Stationary Distributions for Exclusion Processes on Z^d

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The Exclusion Process

1. S is a countable set and $p(x, y)$ are transition probabilities for an irreducible Markov chain on S. Usually, $S = Z^d$.

2. η_t is a continuous time Markov process on $\{0, 1\}^S$ in which particles attempt to move with probabilities $p(x, y)$ after unit exponential holding times, but particles are not allowed to move to occupied sites.

Some History

C. T. MacDonald, J. H. Gibbs and A. C. Pipkin, *Kinetics of biopolymerization on nucleic acid templates*, Biopolymers **6** (1968), 1–25.

F. Spitzer, *Interaction of Markov processes*, Adv. Math. **5** (1970), 246–290.

Number of MathSciNet hits for **exclusion process** = 446.

A probability measure μ on $\{0,1\}^S$ is **stationary** if $\eta_0 \sim \mu$ implies $\eta_t \sim \mu$ for all $t > 0$.

Notation: $\mathcal{I}$ = set of stationary distributions;
$\mathcal{I}_e$ = extreme points of $\mathcal{I}$.
If $\alpha : S \to [0,1]$, ν_α is the product measure with

$$\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha(x).$$

The Symmetric Case: $p(x,y) = p(y,x)$

Main simplification: If μ is any probability measure on $\{0,1\}^S$, then

$$P^\mu[\eta_t(x_1) = 1, ..., \eta_t(x_n) = 1]$$

depends on μ only through

$$\mu\{\eta : \eta(y_1) = 1, ..., \eta(y_n) = 1\}.$$

Also, the **hydrodynamics** are governed by the heat equation

$$u_t = c\Delta u,$$

while if the process has a drift, it is governed by Burger's equation

$$u_t = c \cdot \nabla[u(1-u)].$$

The Stationary Distributions

Let $\mathcal{H} = \{\alpha(\cdot) : 0 \leq \alpha(x) \leq 1, \sum_y p(x,y)\alpha(y) = \alpha(x)\}$. Then

(a) $T_t \nu_\alpha \to \mu_\alpha$ as $t \uparrow \infty$ for all $\alpha \in \mathcal{H}$.

(b) $\mu_\alpha\{\eta(x) = 1\} = \alpha(x)$.

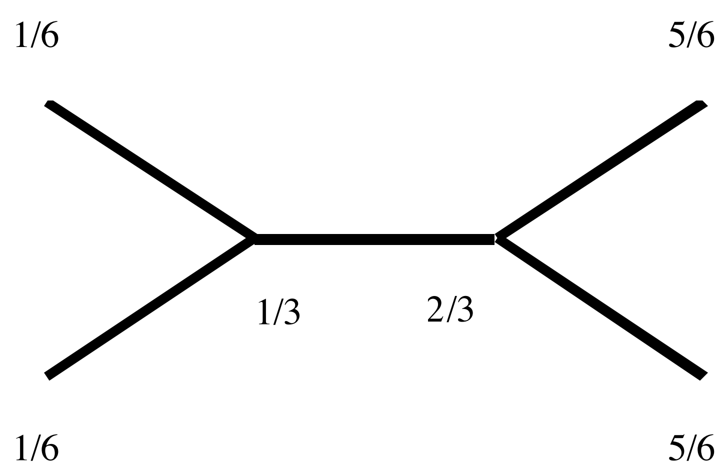
(c) $\mu_\alpha = \nu_\alpha$ if and only if α is constant.

(d) $\mathcal{I}_e = \{\mu_\alpha : \alpha \in \mathcal{H}\}$.

Examples

1. $S = Z^d$, $p(x,y) = p(y-x)$. Then $\mathcal{H}$ consists only of constants, so $\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$, i.e., all the homogeneous product measures.

2. S is a homogeneous tree. Then $\mathcal{H}$ contains many nonconstant α's, and hence $\mathcal{I}_e$ contains many measures that are not product measures. Here is one such α:



One Dimension: $S = Z^1,\ p(x,y) = p(y-x)$

M. Bramson, T. Liggett and T. Mountford, *Characterization of stationary measures for one dimensional exclusion processes*, Ann. Probab. **30** (2002), 1539–1575.

Assumptions

$\sum_x |x|p(x) < \infty$, and then without loss of generality, that $\sum_x xp(x) \geq 0$.

Definitions

A probability measure μ on $\{0,1\}^{Z^1}$ is a *blocking measure* if

$$\mu\left\{\eta : \sum_{x<0} \eta(x) < \infty, \sum_{x>0} [1-\eta(x)] < \infty\right\} = 1.$$

It is a *profile measure* if

$$\lim_{x\to-\infty} \mu\{\eta : \eta(x) = 1\} = 0, \quad \lim_{x\to\infty} \mu\{\eta : \eta(x) = 1\} = 1.$$

Note: Blocking $\Rightarrow$ profile, but not conversely.

Results

1. Either

(i)
$$\mathcal{I}_e = \{\nu_\rho, 0 \le \rho \le 1\}$$

or

(ii)
$$\mathcal{I}_e = \{\nu_\rho, 0 \le \rho \le 1\} \cup \{\mu_n, n \in Z^1\},$$

where μ_n are profile measures and are shifts of each other. If they are blocking measures, they are the limiting distributions as $t \to \infty$ starting from the configuration

$$\cdots 0\ 0\ 0\ 1\ 1\ 1 \cdots$$

$$n$$

2. If $\sum_x xp(x) = 0$, case (i) occurs.

3. If $\sum_x xp(x) > 0$ and $p(\cdot)$ has finite range, then case (ii) occurs and μ_n is a blocking measure.

4. If $\sum_x xp(x) > 0$, $p(x) \downarrow$ and $p(-x) \downarrow$ for $x > 0$, $p(x) \ge p(-x)$ for $x > 0$, and $\sum_{x<0} x^2 p(x) < \infty$, then case (ii) occurs and μ_n is a blocking measure.

5. If $\sum_{x<0} x^2 p(x) = \infty$, there exists no stationary blocking measure.

Relevance of the second moment assumption

For a probability measure ν on $\{0,1\}^{Z^1}$, define the flux across the bond $(n, n+1)$ by

$$\phi_n(\nu) = \sum_{x \le n < y} p(x,y)\nu\{\eta(x) = 1, \eta(y) = 0\}$$
$$- \sum_{x \le n < y} p(y,x)\nu\{\eta(x) = 0, \eta(y) = 1\}.$$

If ν is stationary, $\phi_n(\nu)$ is independent of n, and if it is also profile, it is $\equiv 0$. Summing these identities leads formally to

$$\sum_{n=1}^{\infty} n^2 p(-n) = \sum_{n=1}^{\infty} nM(n)[p(n) - p(-n)],$$

umpiton where

$$M(n) = \sum_x \nu\{\eta(x) = 1, \eta(x+n) = 0\}.$$

Main open questions

1. What happens if $\sum_x xp(x) > 0$ and $\sum_{x<0} x^2 p(x) = \infty$?

2. Is there any example with a stationary profile measure that is not blocking?

One Dimension – Random Environment

L. Chayes and T. M. Liggett, *One dimensional nearest neighbor exclusion processes in inhomogeneous and random environments*, J. Stat. Phys. (2007).

Assumptions

$$p(i, i+1) = p_i \text{ and } p(i, i-1) = q_i = 1 - p_i,$$

where $p_i \in (0,1)$ are independent and identically distributed random variables.

Product Stationary Distributions

$$\nu_\alpha \in \mathcal{I} \text{ if } \alpha(i) = \frac{\pi_i}{1+\pi_i} \text{ where } \pi_i p_i = \pi_{i+1} q_{i+1}.$$

These are extremal iff $\sum_i \alpha(i)[1-\alpha(i)] = \infty$; otherwise they are mixtures of $\mu_n \in \mathcal{I}_e$.

Results

1. $\mu \in \mathcal{I}_e$ is one of the above measures iff $\phi(\mu) = 0$.

2. Let K be the (closed) support of the distribution of p_0. Then there exists $\mu \in \mathcal{I}_e$ with $\phi(\mu) \neq 0$ iff

$$K \subset \left(\frac{1}{2}, 1\right) \text{ or } K \subset \left(0, \frac{1}{2}\right).$$

Higher Dimensions: $S = Z^d,\ p(x,y) = p(y-x)$

M. Bramson and T. Liggett, *Exclusion processes in higher dimensions: Stationary measures and convergence*, Ann. Probab. **33** (2005), 2255–2313.

Some Results

1. **Product measures:** $\nu_\alpha \in \mathcal{I}$ if and only if there is a $v \in R^d$ so that

$$p(x) = e^{<x,v>} p(-x)$$

for all x such that $< x, v > \neq 0$, where

$$\pi(x) = \pi(0) e^{<x,v>} \text{ and } \alpha(x) = \frac{\pi(x)}{1+\pi(x)}.$$

An Example on Z^2

Suppose $p(\cdot)$ is nearest neighbor:

$$\begin{array}{ccc} & p_2 & \\ & \uparrow & \\ q_1 \longleftarrow & & \longrightarrow p_1 \\ & \downarrow & \\ & q_2 & \end{array}$$

The stationary product measures with density 1/2 at (0,0) are exactly the following four:

$$\pi(x_1,x_2)\equiv 1, \quad \pi(x_1,x_2)=\left(\frac{p_1}{q_1}\right)^{x_1},$$

$$\pi(x_1,x_2)=\left(\frac{p_2}{q_2}\right)^{x_2}, \quad \pi(x_1,x_2)=\left(\frac{p_1}{q_1}\right)^{x_1}\left(\frac{p_2}{q_2}\right)^{x_2}.$$

The first one is homogeneous, the last one is reversible (i.e., satisfies detailed balance), and the other two are neither!

Terminology: The last three measures above are v-*profile* with (if $p_1 > q_1, p_2 > q_2$, for example) respectively

$$v=(1,0), \quad v=(0,1), \quad \text{and } v=\left(\log\frac{p_1}{q_1},\log\frac{p_2}{q_2}\right).$$

2. Suppose $v \in Z^d\backslash\{0\}$. If μ is a v-profile stationary distribution, then

$$\left\langle \sum_x xp(x), v\right\rangle > 0.$$

3. If μ_1 and μ_2 are extremal stationary v-profile measures, then either $\mu_1 \le \mu_2$ or $\mu_2 \le \mu_1$.

4. Let $\pi_c(x) = ce^{<x,v>}$ and suppose that

$$p(x) = e^{<x,v>} p(-x)$$

for all x such that $< x, v > \neq 0$. Then the set of extremal stationary v-profile measures is exactly $\{\nu_{\alpha_c}, c > 0\}$.

Note: This is a one *continuous* parameter family!

5. Suppose

(a) $p(x) \geq p(-x)$ whenever $< x, v >> 0$,

and

(b) $\sum_{x:<x,v><0} < x, v >^2 p(x) < \infty$.

Then extremal stationary v-profile $\Rightarrow$ v-blocking.

Some Open Problems

Assume throughout that $d > 1$ and $p(\cdot)$ is finite range.

1. Are all extremal inhomogeneous stationary distributions v-profile for some $v \in R^d$ satisfying $< \sum_x xp(x), v >> 0$?

2. For each $v \in R^d$ satisfying $< \sum_x xp(x), v >> 0$, is there a continuous one parameter family of extremal v-profile stationary distributions?

3. If $\sum_x xp(x) = 0$, are all stationary distributions exchangeable?

Main Embarassments

We can't even answer the following questions:

4. Is there any example of a $p(\cdot)$ for which there is some extremal, stationary distribution that is not a product measure?

5. Is the set of $v \in R^d$ for which there exists a v-profile stationary distribution convex?

Especially in the Example on Z^2

$$\begin{array}{ccc} & p_2 & \\ & \uparrow & \\ q_1 \longleftarrow & & \longrightarrow p_1 \\ & \downarrow & \\ & q_2 & \end{array}$$

We have v-profile stationary measures for only the three values of v for which they are product measures!