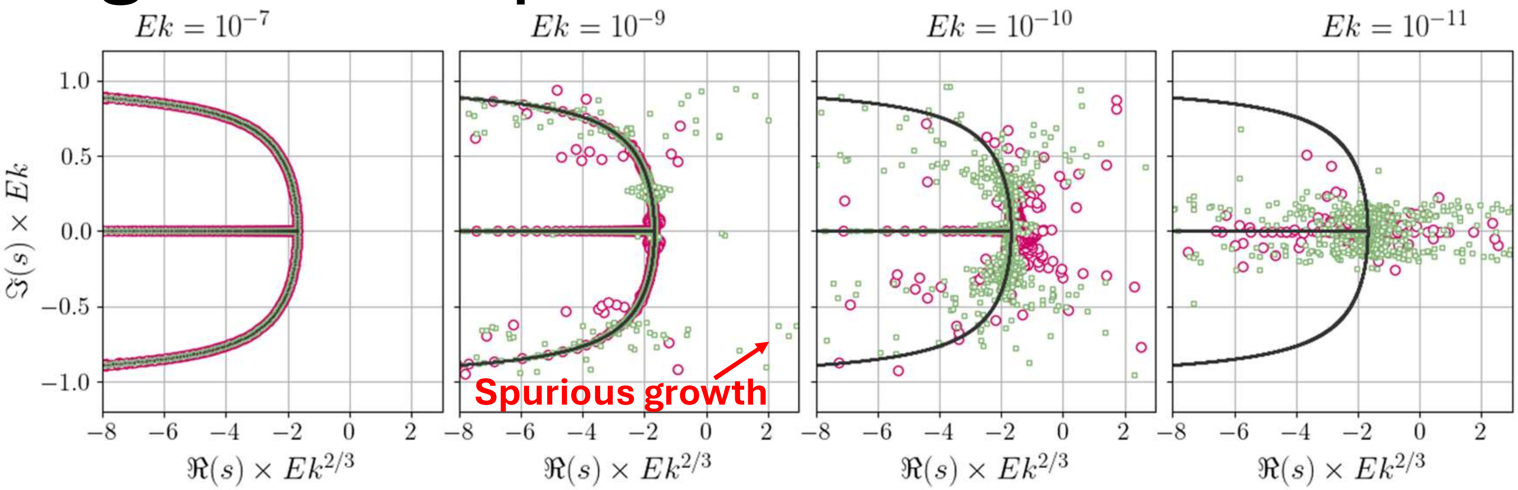
$$\kappa(\mathbf{L}_I) = \|\mathbf{L}_I\|_2 \, \|\mathbf{L}_I^{-1}\|_2$$

Large κ indicates sensitivity to errors in right-hand side



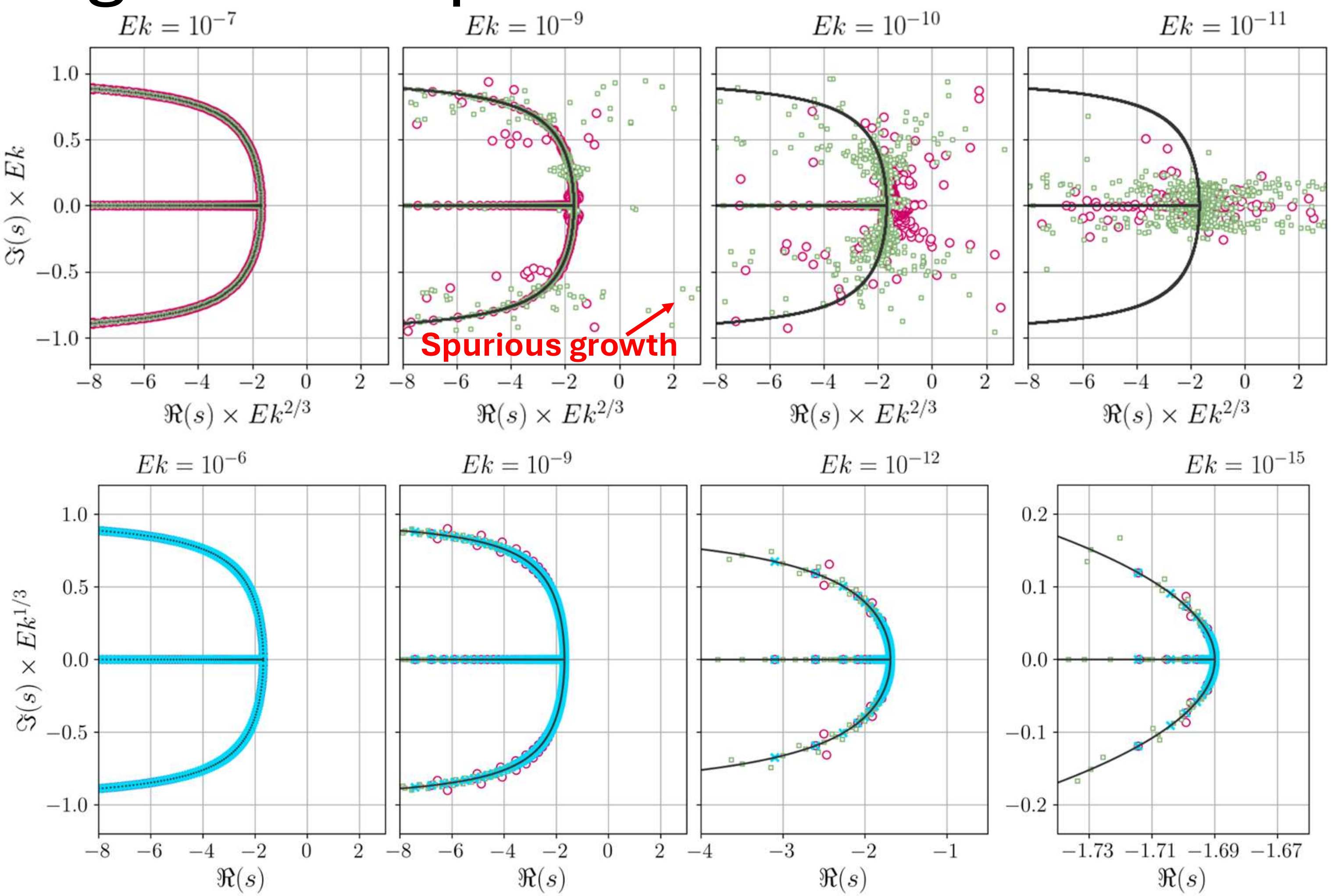
RiNSE formulation ensures adequate preconditioning

Eigenvalue spectrum



Unrescaled Boussinesq-NS system
For $n_z = 256, 512$,
$k_\perp = 1.3 Ek^{1/3}$,
$\Pr = 1, \widetilde{Ra} = 5$

Eigenvalue spectrum



Unrescaled Boussinesq-NS system
For $n_z = 256, 512$
$k_\perp = 1.3Ek^{1/3}$,
$Pr = 1, \widetilde{Ra} = 5$

RiNSE

Correct spectrum recovered, no spurious instability

Nonlinear, direct numerical simulations

Integrate the RiNSE with **stress-free BC** from small-amplitude noise

- over a wide range of Ekman numbers

$$\mathbf{10^{-1} \leq Ek \leq 10^{-15}}$$

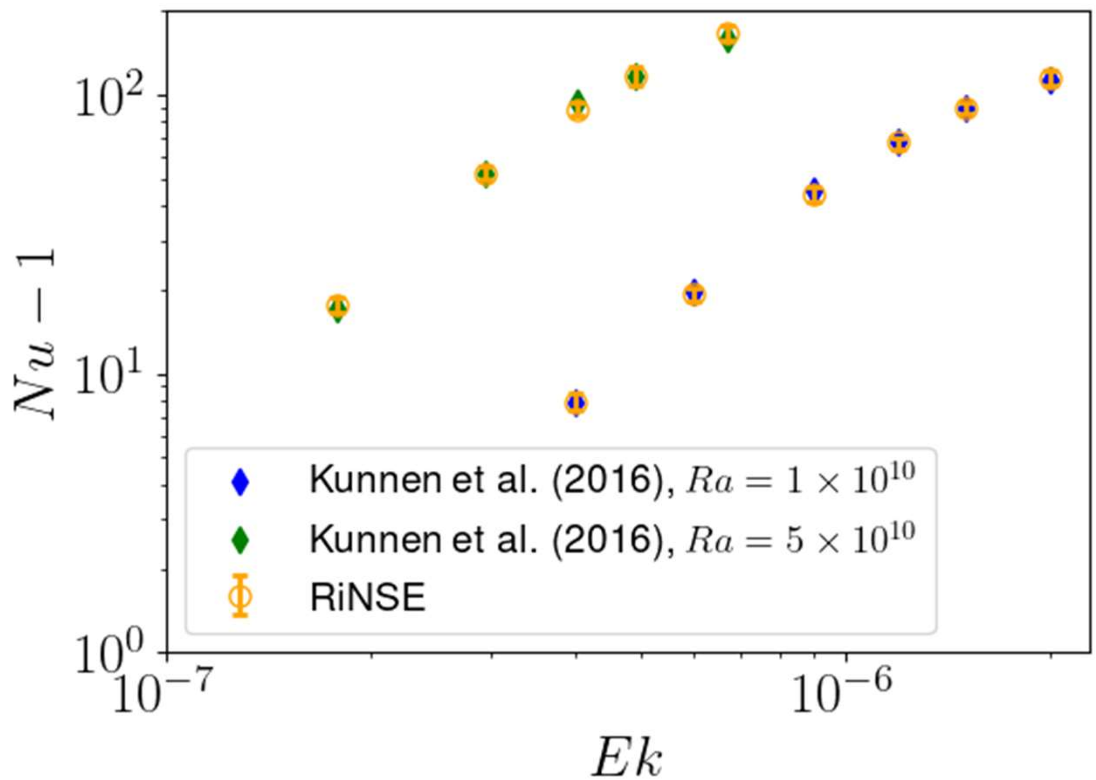
- and

$$\mathbf{Pr = 1}$$

- in a **slender domain** of horizontal extent $L = 10\ell_c \approx 48.2\mathrm{Ek}^{1/3}H$
- Resolve thermal boundary layers in vertical, energy dissipation scale in horizontal

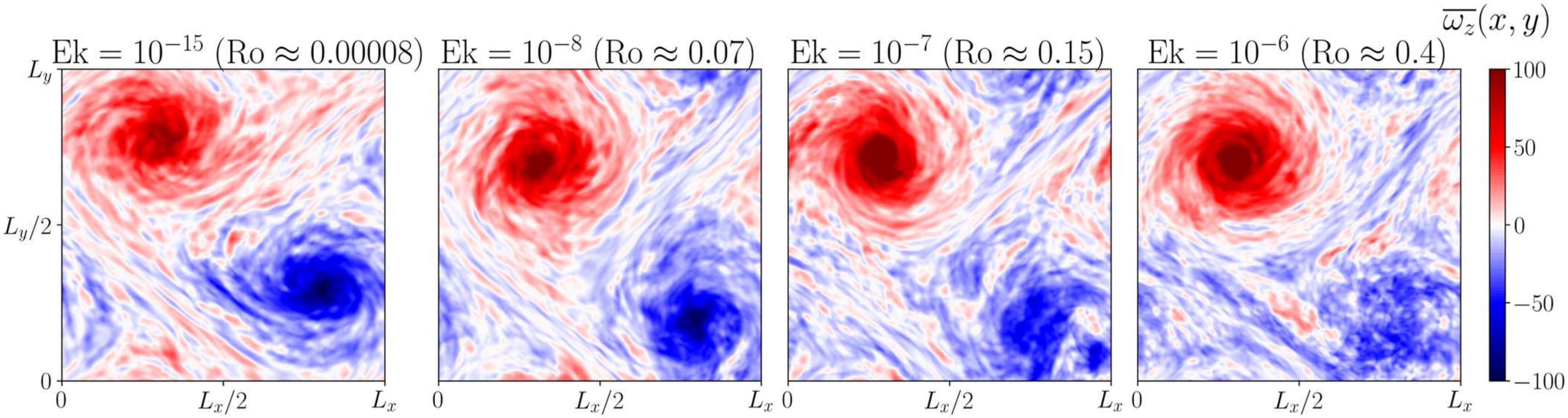
Spoiler:
Will present evidence for convergence of full governing equations towards the reduced NHQGE at small Ek.

Validation of RiNSE at moderate Ek



RiNSE reproduces Nusselt numbers from the literature

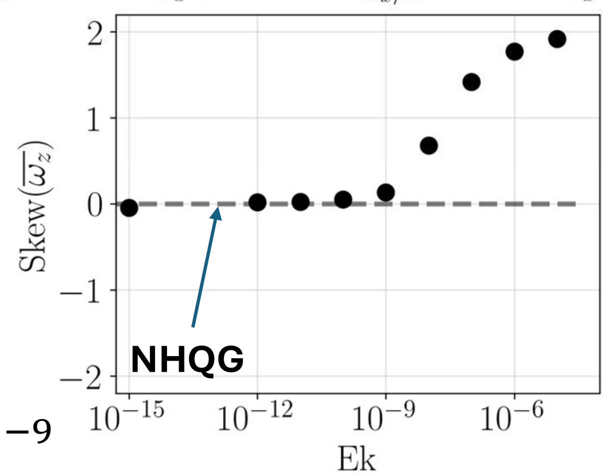
Cyclone anti-cyclone symmetry breaking



Vertically averaged vertical vorticity:

$$\overline{\omega_z}(x, y) \equiv \int_0^1 (\partial_x v - \partial_y u) dz$$

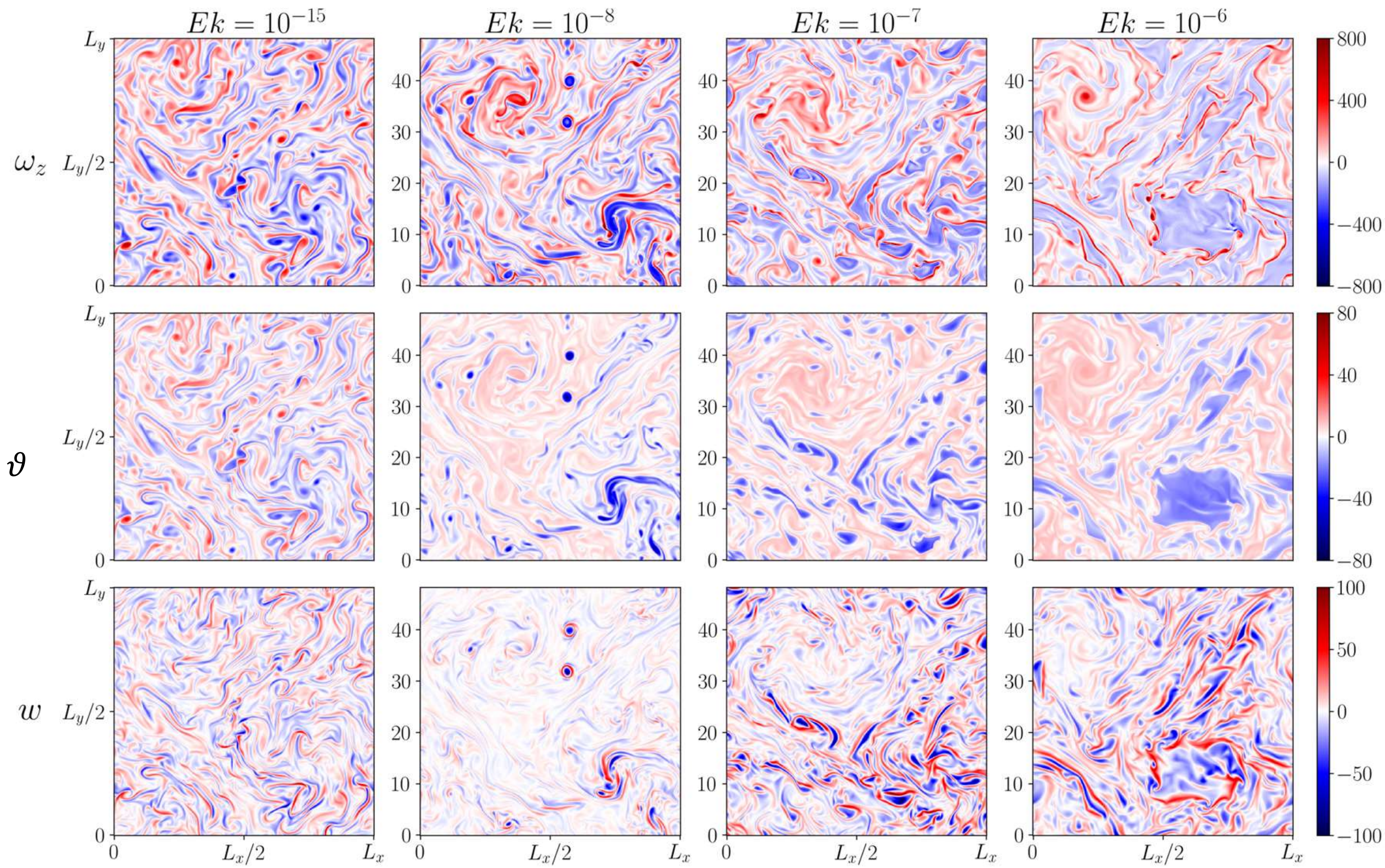
Data shown for $\widetilde{Ra} = 120$.



Symmetries of NHQG are respected at $Ek \lesssim 10^{-9}$

Boundary layer flow morphology ($z < \delta_\theta$)

$\boxed{\widetilde{Ra} = 120}$



Boundary layer flow morphology ($z < \delta_\theta$)

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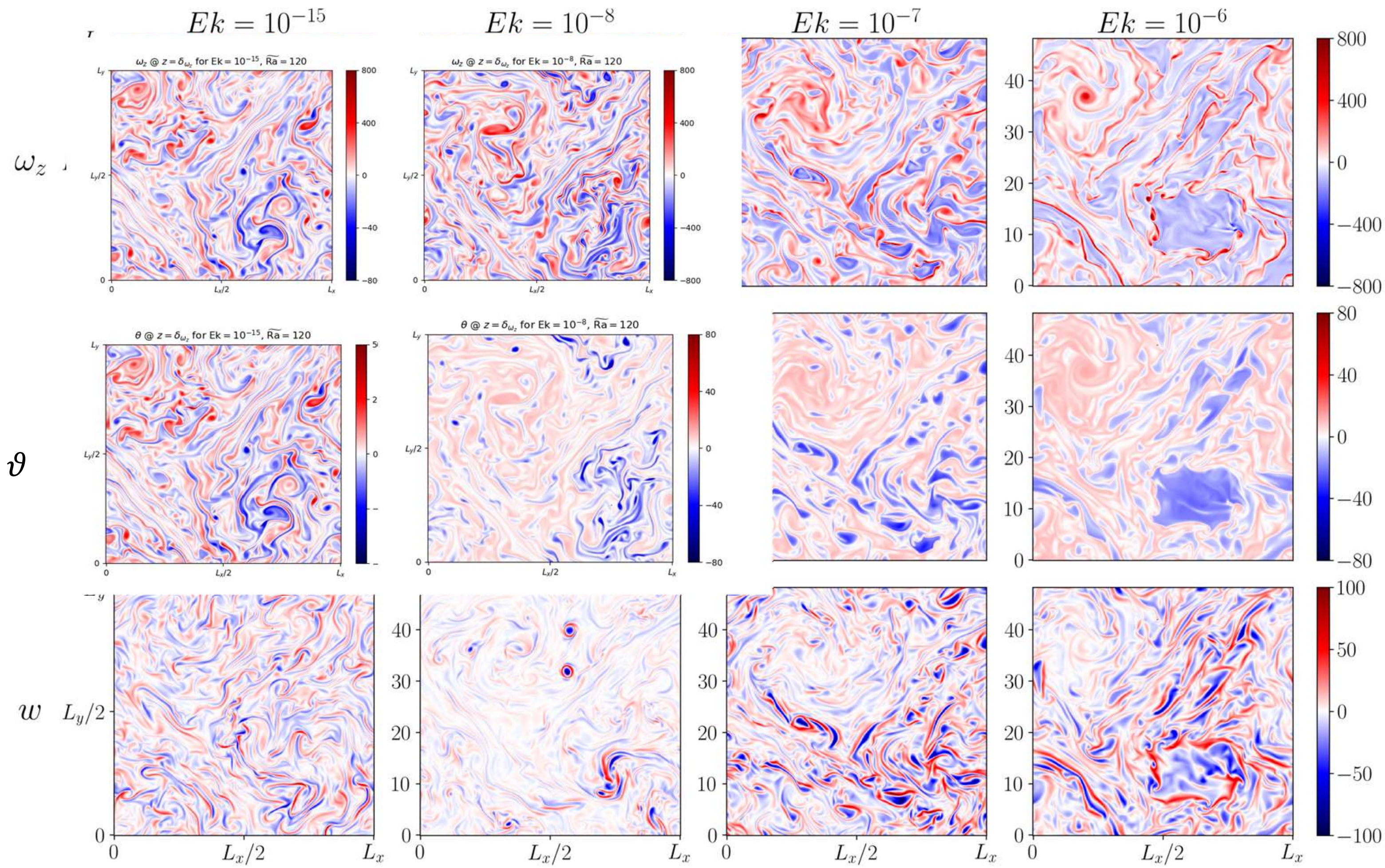

Boundary layer flow morphology ($z < \delta_\theta$)

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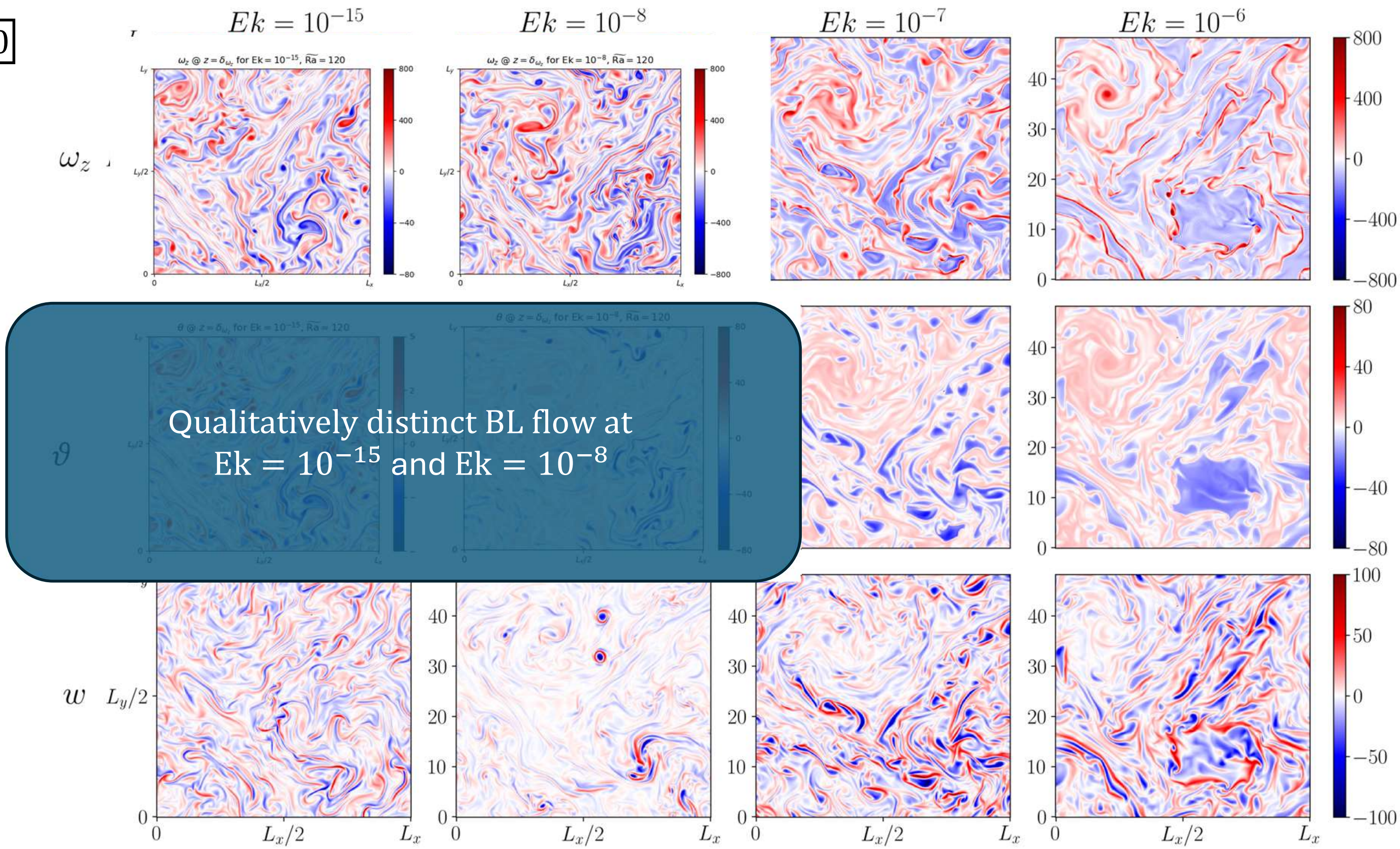
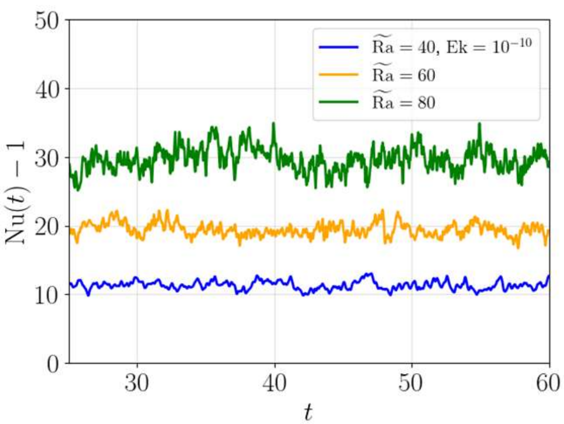


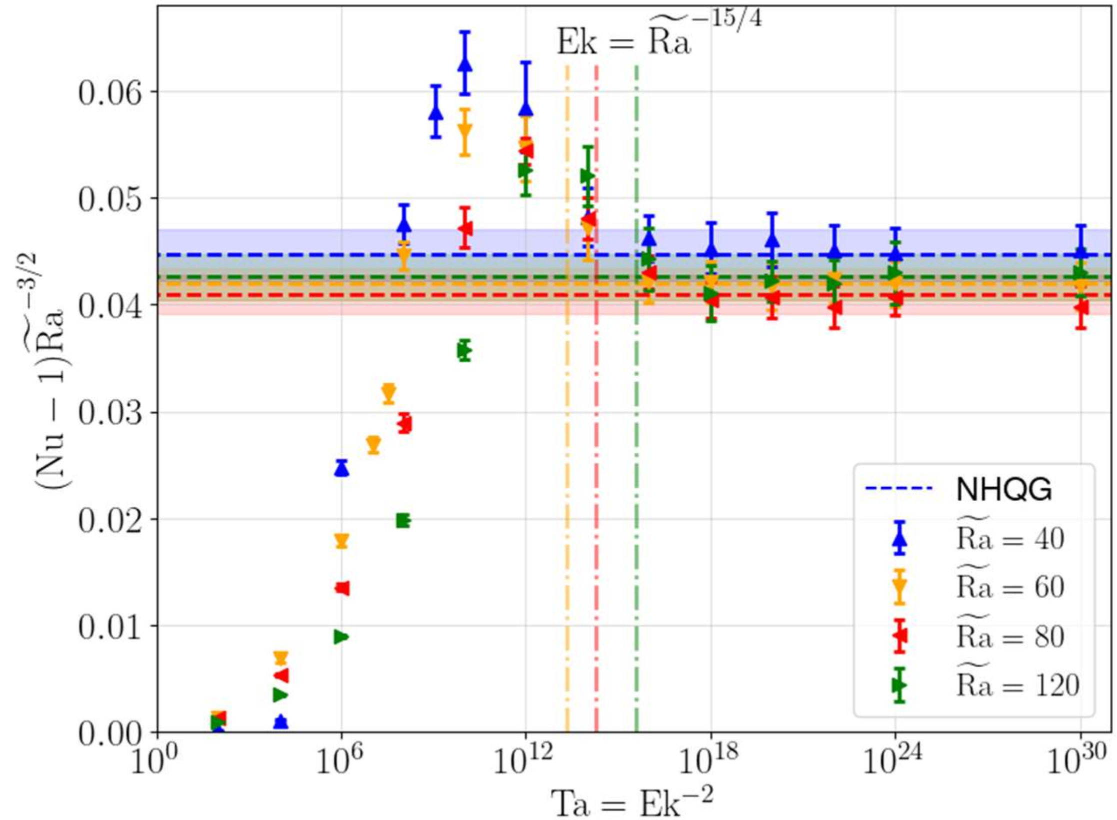
Flow statistics: Nusselt number

[Julien et al. Phys. Rev. Lett. 2012]

Measure time series of Nu

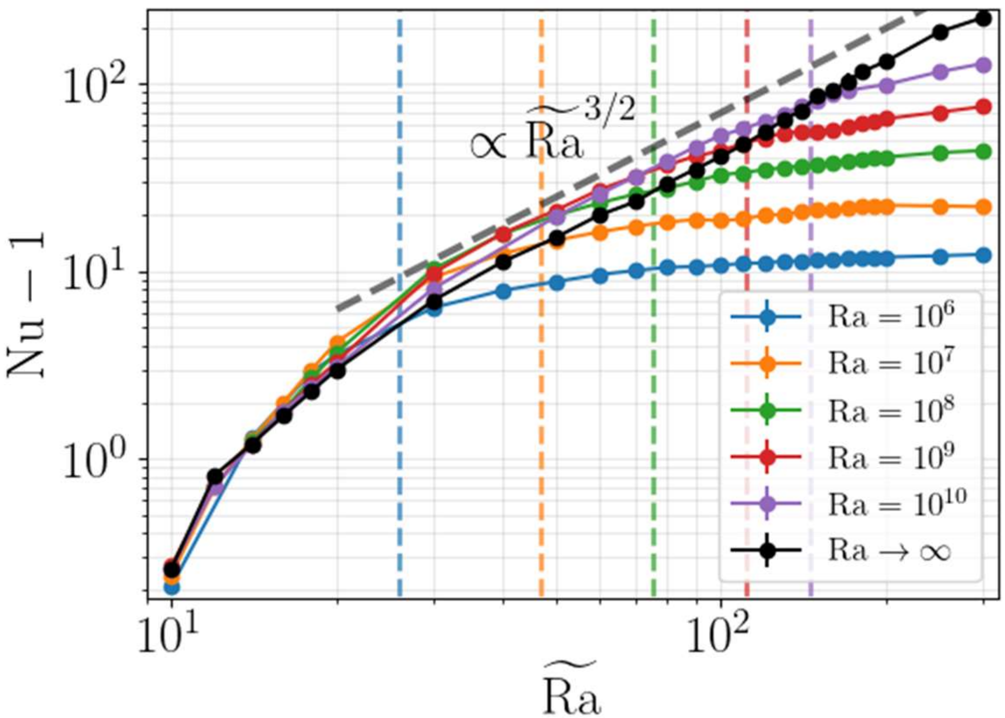


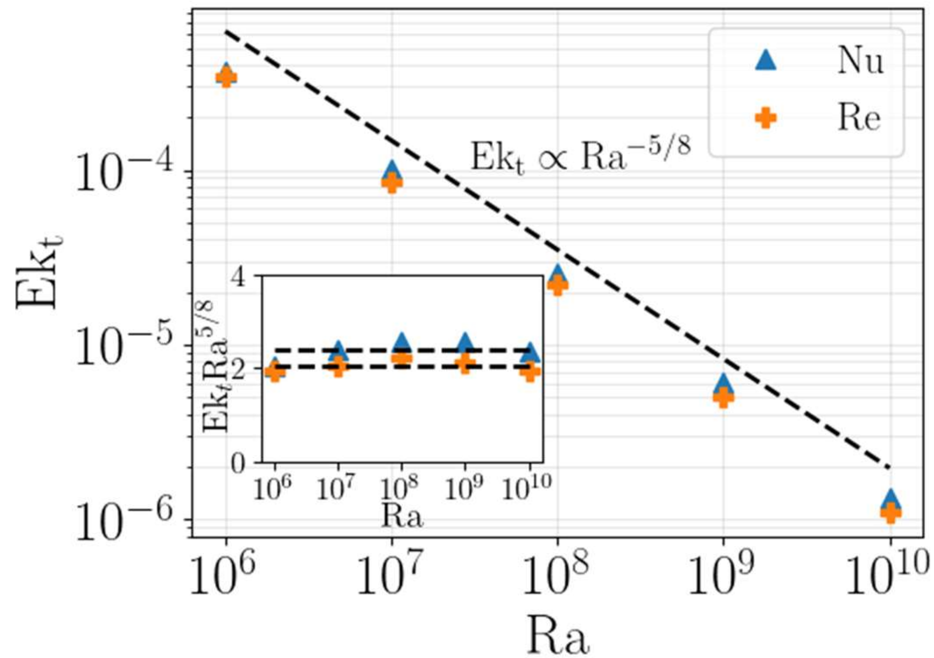
Average,
Rescale



Global statistics in agreement within one standard deviation

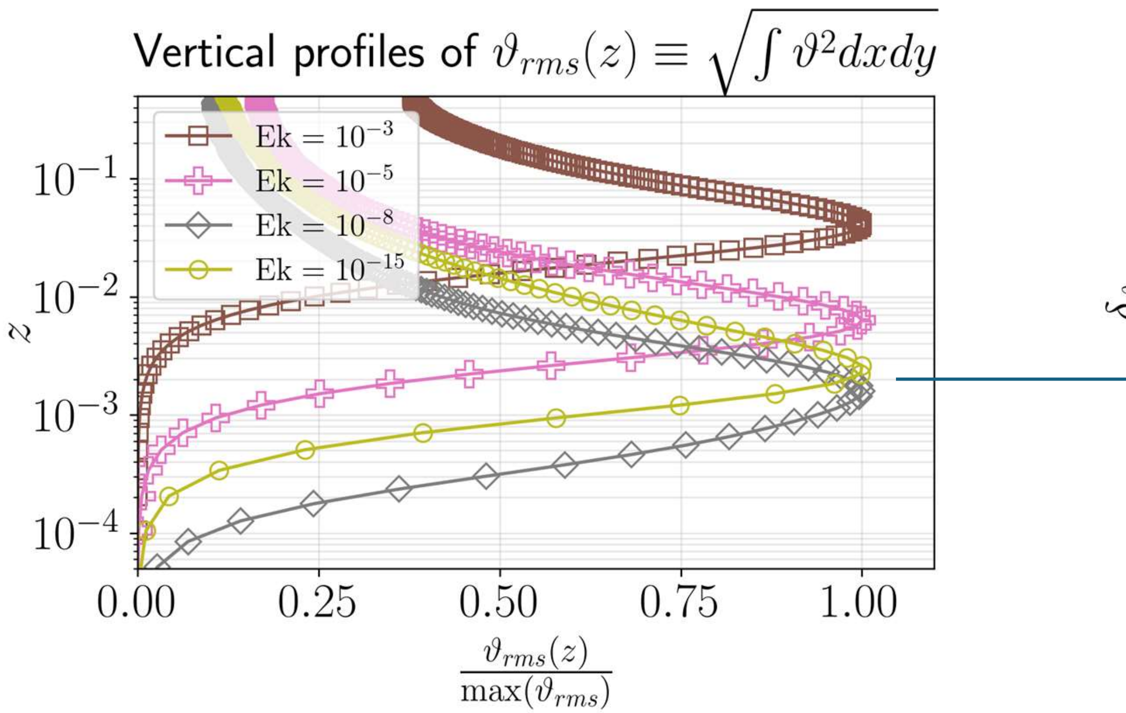
Alternative parameter space cut: $\mathrm{Ra} = const.$

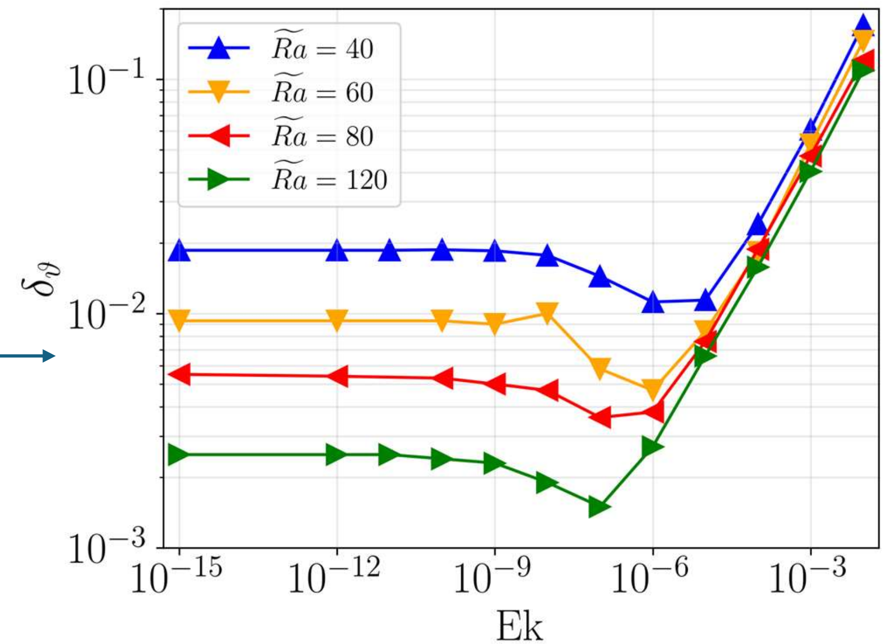




Confirms predicted scaling law
[Julien et al. PRL 2012]

Thermal boundary layer depth



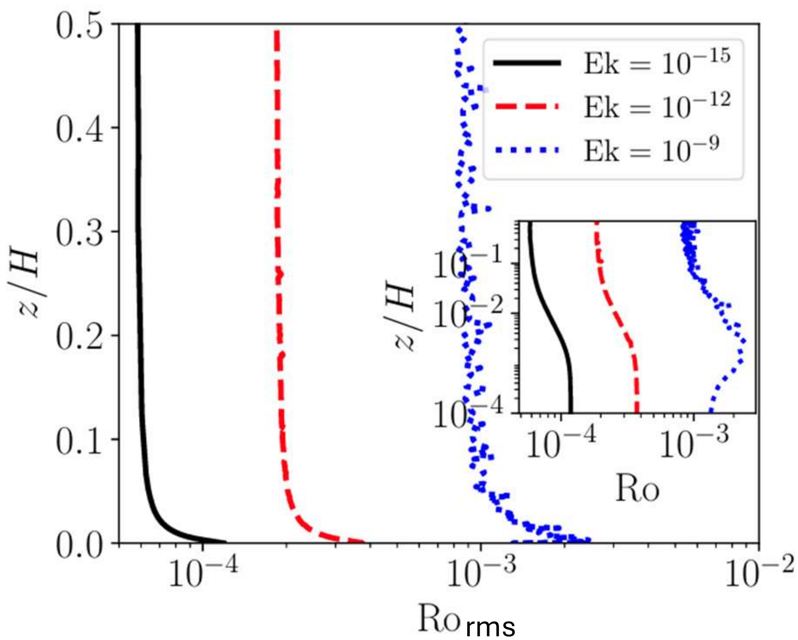


Denote by δ_ϑ the location of the local maximum of $\vartheta_{rms}(z)$

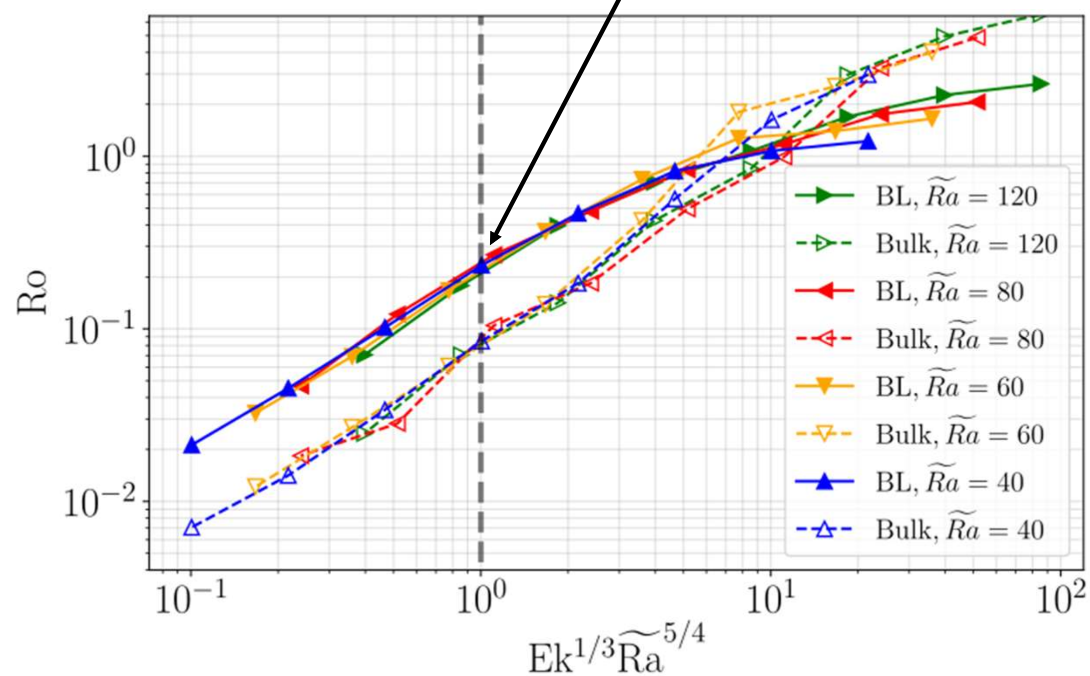
Boundary layer depth converges to NHQG value

A posteriori Rossby number

$$\mathrm{Ro} = \overline{|\mathbf{u}\cdot\nabla\mathbf{u}|}/(2\Omega|\mathbf{u}_\perp|)$$

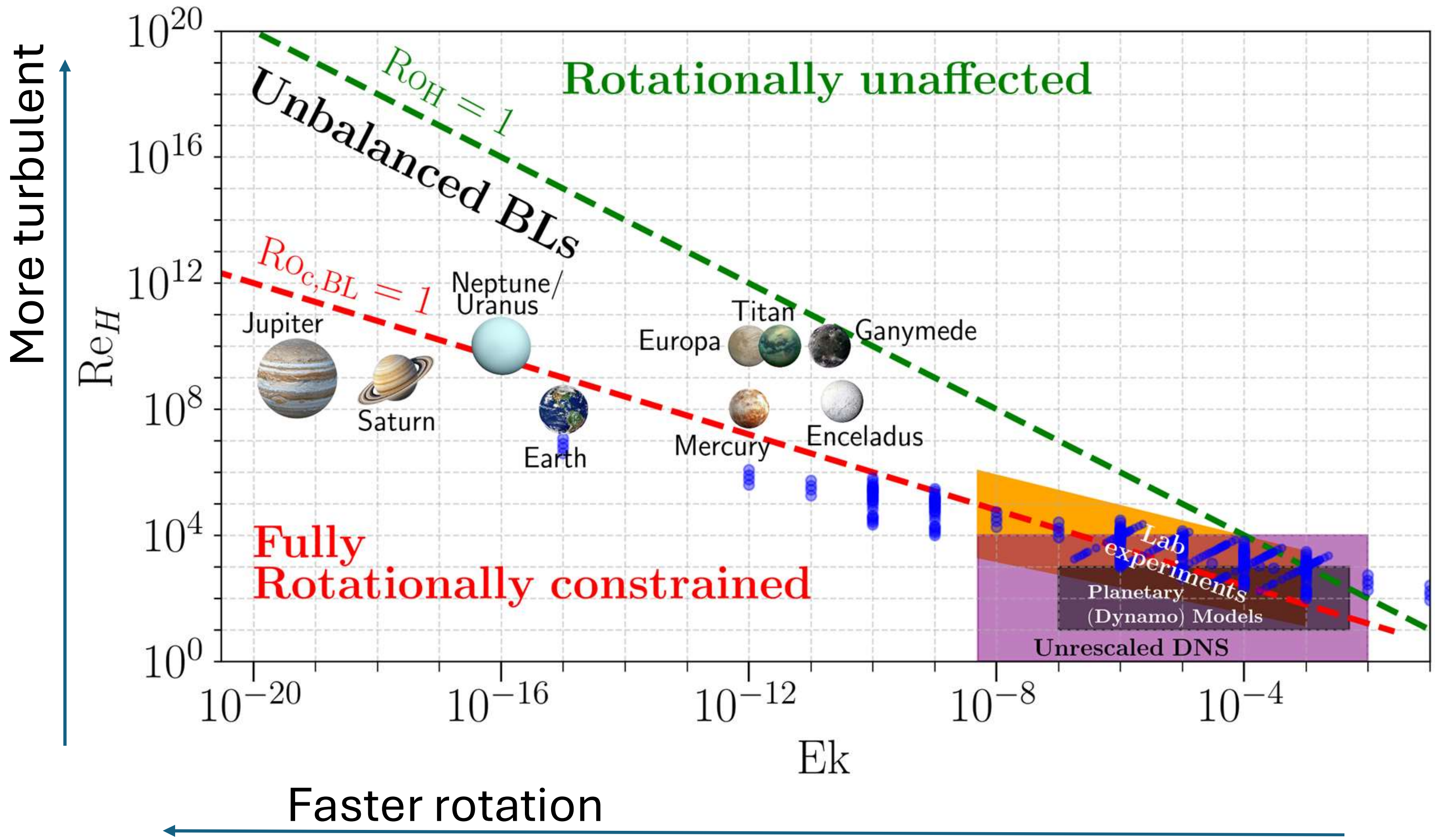


Predicted loss of rotational support in BL



Rossby number self-similar when plotted versus $\mathrm{Ek}\,\widetilde{Ra}^{5/4}$

Parameter space



Simulations at Ek
6 orders of magnitude beyond state of the art

NB: based on $Re_H = Ro_c^2/Ek$
Also $Re_H = \frac{1}{Ek^{1/3}} Re_\ell$

Summary

- Described NHQG equations arising from asymptotic expansion
- Introduced RiNSE formulation based on NHQG scaling laws
- Performed numerical simulations at Ek down to 10^{-15}

(six orders of magnitude smaller than previous state of the art)

- **Numerical evidence points to convergence of RiNSE to NHQGE** at small Ek based on symmetry, global statistics, small-scale structure
- Threshold value below which convergence is observed $\mathrm{Ek} \approx \widetilde{\mathrm{Ra}}^{-\frac{15}{4}}$

confirming theoretical predictions by [Julien+al. GAFD 2012]

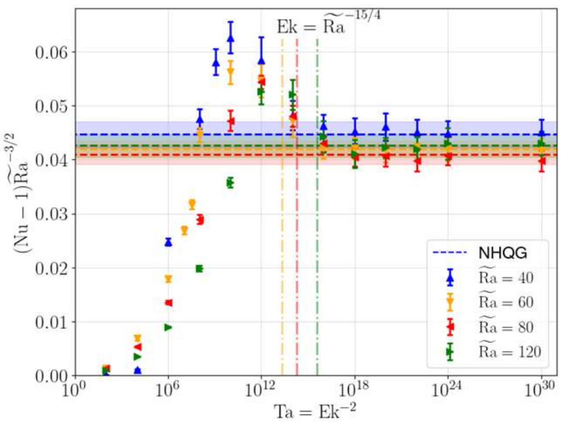
Under review: Julien et al. arXiv:2410.02702,
van Kan et al. arXiv:2409.08536

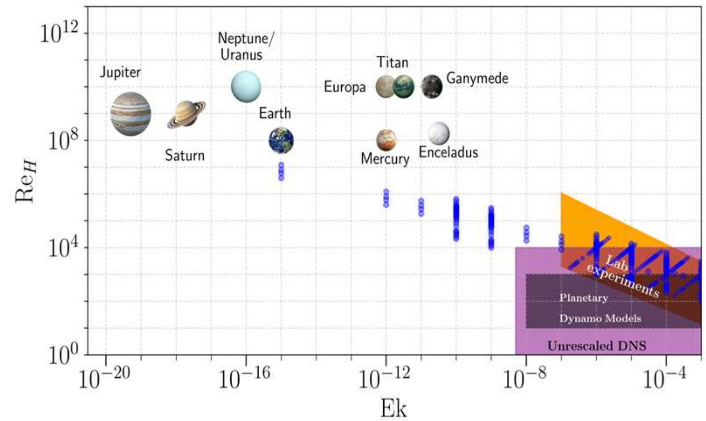
Outlook

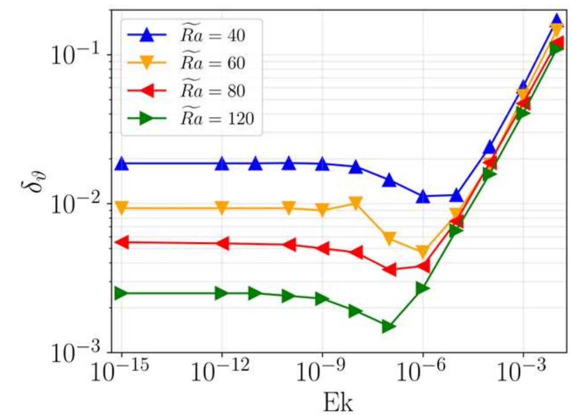
- RiNSE opens the door to exploration of **unprecedented parameter regimes**

Important additional physics should be included, such as

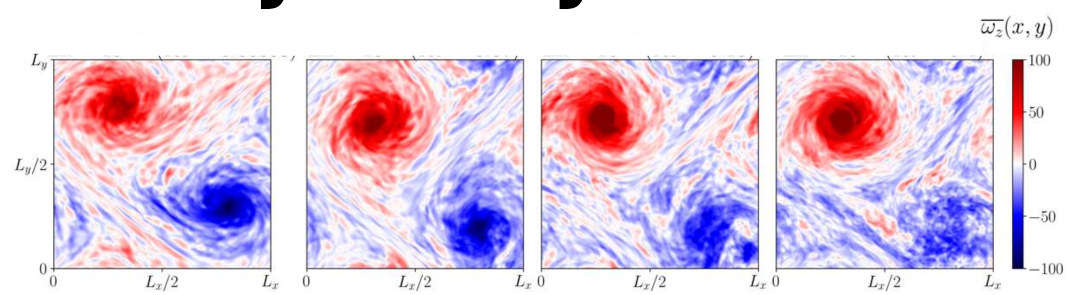
  - Prandtl numbers different from unity
  - Alternative boundary conditions (e.g., no-slip walls → Ekman pumping)
  - Internal heating
  - Latitudinal variations
  - Magnetic fields
  - …







Thank you for your attention!



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