Random Averaging

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Talk, papers available from: http://cnls.lanl.gov/~ebn

Plan

I. Averaging
II. Restricted averaging
III. Diffusive averaging
IV. Orientational averaging

Themes

1. Scaling and multiscaling
2. Cascades
3. Pattern formation and bifurcations
4. Phase transitions and synchronization

1. Averaging

The basic averaging process

- N identical particles (grains, billiard balls)
- Each particle carries a number (velocity) v_i
- Particles interact in pairs (collision)
- Both particles acquire the average (inelastic)

$$(v_1, v_2) \to \left(\frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2} \right)$$

Melzak 76

Conservation laws & dissipation

- Total number of particles is conserved
- Total momentum is conserved

$$\sum_{i=1}^{N} v_i = \text{constant}$$

- Energy is dissipated in each encounter $E_i = \frac{1}{2} v_i^2$

$$\Delta E = \frac{1}{4}(v_1 - v_2)^2$$

We expect the velocities to shrink

Some details

- Dynamic treatment

  Each particle collides once per unit time

- Random interactions

  The two colliding particles are chosen randomly

- Infinite particle limit is implicitly assumed

$$N \to \infty$$

- Process is galilean invariant $x \to x + x_0$

  Set average velocity to zero $\langle x \rangle = 0$

The temperature

- Definition

$$T = \langle v^2 \rangle$$

- Time evolution = exponential decay

$$\frac{dT}{dt} = -\lambda T$$

$$T = T_0 e^{-\lambda t}$$
$$\lambda = \frac{1}{2}$$

- All energy is eventually dissipated
- Trivial steady-state

$$P(v) \to \delta(v)$$

The moments

- Kinetic theory

$$\frac{\partial P(v,t)}{\partial t} = \iint dv_1 dv_2 P(v_1,t) P(v_2,t) \left[\delta\left(v - \frac{v_1+v_2}{2}\right) - \delta(v - v_1)\right]$$

- Moments of the distribution

$$M_n = \int dv\, v^n P(v,t)$$

$$\begin{aligned} M_0 &= 1 \\ M_{2n+1} &= 0 \end{aligned}$$

- Closed nonlinear recursion equations

$$\frac{dM_n}{dt} + \lambda_n M_n = 2^{-n} \sum_{m=2}^{n-2} \binom{n}{m} M_m M_{n-m}$$

- Asymptotic decay

$$\lambda_n < \lambda_m + \lambda_{n-m}$$

$$M_n \sim e^{-\lambda_n t} \qquad \text{with} \qquad \lambda_n = 1 - 2^{-(n-1)}$$

Multiscaling

- Nonlinear spectrum of decay constants

$$\lambda_n = 1 - 2^{-(n-1)}$$

- Spectrum is concave, saturates

$$\lambda_n < \lambda_m + \lambda_{n-m}$$

- Each moment has a distinct behavior

$$\frac{M_n}{M_m M_{n-m}} \to \infty \qquad \text{as} \qquad t \to \infty$$

Multiscaling Asymptotic Behavior

The Fourier transform

- The Fourier transform $F(k) = \int dv\, e^{ikv} P(v,t)$
- Obeys closed, nonlinear, nonlocal equation

$$\frac{\partial F(k)}{\partial t} + F(k) = F^2(k/2)$$

- Scaling behavior, scale set by second moment

$$F(k,t) \to f\left(ke^{-\lambda t}\right) \qquad \lambda = \frac{\lambda_2}{2} = \frac{1}{4}$$

- Nonlinear differential equation

$$-\lambda\, z\, f'(z) + f(z) = f^2(z/2) \qquad \begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \end{aligned}$$

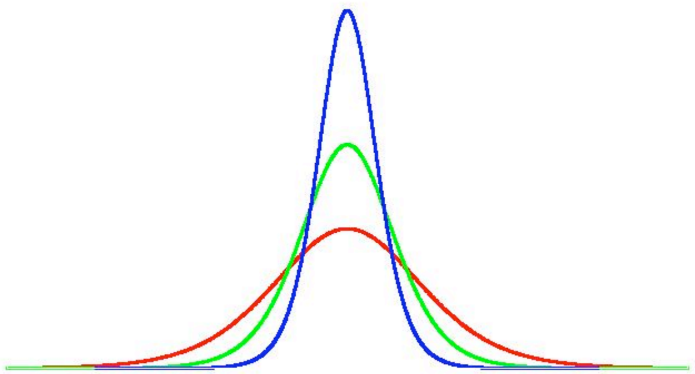
- Solution

$$f(z) = (1 + |z|)e^{-|z|}$$

The velocity distribution

- Self-similar form

$$P(v,t) \to e^{\lambda t} p\left(v e^{\lambda t}\right)$$



- Obtained by inverse Fourier transform

$$p(w) = \frac{2}{\pi} \frac{1}{\left(1+w^2\right)^2}$$

- Power-law tail

$$p(w) \sim w^{-4}$$

1. Temperature is the characteristic velocity scale
2. Multiscaling is consequence of diverging moments of the power-law similarity function

Stationary Solutions

- Stationary solutions do exist!

$$F(k) = F^2(k/2)$$

- Family of exponential solutions

$$F(k) = \exp(-kv_0)$$

- Lorentz/Cauchy distribution

$$P(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2}$$

How is a stationary solution consistent with energy dissipation?

Extreme Statistics

- Large velocities, cascade process

$$v \to \left(\frac{v}{2}, \frac{v}{2}\right)$$

$$(v_1, v_2) \to \left(\frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2}\right)$$

- Linear evolution equation

$$\frac{\partial P(v)}{\partial t} = 4P\left(\frac{v}{2}\right) - P(v)$$

- Steady-state: power-law distribution

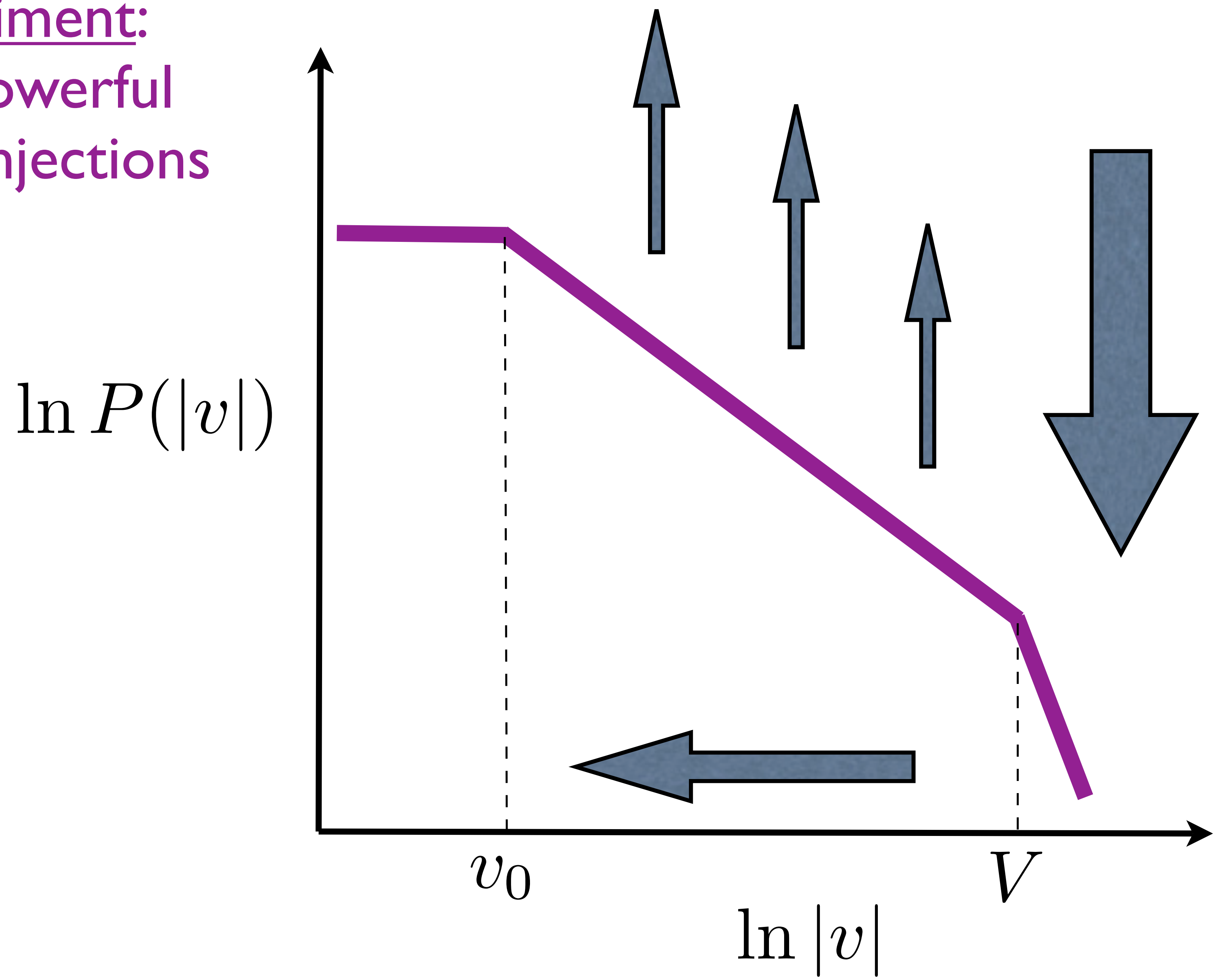
$$P(v) \sim v^{-2}$$

$$4P\left(\frac{v}{2}\right) = P(v)$$

- Divergent energy, divergent dissipation rate

Injection, Cascade, Dissipation

<u>Experiment:</u>
rare, powerful
energy injections



<u>Lottery MC:</u>
award one particle
all dissipated energy

Injection selects the typical scale!

I. Conclusions

- Moments exhibit multiscaling
- Distribution function is self-similar
- Power-law tail
- Stationary solution with infinite energy exists
- Driven steady-state
- Energy cascade

11. Restricted Averaging

The compromise process

- Opinion measured by a continuum variable

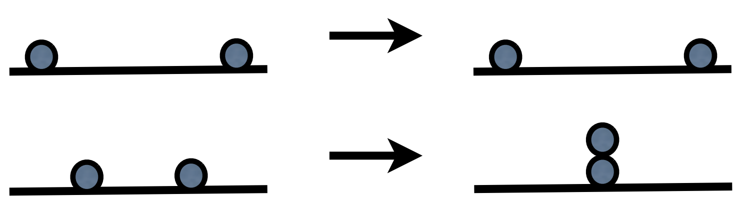
$$-\Delta < x < \Delta$$

- Compromise: reached by pairwise interactions

$$(x_1, x_2) \rightarrow \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2} \right)$$

- Conviction: restricted interaction range

$$|x_1 - x_2| < 1$$



- Minimal, one parameter model
- Mimics competition between compromise and conviction

Weisbuch 2001

Problem set-up

- Given uniform initial (un-normalized) distribution

$$P_0(x) = \begin{cases} 1 & |x| < \Delta \\ 0 & |x| > \Delta \end{cases}$$

- Find final distribution

$$P_\infty(x) = ?$$

- Multitude of final steady-states

$$P_0(x) = \sum_{i=1}^{N} m_i \, \delta(x - x_i) \qquad |x_i - x_j| > 1$$

- Dynamics selects one (deterministically)

Multiple localized clusters

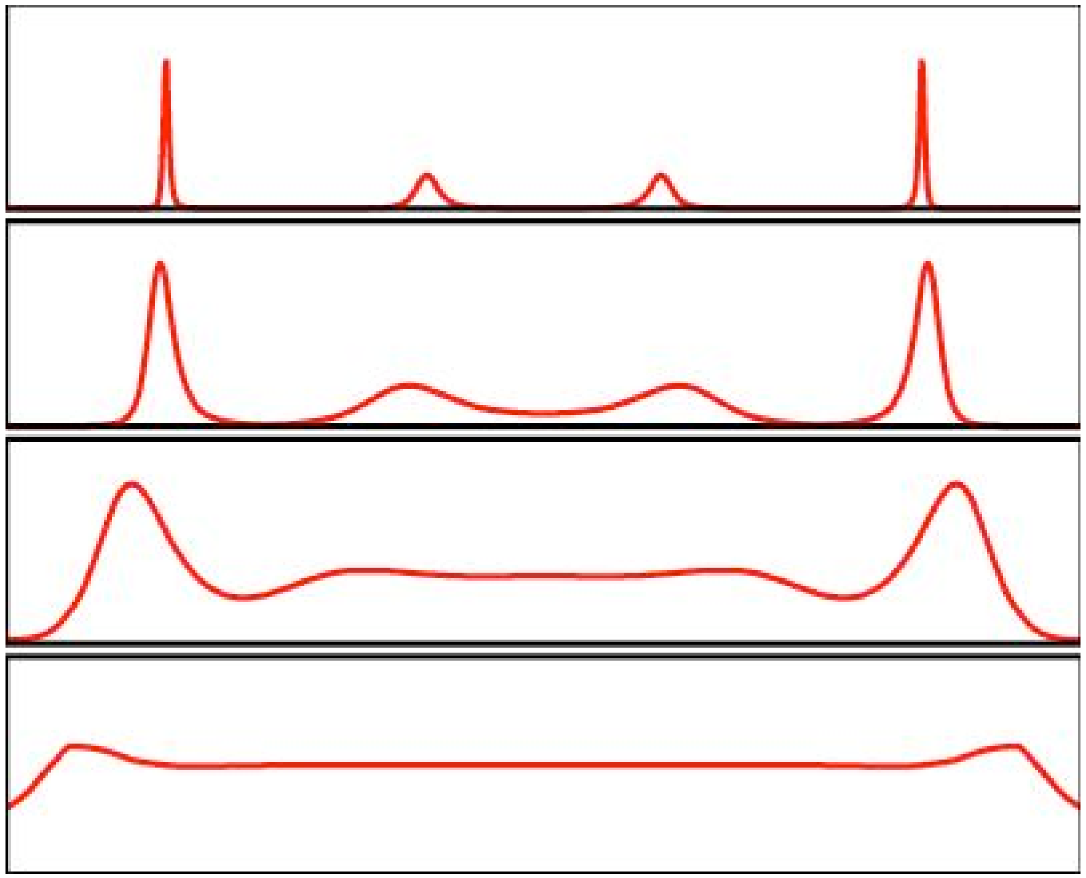
Numerical methods, kinetic theory

- Same master equation, restricted integration

$$\frac{\partial P(x,t)}{\partial t} = \iint\limits_{|x_1-x_2|<1} dx_1 dx_2 P(x_1,t)P(x_2,t)\left[\delta\left(x - \frac{x_1+x_2}{2}\right) - \delta(x-x_1)\right]$$

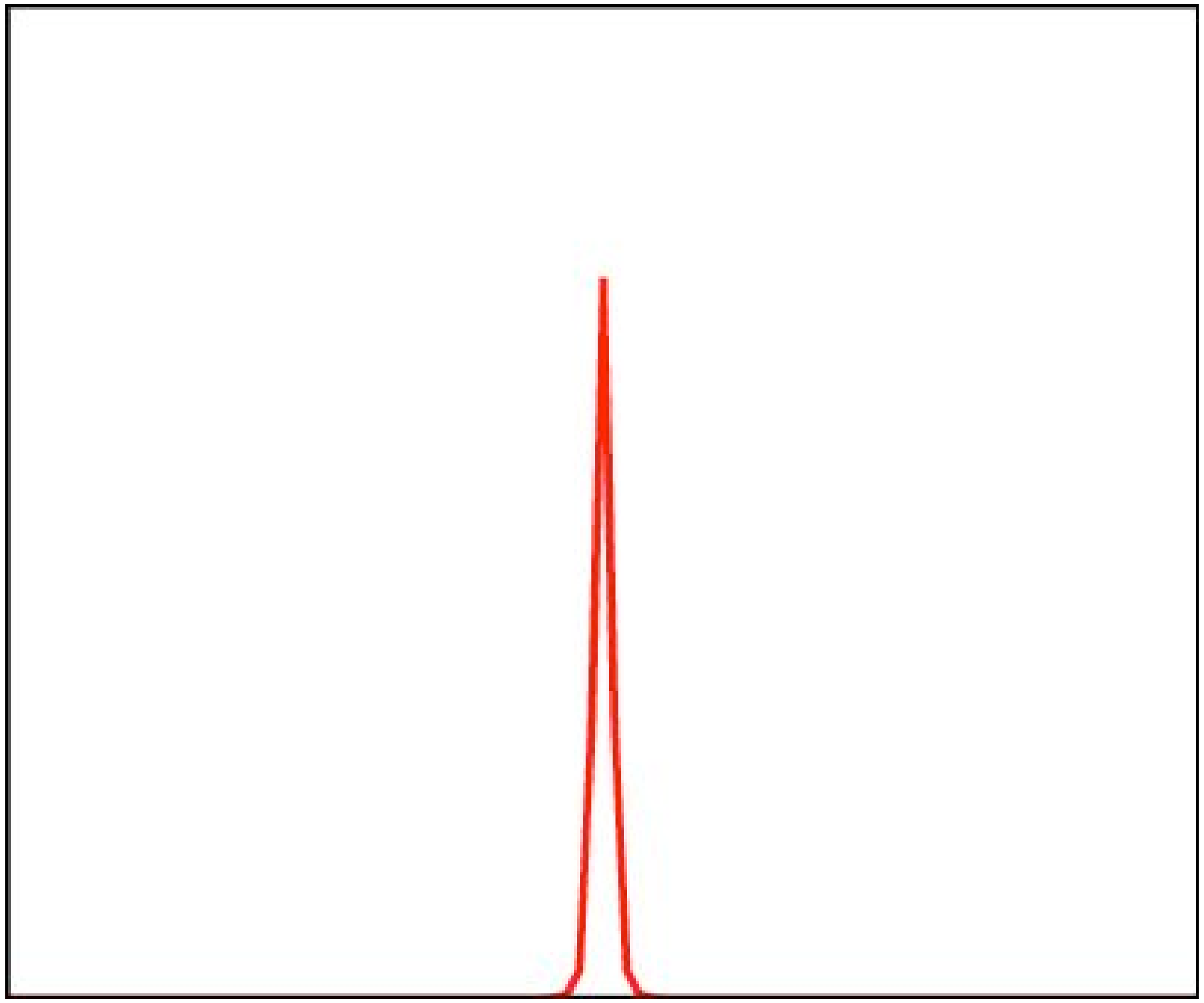
Direct Monte Carlo simulation of stochastic process

- Numerical integration of rate equations

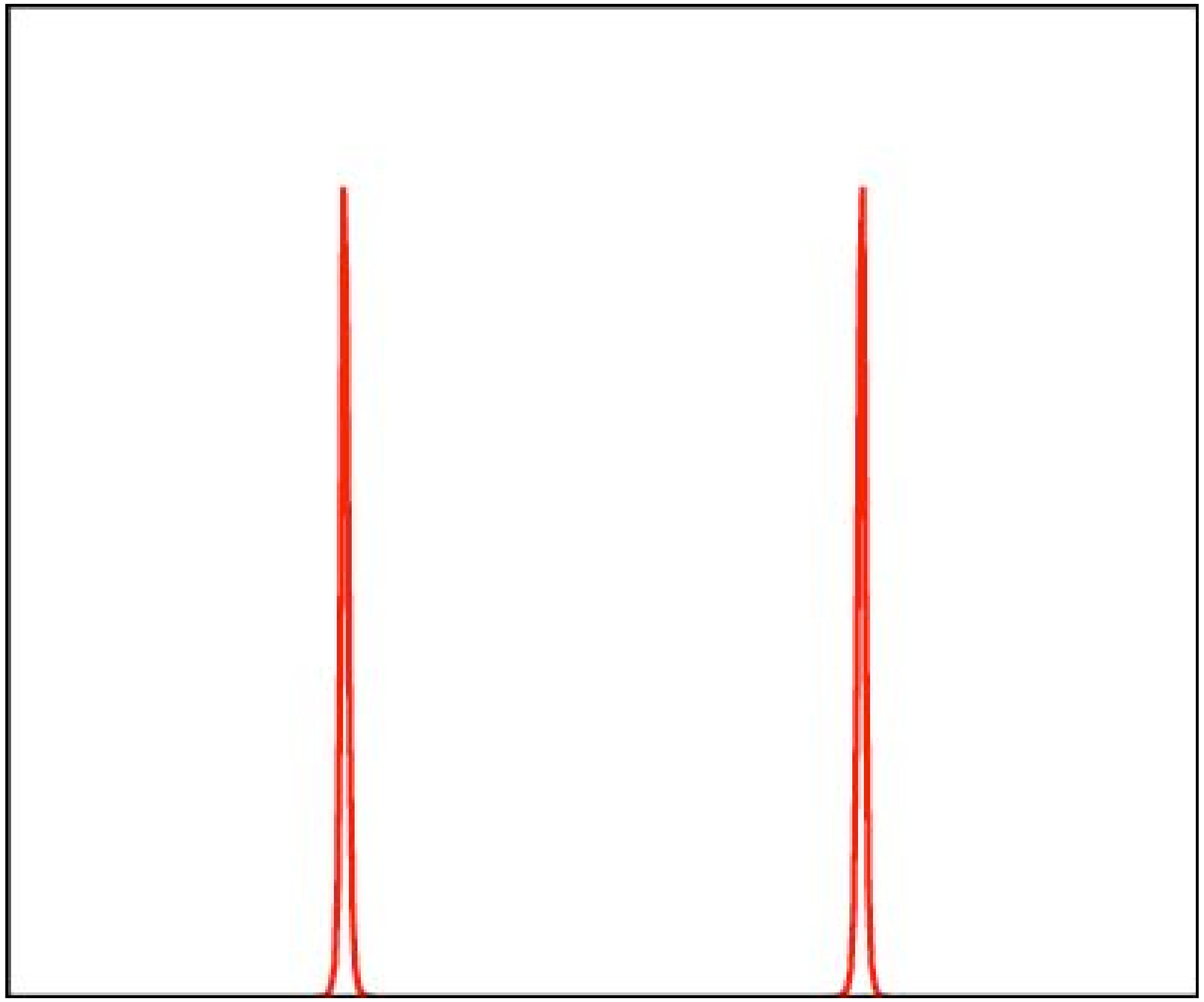


Rise and fall of central party

$0 < \Delta < 1.871$



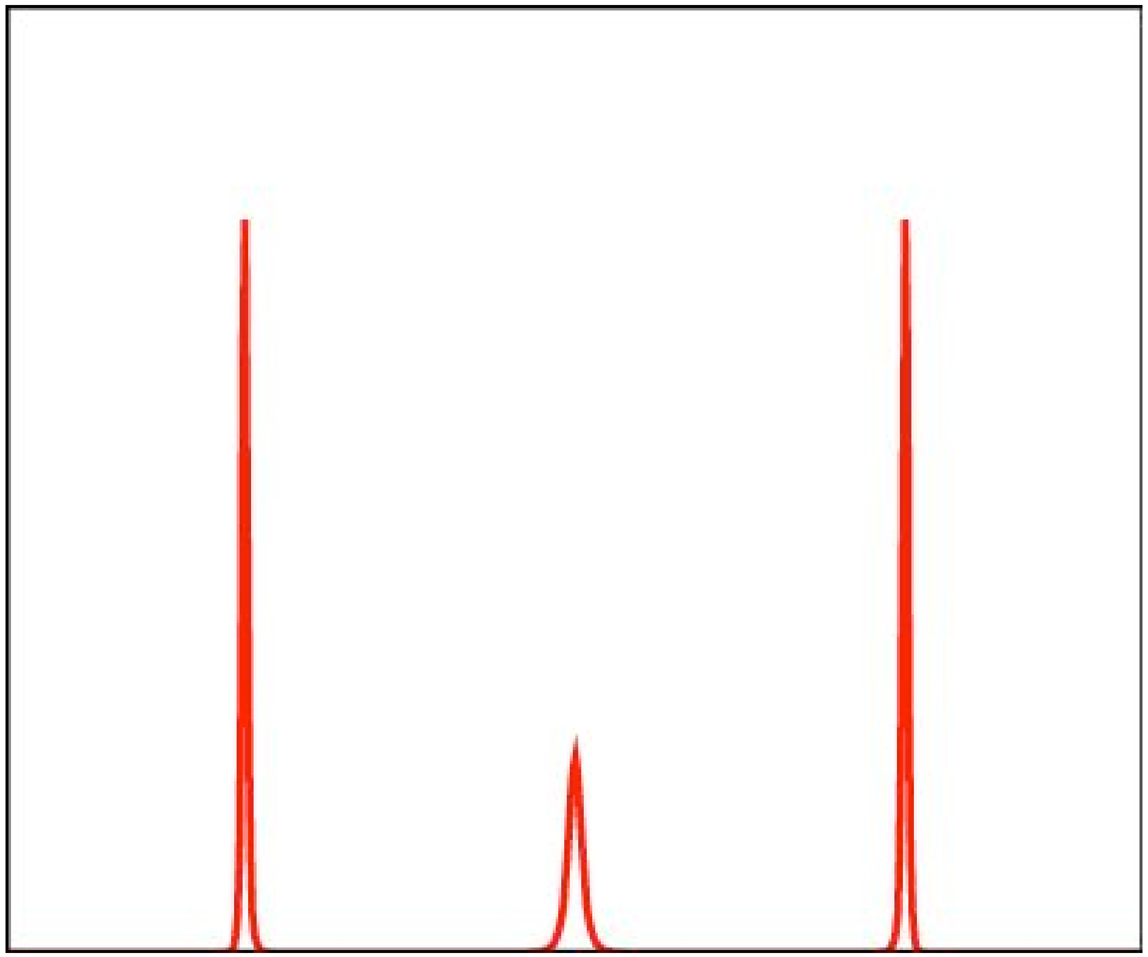
$1.871 < \Delta < 2.724$



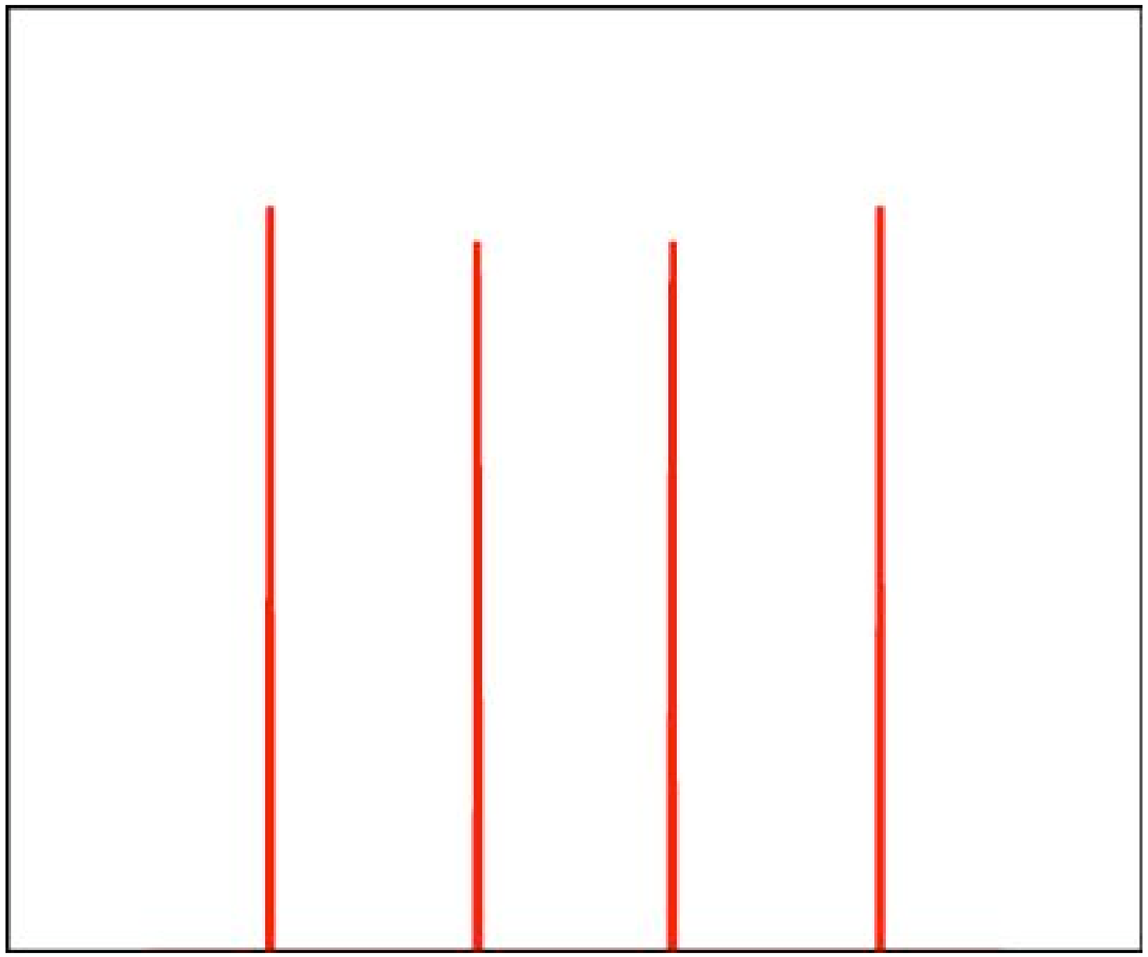
Central party may or may not exist!

Resurrection of central party

$2.724 < \Delta < 4.079$

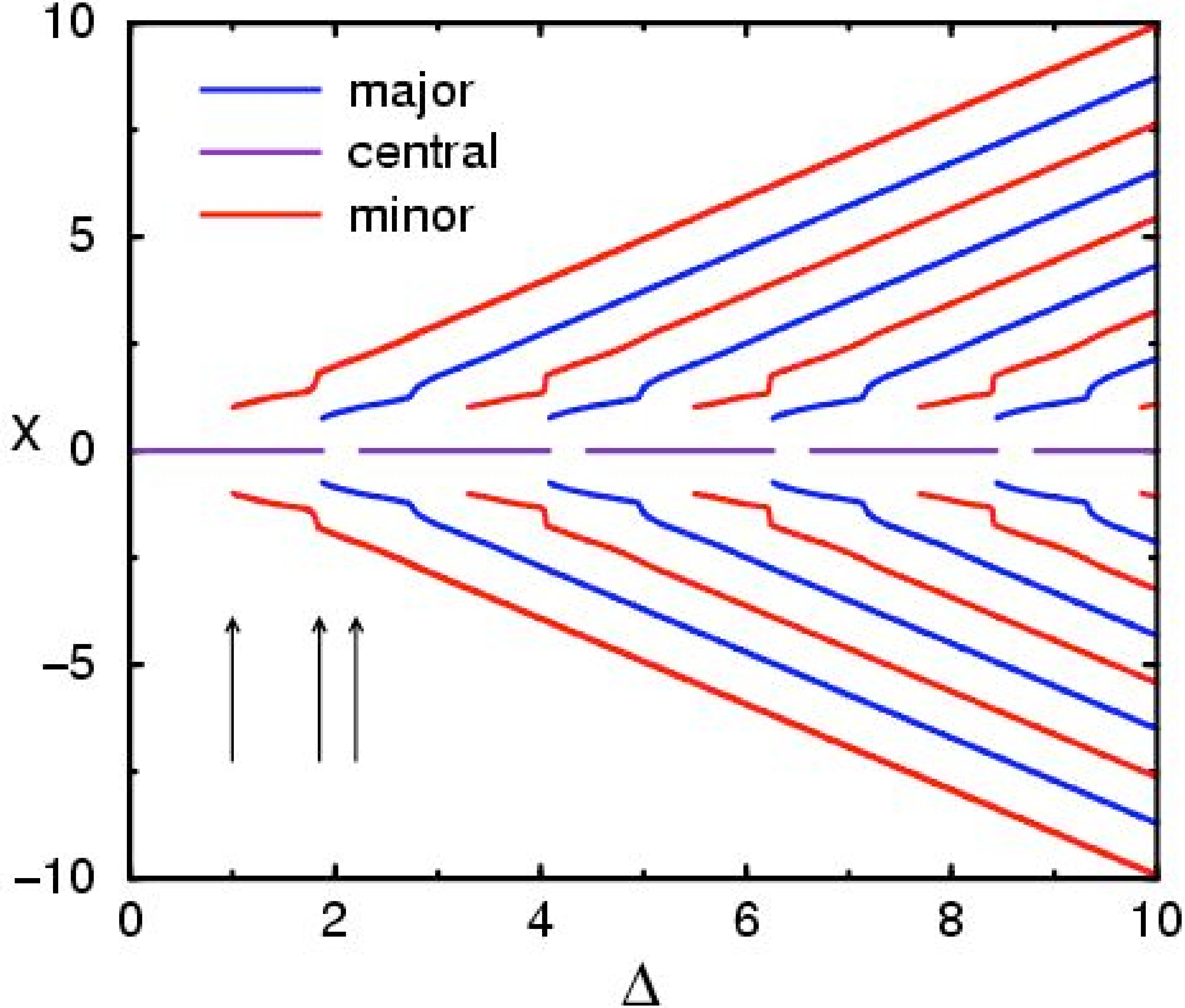


$4.079 < \Delta < 4.956$



Parties may or may not be equal in size

Bifurcations and Patterns



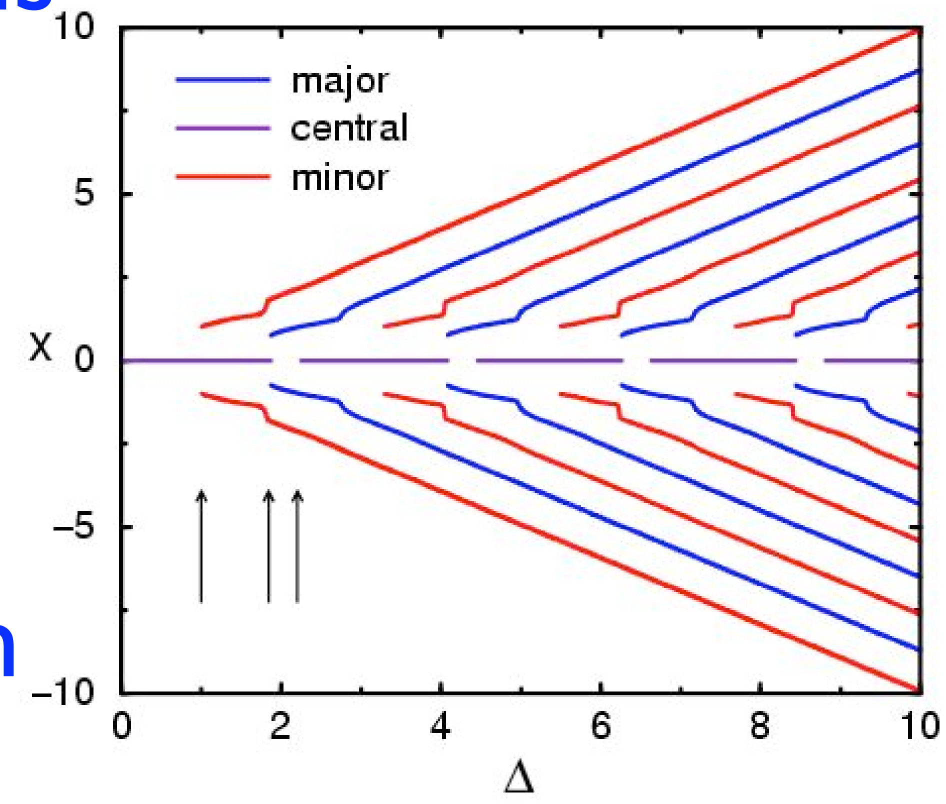
Self-similar structure, universality

- Periodic sequence of bifurcations

  1. Nucleation of minor cluster branch

  2. Nucleation of major cluster brunch

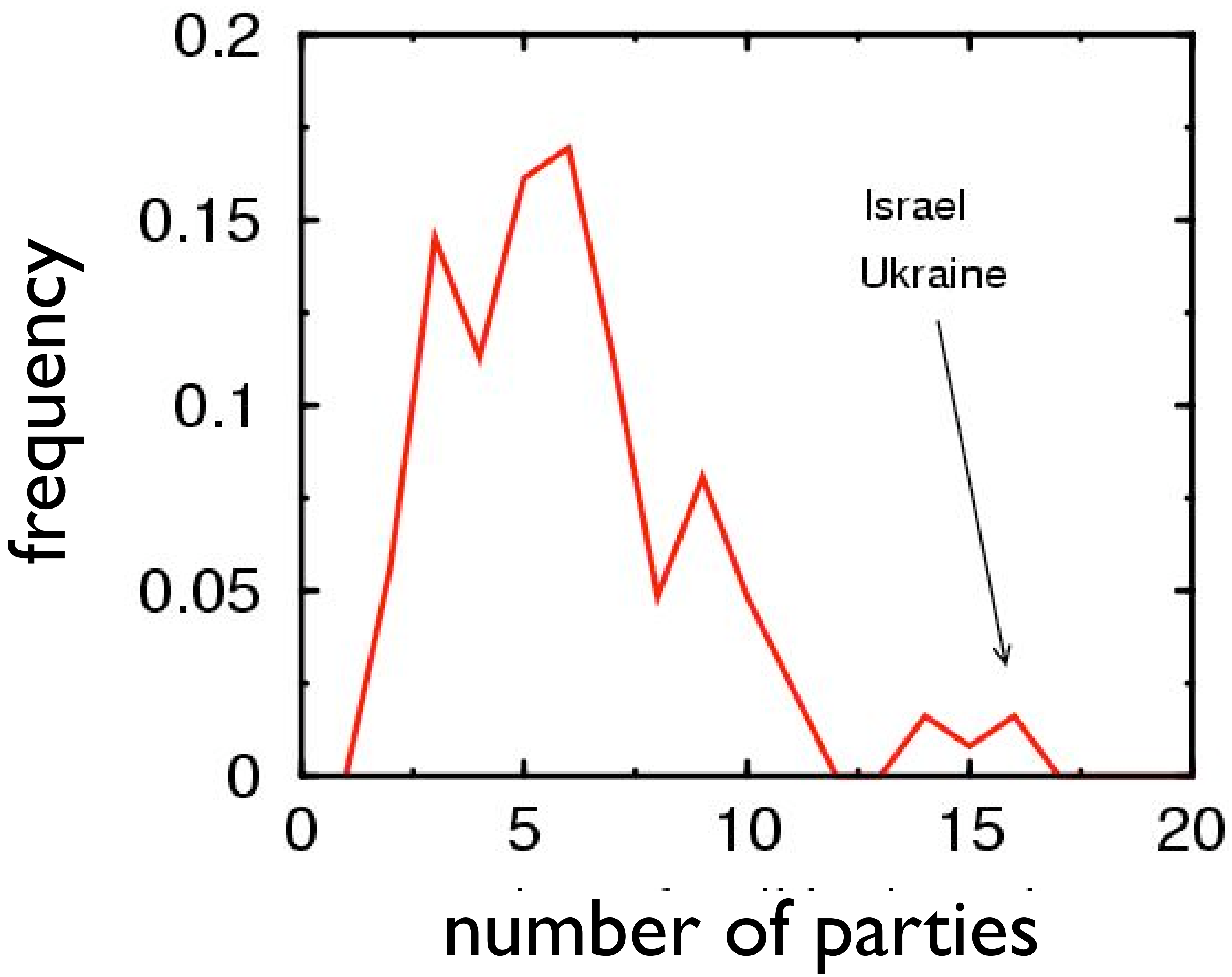
  3. Nucleation of central cluster



- Alternating major-minor pattern
- Clusters are equally spaced
- Period L gives major cluster mass, separation

$$x(\Delta) = x(\Delta) + L \qquad L = 2.155$$

How many political parties?



- Data: CIA world factbook 2002
- 120 countries with multi-party parliaments
- Average=5.8; Standard deviation=2.9

Cluster mass

- Masses are periodic

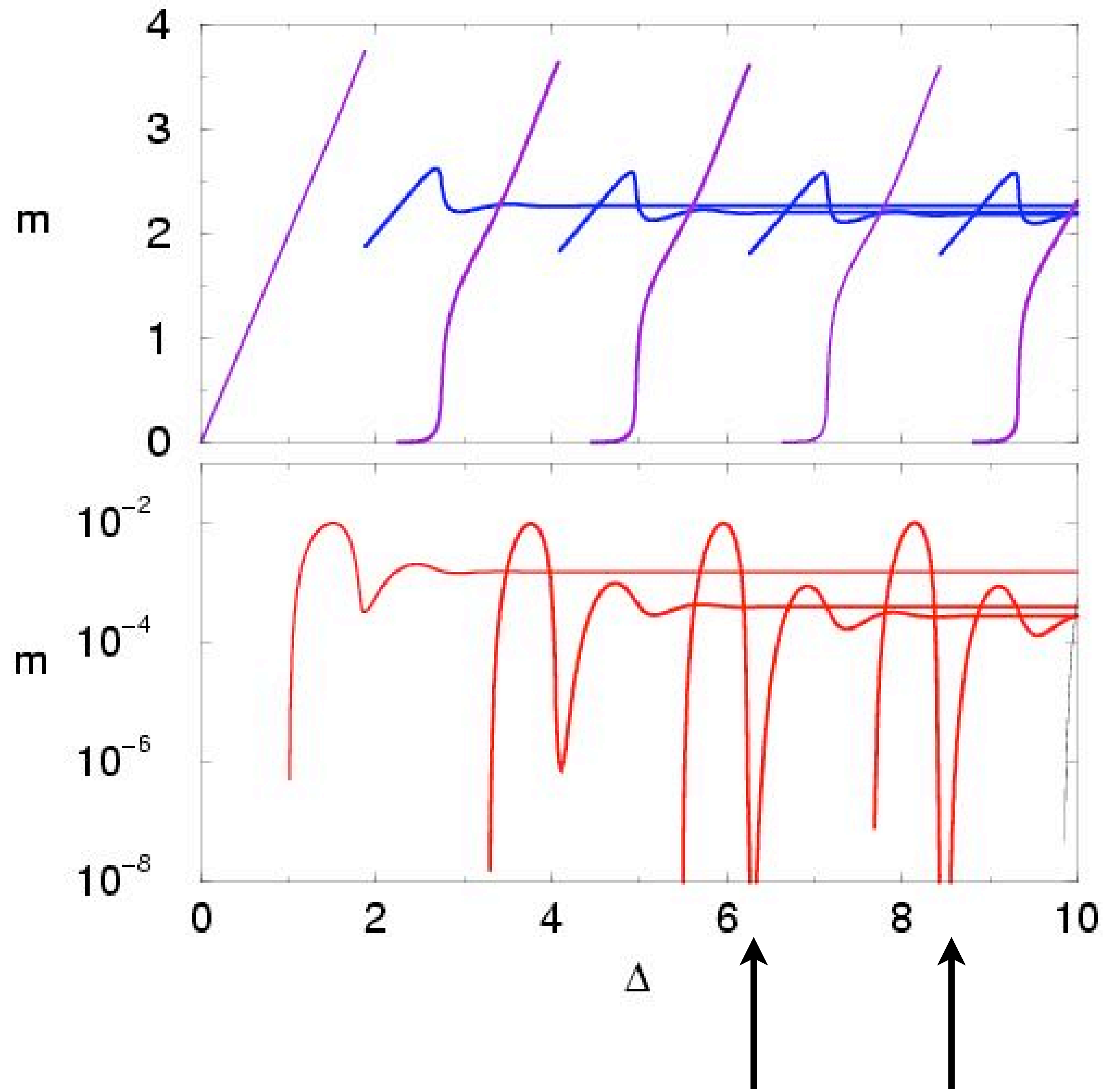
$$m(\Delta) = m(\Delta + L)$$

- Major mass

$$M \to L = 2.155$$

- Minor mass

$$m \to 3 \times 10^{-4}$$



Why are the minor clusters so small?

gaps?

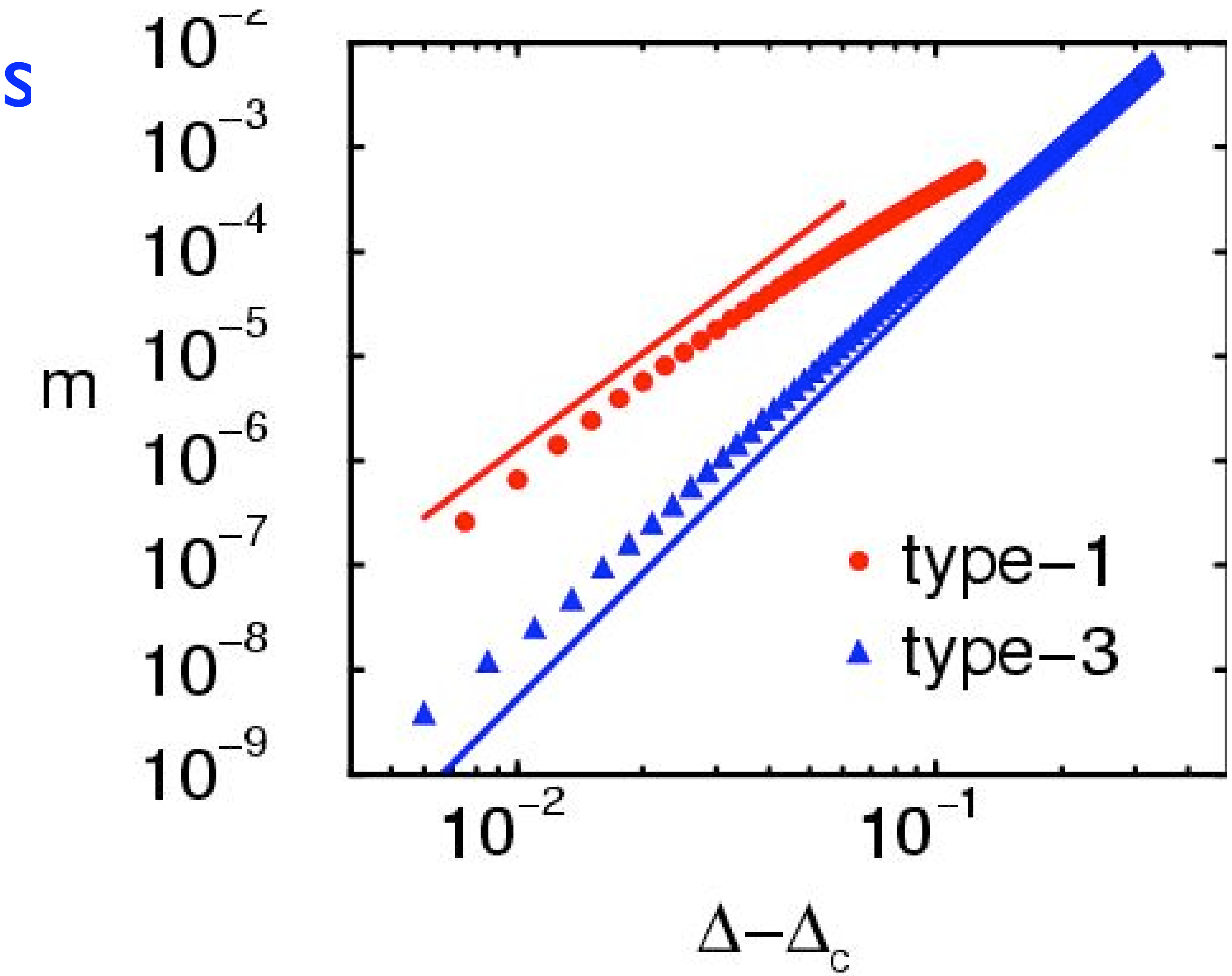
Scaling near bifurcation points

- Minor mass vanishes

$$m \sim (\Delta - \Delta_c)^{\alpha}$$

- Universal exponent

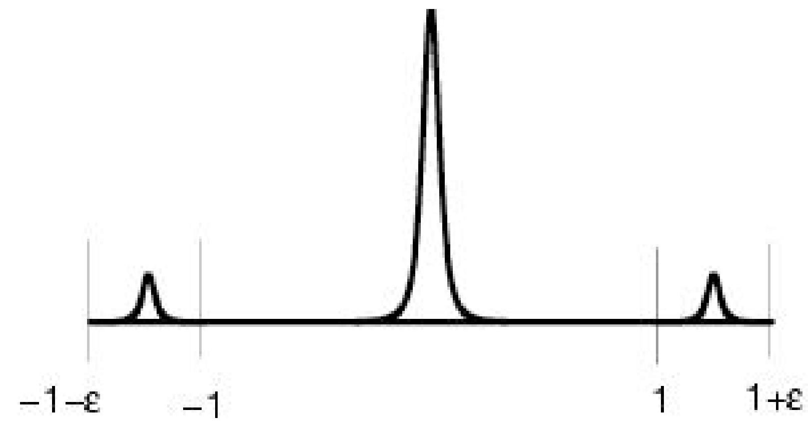
$$\alpha = \begin{cases} 3 & \text{type1} \\ 4 & \text{type3} \end{cases}$$



L-2 is the small parameter
explains small saturation mass

Heuristic derivation of exponent

- Perturbation theory $\Delta = 1 + \epsilon$
- Major cluster $x(\infty) = 0$
- Minor cluster $x(\infty) = \pm(1 + \epsilon/2)$



- Rate of transfer from minor cluster to major cluster

$$\frac{dm}{dt} = -m\,M \longrightarrow m \sim \epsilon\, e^{-t}$$

- Process stops when

$$x \sim e^{-t_f/2} \sim \epsilon \qquad \langle x^2 \rangle \sim e^{-t}$$

- Final mass of minor cluster

$$m(\infty) \sim m(t_f) \sim \epsilon^3 \qquad \alpha = 3$$

Pattern selection

- Linear stability analysis

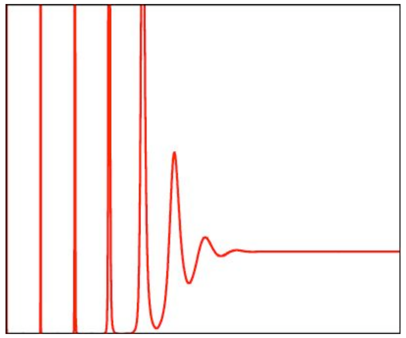
$$P - 1 \propto e^{i(kx+wt)} \quad \Longrightarrow \quad w(k) = \frac{8}{k}\sin\frac{k}{2} - \frac{2}{k}\sin k - 2$$

- Fastest growing mode

$$\frac{dw}{dk} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 2.2515$$

- Traveling wave (FKPP saddle point analysis)

$$\frac{dw}{dk} = \frac{\mathrm{Im}(w)}{\mathrm{Im}(k)} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 2.0375$$



Patterns induced by wave propagation from boundary
However, emerging period is different

$$2.0375 < L < 2.2515$$

Pattern selection is intrinsically nonlinear

II. Conclusions

- Clusters form via bifurcations
- Periodic structure
- Alternating major-minor pattern
- Central party does not always exist
- Power-law behavior near transitions
- Nonlinear pattern selection

III. Diffusive Averaging

Diffusive Forcing

Two independent competing processes

1. Averaging (nonlinear)

$$(v_1, v_2) \to \left(\frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2}\right)$$

2. Random uncorrelated white noise (linear)

$$\frac{dv_j}{dt} = \eta_j(t) \qquad \langle \eta_j(t)\eta_j(t')\rangle = 2D\delta(t - t')$$

- Add diffusion term to equation (Fourier space)

$$(1 + Dk^2)F(k) = F^2(k/2)$$

System reaches a nontrivial steady-state
Energy injection balances dissipation

Infinite product solution

- Solution by iteration

$$F(k) = \frac{1}{1+Dk^2}F^2(k/2) = \frac{1}{1+Dk^2}\frac{1}{(1+D(k/2)^2)^2}F^4(k/4) = \cdots$$

- Infinite product solution

$$F(k) = \prod_{i=0}^{\infty}\left[1 + D(k/2^i)^2\right]^{-2^i}$$

- Exponential tail $v \to \infty$

$$P(v) \propto \exp\left(-|v|/\sqrt{D}\right)$$

$$P(k) \propto \frac{1}{1+Dk^2} \propto \frac{1}{k - i/\sqrt{D}}$$

- Also follows from

$$D\frac{\partial^2 P(v)}{\partial v^2} = -P(v)$$

Non-Maxwellian distribution/Overpopulated tails

Cumulant solution

- Steady-state equation

$$F(k)(1 + Dk^2) = F^2(k/2)$$

- Take the logarithm $\psi(k) = \ln F(k)$

$$\psi(k) + \ln(1 + Dk^2) = 2\psi(k/2)$$

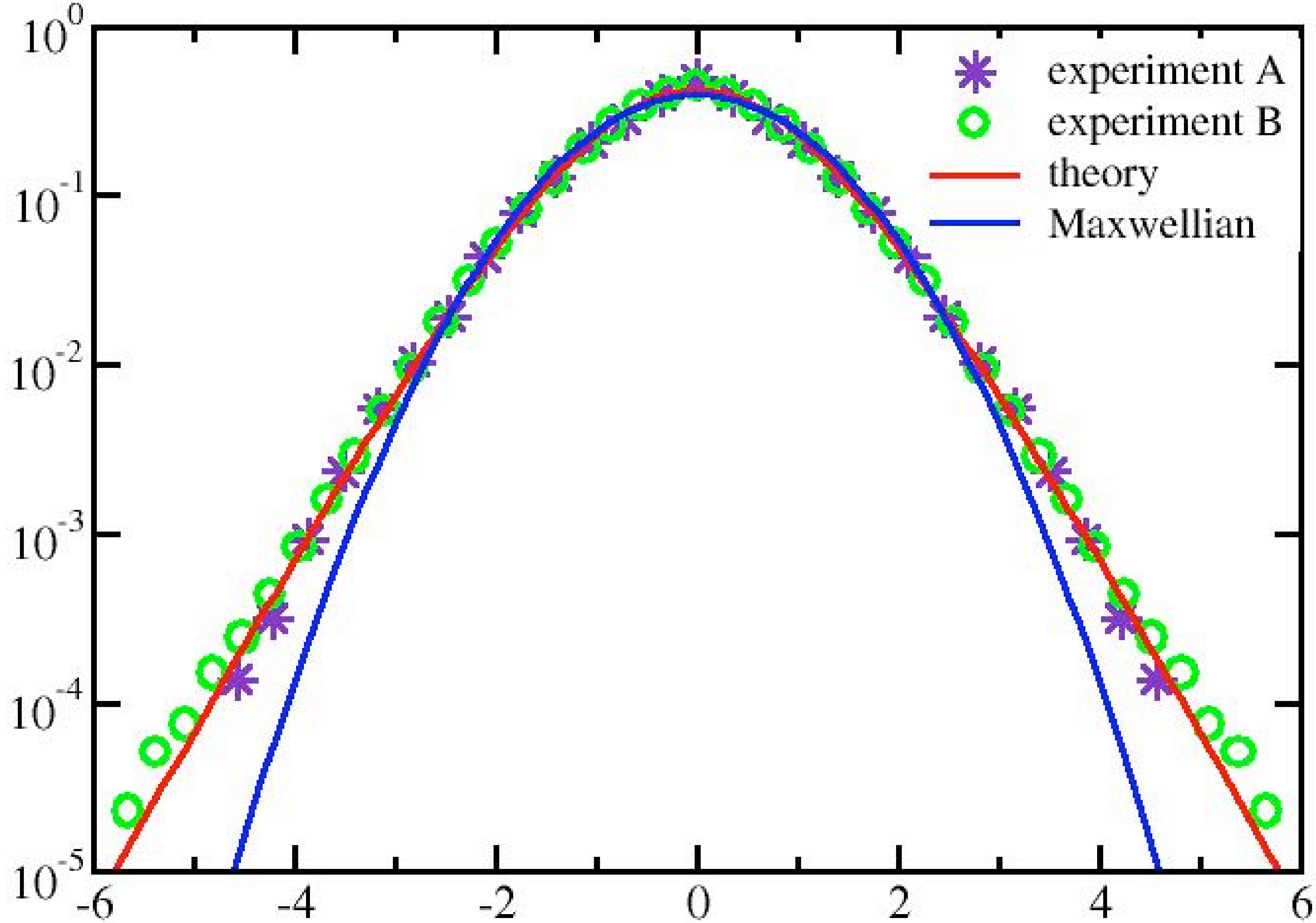
- Cumulant solution

$$F(k) = \exp\left[\sum_{n=1}^{\infty} \psi_n (-Dk^2)^n / n\right]$$

- Generalized fluctuation-dissipation relations

$$\psi_n = \lambda_n^{-1} = \left[1 - 2^{1-n}\right]^{-1}$$

Experiment



"A shaken box of marbles"

Menon 01
Aronson 05

III. Conclusions

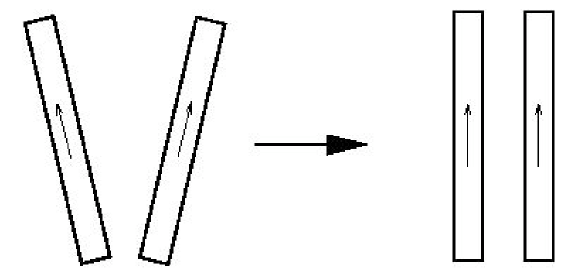
- Nonequilibrium steady-states
- Energy pumped and dissipated by different mechanisms
- Overpopulation of high-energy tail with respect to equilibrium distribution

IV. Orientational Averaging

Orientational Averaging

- Each rod has an orientation

$$0 \le \theta \le \pi$$



- Alignment by pairwise interactions

$$(\theta_1, \theta_2) \to \begin{cases} \left(\frac{\theta_1+\theta_2}{2}, \frac{\theta_1+\theta_2}{2}\right) & |\theta_1 - \theta_2| < \pi \\ \left(\frac{\theta_1+\theta_2+2\pi}{2}, \frac{\theta_1+\theta_2+2\pi}{2}\right) & |\theta_1 - \theta_2| > \pi \end{cases}$$

- Diffusive wiggling

$$\frac{d\theta_j}{dt} = \eta_j(t) \qquad \langle \eta_j(t)\eta_j(t')\rangle = 2D\delta(t-t')$$

- Kinetic theory

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial \theta^2} + \int_{-\pi}^{\pi} d\phi\, P\left(\theta - \frac{\phi}{2}\right) P\left(\theta + \frac{\phi}{2}\right) - P.$$

Fourier analysis

- Fourier transform

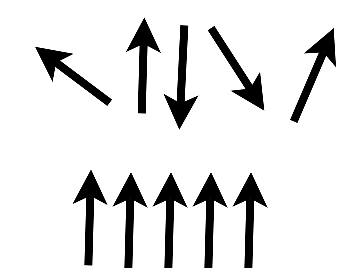
$$P_k = \langle e^{-ik\theta} \rangle = \int_{-\pi}^{\pi} d\theta e^{-ik\theta} P(\theta) \qquad P(\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} P_k e^{ik\theta}$$

- Order parameter

$$R = |\langle e^{i\theta} \rangle| = |P_{-1}|$$

- Probes state of system

$$R = \begin{cases} 0 & \text{disordered state} \\ 1 & \text{perfectly ordered state} \end{cases}$$

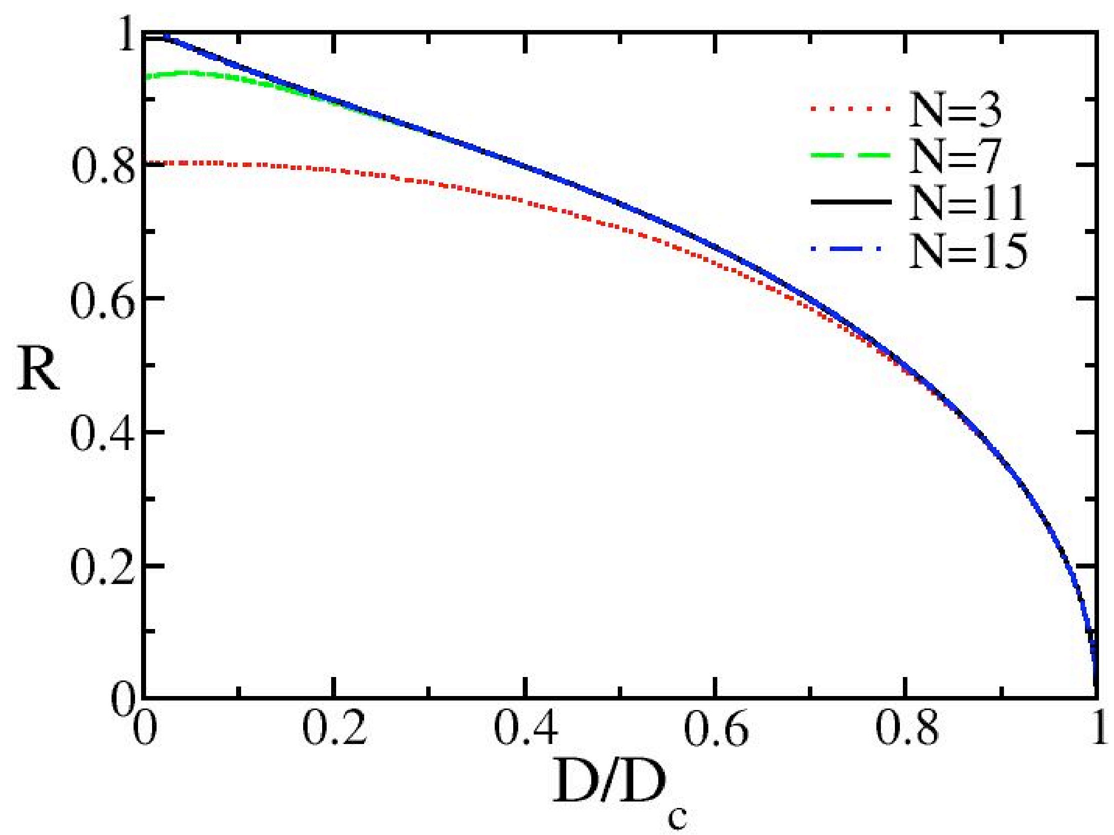


- Closed equation for Fourier modes

$$P_k = \sum_{i+j=k} G_{i,j} P_i P_j \qquad G_{i,j} = 0 \quad \text{when} \quad |i-j| = 2n$$

Nonequilibrium phase transition

- Critical diffusion constant $D_c = \dfrac{4}{\pi} - 1$
- Subcritical: ordered phase $R > 0$
- Supercritical: disordered phase $R = 0$
- Critical behavior $R \sim (D_c - D)^{1/2}$

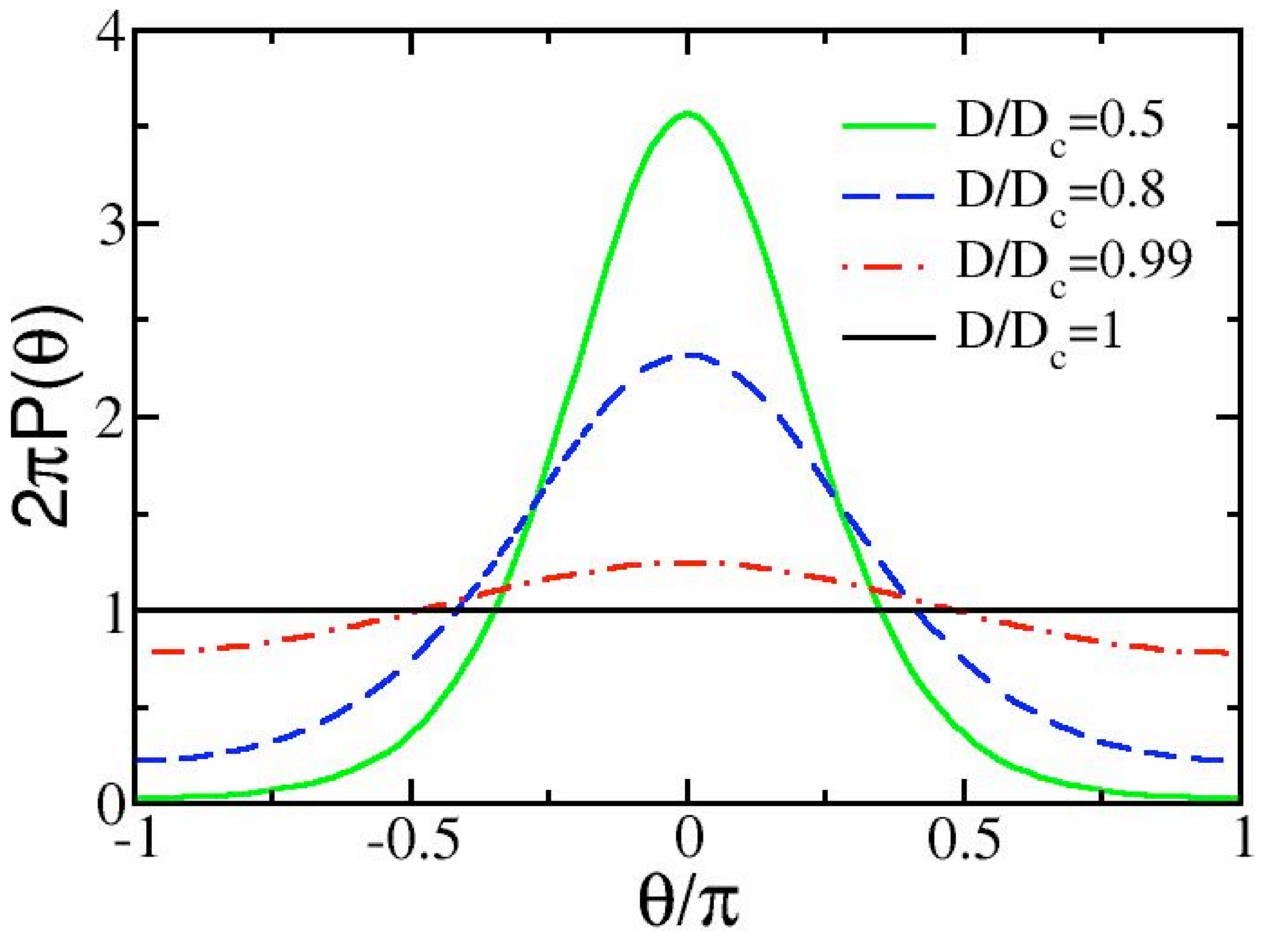


Distribution of orientation

- Fourier modes decay exponentially with R

$$P_k \sim R^k$$

- Small number of modes sufficient



Partition of Integers

- Iterate the Fourier equation

$$P_k = \sum_{i+j=k} G_{i,j} P_i P_j = \sum_{i+j=k} \sum_{l+m=j} G_{i,j} G_{l,m} P_i P_l P_m = \cdots$$

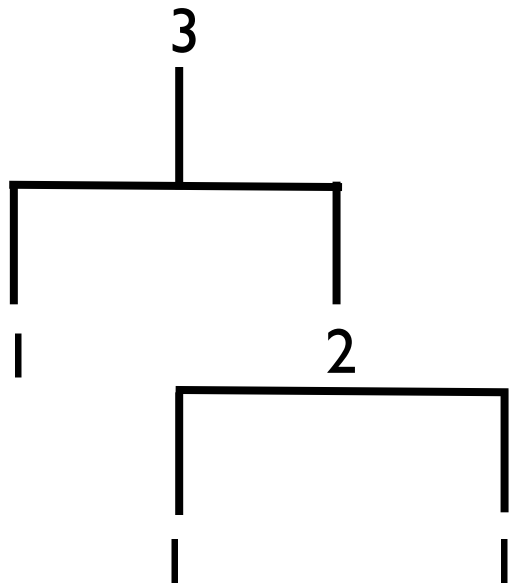
- Series solution

$$R = r_3 R^3 + r_5 R^5 + \cdots$$

$$r_3 = G_{1,2} G_{1,1}$$

Partition rules

$$\begin{aligned} k &= i + j \\ i &\neq 0 \\ j &\neq 0 \\ G_{i,j} &\neq 0 \end{aligned}$$



Experiments

“A shaken dish of toothpicks”

IV. Conclusions

- Nonequilibrium phase transition
- Weak noise: ordered phase (nematic)
- Strong noise: disordered phase
- Solution relates to iterated partition of integers
- Only when Fourier spectrum is discrete: exact solution possible for arbitrary averaging rates