A Mathematical Theory of Ramified Transport

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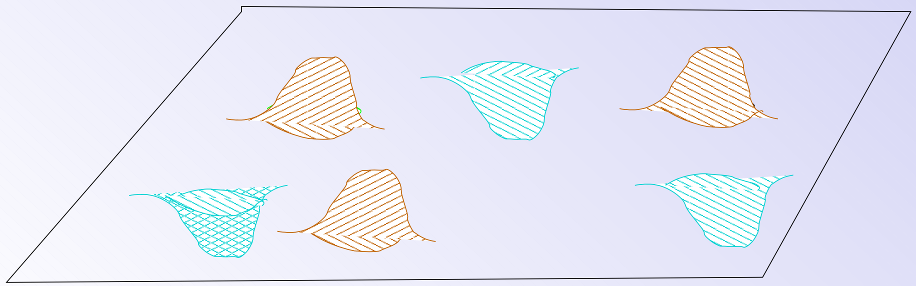
April 18, 2007

Monge's Transport Problem

How do you best move a given pile of sand to fill a given hole of the same volume?



Pile of Sand: a positive Radon measure μ^+ on a compact convex subset $X \subset \mathbb{R}^m$.

Hole: another positive Radon measure μ^- on X.

Same Volume: $0 < \mu^+(X) = \mu^-(X) < +\infty$

move: a Borel, one-to-one map $\psi : X \to X$

fill: $\psi_{\#}\mu^+ = \mu^-$ (i.e. $\mu^-(A) = \psi_{\#}\mu^+(A) = \mu^+(\psi^{-1}(A))$).

best: minimum total "work"

Work or cost of ψ: $I(\psi) = \int_X |x - \psi(x)|\, d\mu^+(x)$.

Monge's problem (1781): Minimize the cost

$$I[\psi] := \int_X |x - \psi(x)| \, d\mu^+(x)$$

among all "transport maps" in

$$\mathcal{A} = \left\{ \psi : X \to X \text{ Borel, one-to-one, } \psi_\#(\mu^+) = \mu^- \right\}.$$

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Or in more general case, minimize

$$I[\psi] := \int_X c(x, \psi(x))\, d\mu^+(x)$$

for some given cost function $c : X \times X \to [0, +\infty)$. For instance, $c(x, y) = |x - y|^p$ for some $p > 0$.

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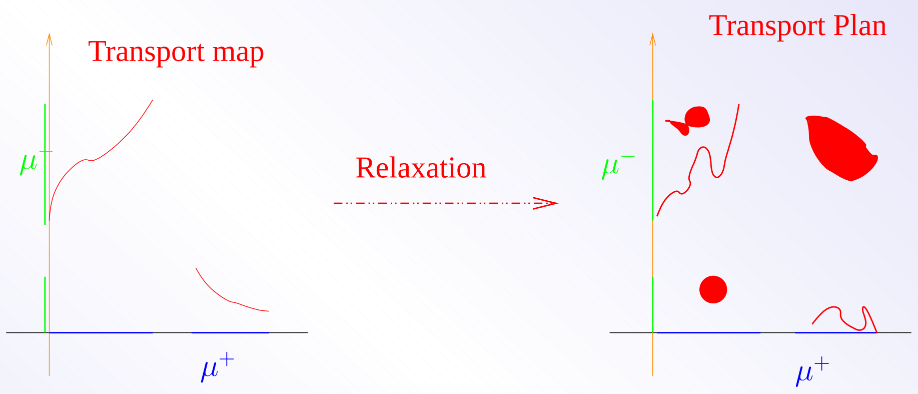
Technical Difficulties:

- Highly nonlinear structure of I.
- No solution when $X = [-1, 1]$, $\mu^+ = \delta_0$, $\mu^- = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$.

Kantorovich (1940's)

Transform it into a linear problem on a convex set.



Minimize

$$J(\gamma) := \int_{X \times X} c(x, y)\, d\gamma(x, y)$$

in the class of transport plans

$$\mathcal{M} = \{\gamma \in P(X \times X) \,|\, \pi_{1\#}\gamma = \mu^+,\ \pi_{2\#}\gamma = \mu^-\}.$$

Existence: from a simple compactness argument of probability measures.

Wassenstein distances on $P(X)$

Definition. *Given* $p \in (0, +\infty)$ *(usually* $[1, +\infty)$*), for any two probability measures* $\mu^+, \mu^- \in P(X)$*, define*

$$W_p(\mu^+, \mu^-) := \left(\min_{\gamma \in \mathcal{M}} \int_{X \times X} |x - y|^p \, d\gamma(x, y) \right)^{\min(1,1/p)}.$$

distance between measures= minimal cost

Proposition. W_p *is a distance on* $P(X)$ *and metrizes the weak * topology of* $P(X)$.

Applications: Monge-Kantorovich problem has many applications in Economic (Nobel Prize in 1975); Fluid Mechanics; PDE; Optimization; meteorology and oceanography; surface reconstruction; $\cdots$.

A Partial list of experts: Ambrosio, Brenier, Caffarelli, Evans, Feldman, Gangbo, Kinderleher, McCann, Otto, Trudinger, Wang, $\cdots$.

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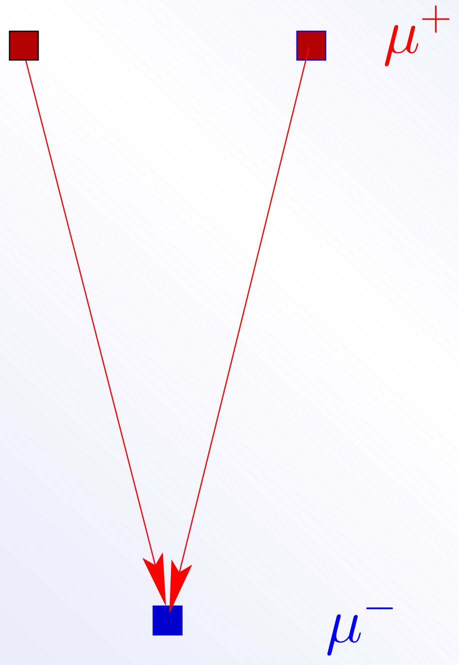
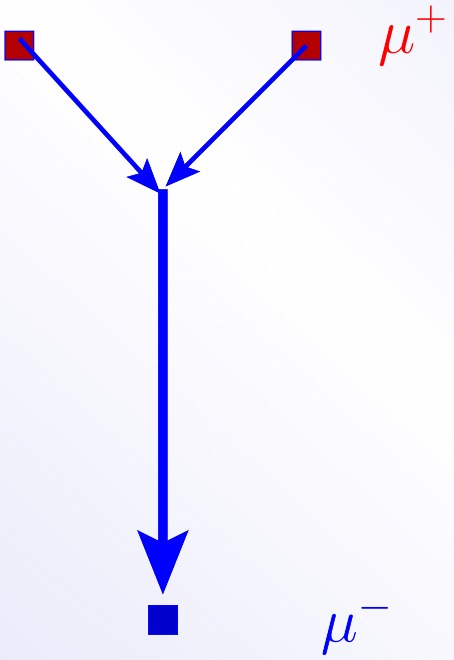


First Attempt: Move them directly to their destination.

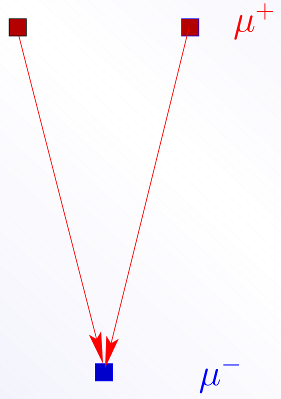
But, should we always define cost as an integral of a cost function $c(x, y)$?

Answer: Not always.

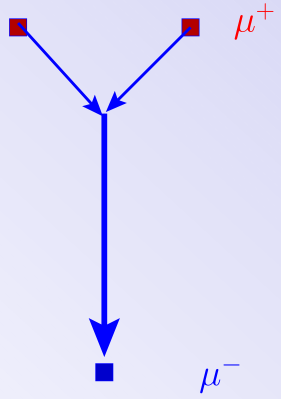
Example: What is the best way to ship two items from nearby cities to the same destination far away.



Another way: put them on the same truck and transport together!



A V-shaped path



A Y-shaped path

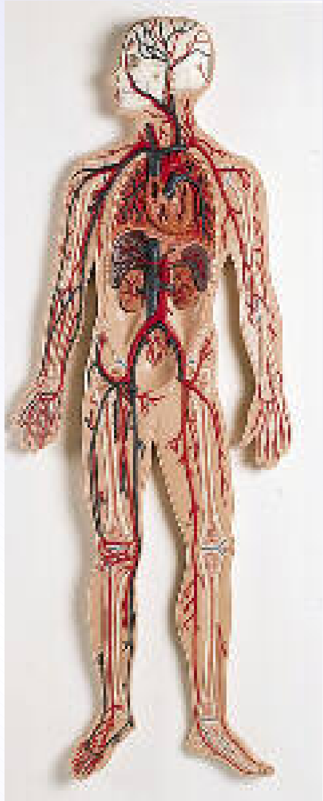
Answer: Transporting two items together might be cheaper than the total cost of transporting them separately. As a result,

- A "Y shaped" path is preferable to a "V shaped" path.
- Here, the cost is naturally given by the actual transport "path", while the transport maps for both types are trivially same. Knowing only maps is not enough here.

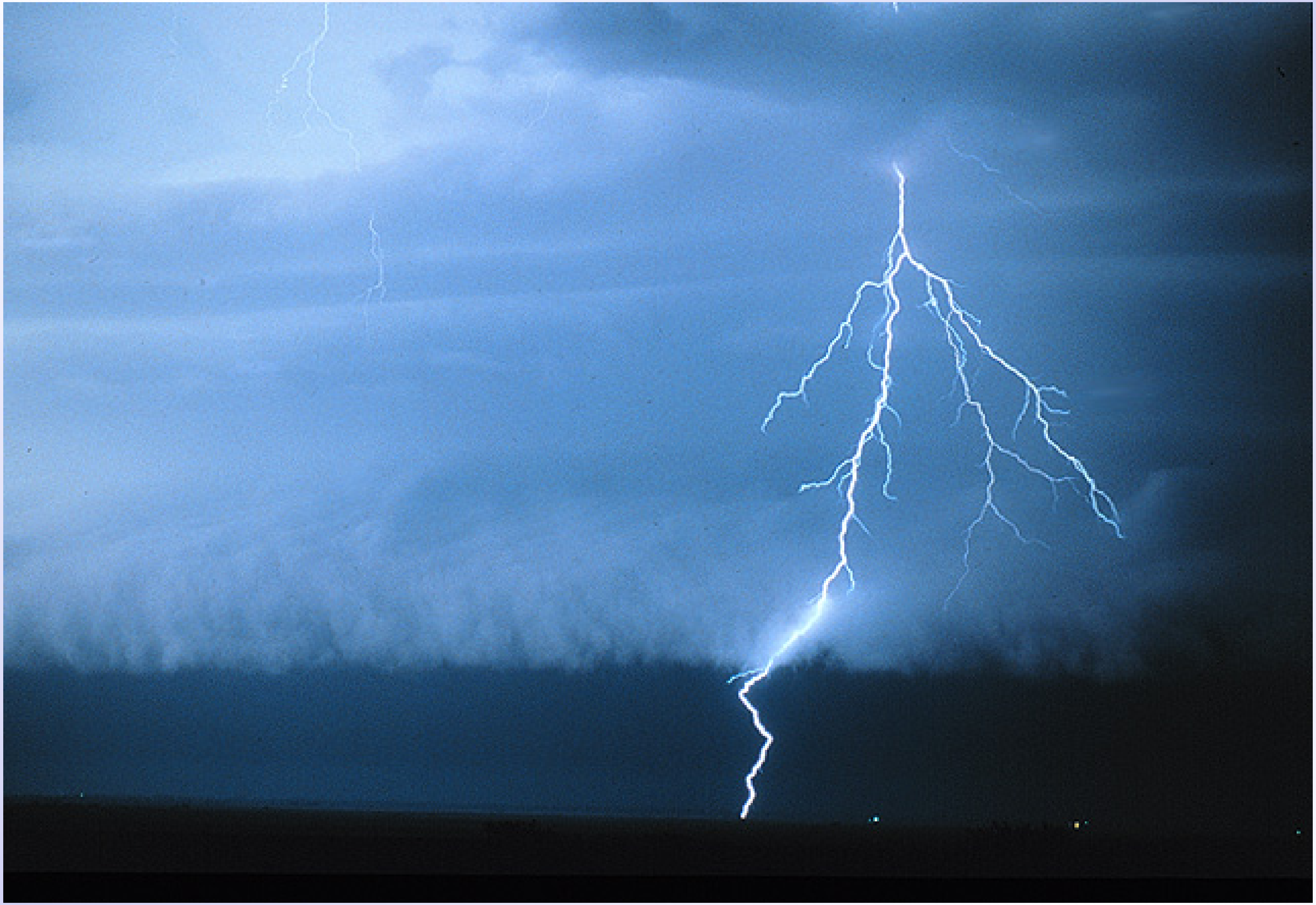
In general, a ramified structure might be more efficient than a "linear" structure consisting of straight lines.

Examples of Ramified Structures

- Trees
- Circulatory systems
- Cardiovasular systems
- Railways, Airlines
- Electric power supply
- River channel networks
- Post office mailing system
- Urban transport network
- Marketing
- Ordinary life
- Communications
- Superconductor



Conclusion: Ramified structures are very common in living and non-living systems. It deserves a more general theoretic treatment.



Problem: Given two arbitrary probability measures μ^+ and $\mu^- \in P(X)$ on a convex compact subset $X \subset \mathbb{R}^m$, find an optimal path transporting μ^+ to μ^-.

Need:

- A class of "transport paths".
  - Broad enough to ensure the existence of optimal transport paths;
- A reasonable cost functional on the category.
  - Optimal transport paths should allow some parts overlap in a cost efficient fashion. Should be "Y-shaped" rather than "V shaped".
  - Nice regularity of optimal transport paths.

Idea: figuring out simple cases first!

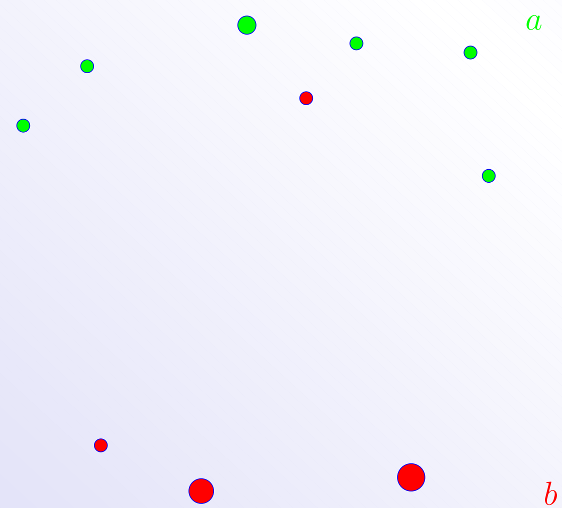
Atomic measures

An atomic measure is a (finite) sum of Dirac measures with positive multiplicities.

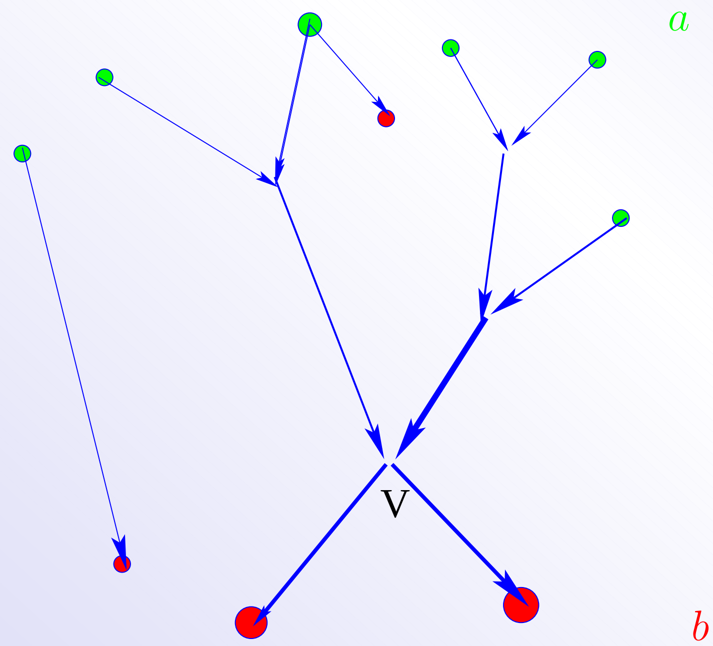
$$a = \sum_i a_i \delta_{x_i}$$

for some $x_i \in X$ and $a_i > 0$. Let $\mathcal{A}(X)$ be the space of all atomic measures on X.

Question: What is a transport path between two atomic probability measures a and b?



Transport atomic measures



A transport path from a to b is a weighted directed graph

$$G = \{V(G), E(G), w : E(G) \to (0, +\infty)\}$$

satisfying Kirchhoff's laws (for eletrical circuits):

$$\sum_{v=e^-} w(e) = \sum_{v=e^+} w(e)$$

for any interior vertex v.

Notation: For atomic measures $a, b \in P(X)$, let

$Path(a, b)$ be the family of all transport paths from a to b.

Cost Functionals

Note that in general the space $\mathrm{Path}(a, b)$ might be very large.

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Answer: For each $G = \{V(G), E(G), w : E(G) \to (0, +\infty)\}$, define the $\mathbf{M}_\alpha$ mass of G by

$$\mathbf{M}_\alpha(\mathbf{G}) := \sum_{\mathbf{e}} \mathbf{w}(\mathbf{e})^{\alpha} \, \text{length}(\mathbf{e})$$

for some $\alpha \in [0, 1)$.

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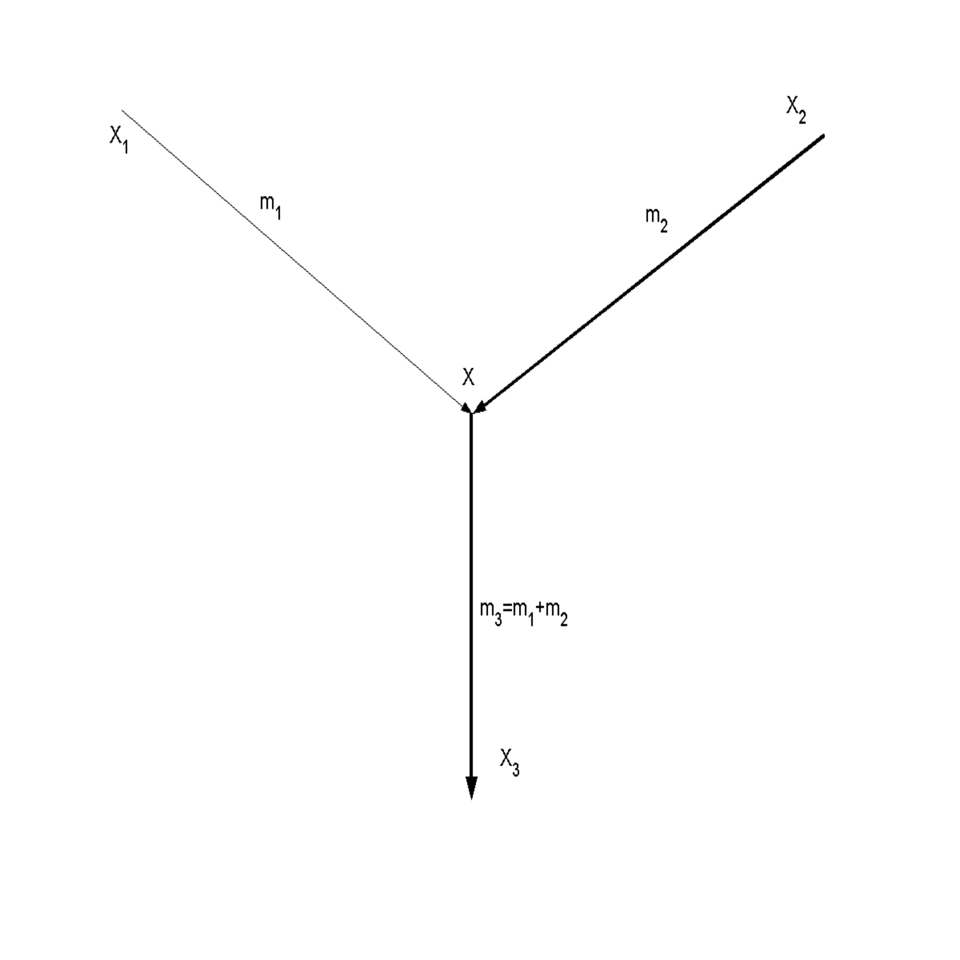
Result: an $\mathbf{M}_\alpha$ mass minimizer is indeed "Y-shaped" or "ramified".

Example 1: Two points to one point



It satisfies a balance equation:

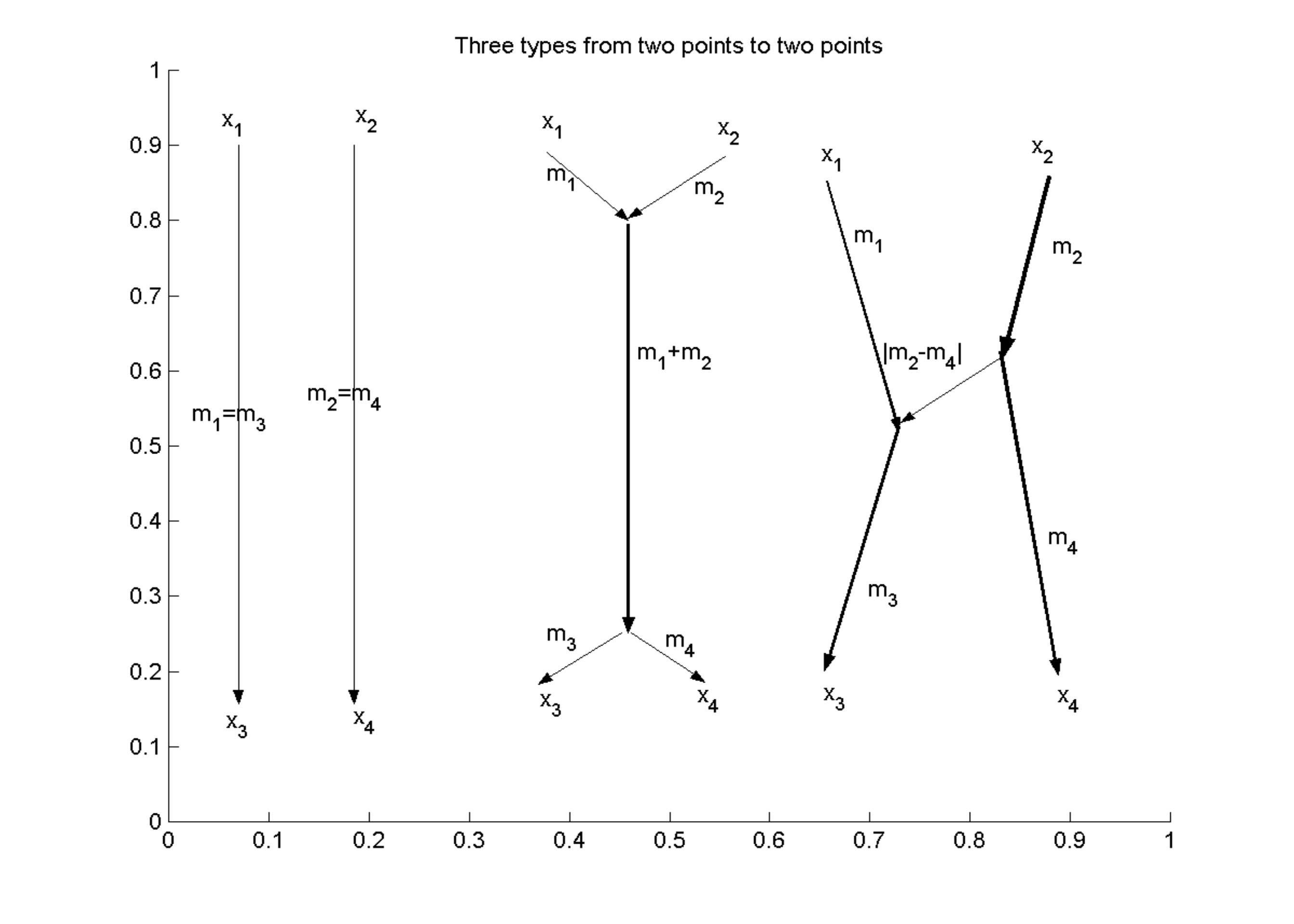
$$\sum_{i=1}^{3} m_i^{\alpha} \vec{n_i} = \vec{0}.$$

Using this equation, we have a formula to calculate the angles.

In particular, if $\alpha = 0$, then the angles are 120^o.

Also, if $\alpha = 1/2$, then the top angle must be 90^o.

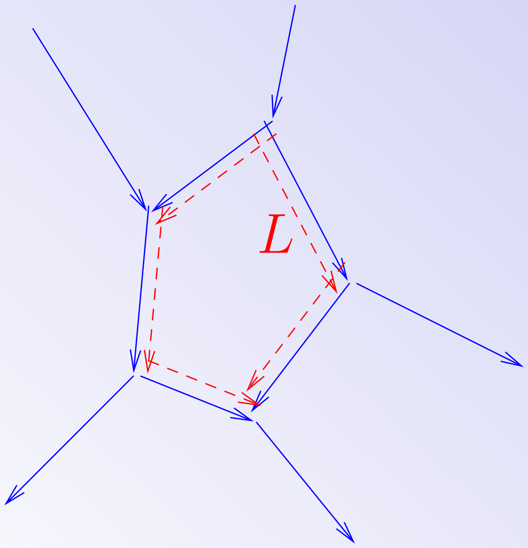
Two points to two points



Some lemmas (Xia, 2001)

Lemma. *For any $G \in Path(a,b)$, there exists a $\tilde{G} \in Path(a,b)$ such that $\tilde{G}$ contains no cycles and*

$$\mathbf{M}_\alpha\left(\tilde{\mathbf{G}}\right) \leq \mathbf{M}_\alpha\left(\mathbf{G}\right).$$



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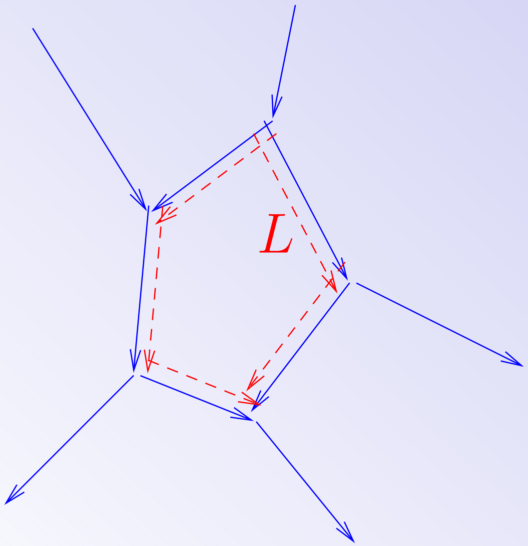

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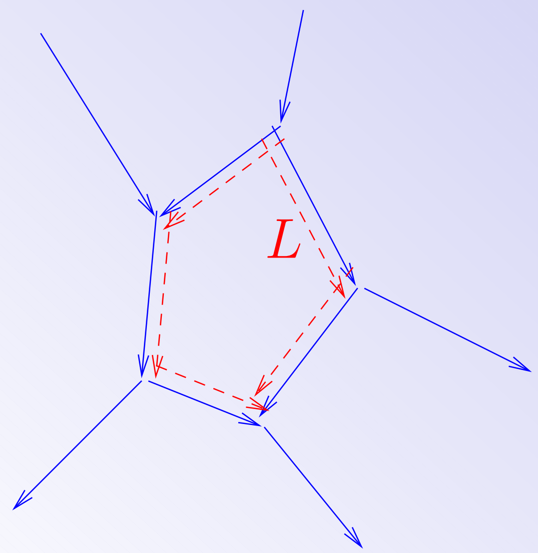

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Lemma. *If G contains no cycles, then $0 < w(e) \leq 1$ for any $e \in E(G)$. Thus*

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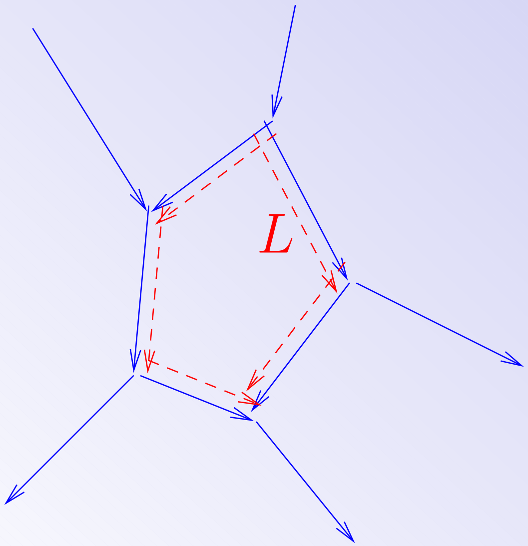

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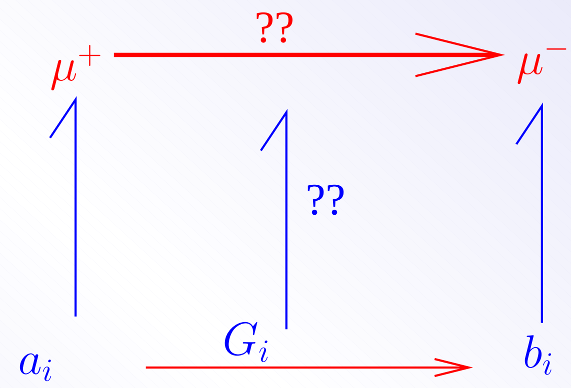
Now, given any two probability measures μ^+ and μ^-, what is a transport path from μ^+ to μ^-?

μ^+  μ^-

Transport general probability measures



Idea:

- Approximate μ^+, μ^- by atomic measures a_i, b_i;
- Transport a_i to b_i by a graph G_i;
- The limit T of G_i (in a suitable sense) is a transportation of μ^+ to μ^-.

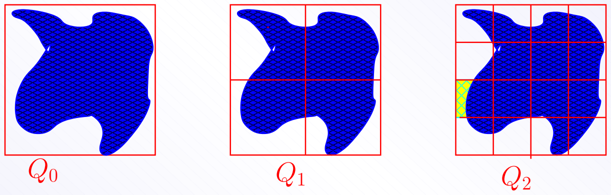
The sequence of triples $\{a_i, b_i, G_i\}$ is called an approximating graph sequence of T.

Dyadic approximation of Radon measures

Assume $X \subset Q$, a cube in $\mathbf{R}^m$ of the edge length d, with center c. Let

$$Q_i = \{Q_i^h : h \in \mathbf{Z}^m \cap [0, 2^i)^m\}$$

be a partition of Q into smaller cubes of edge length $\frac{d}{2^i}$.



For any Radon measure μ on X, let

$$A_i(\mu) = \sum_h \mu(Q_i^h)\delta_{c_i^h}$$

where c_i^h is the center of Q_i^h. Then, $A_i(h)$ converges to μ weakly as measures. This is called "Dyadic approximation of μ".

How to take limits of G_i's ? ——Duality!!

Answer: View each G_i as a 1 dimensional normal current with $\partial G_i = b_i - a_i$.

Let $U \subset \mathbf{R}^m$ be any open set.

- $\mathcal{D}^n(U)$: C^∞ differential $n-$forms in U with compact support.
- An n-current is an element of the dual space $\mathcal{D}_n(U)$ of $\mathcal{D}^n(U)$. i.e. an $n-$current is a continuous linear functional on $\mathcal{D}^n(U)$. Thus, $0-$currents are just distributions.
- For any $T \in \mathcal{D}_n(U)$, its boundary $\partial T \in \mathcal{D}_{n-1}(U)$ is given by

$$\partial T(\psi) = T(d\psi), \forall \psi \in \mathcal{D}^{n-1}(U).$$

- The mass of $T \in \mathcal{D}_n(U)$ is given by

$$\mathbf{M}(T) = \sup\{T(\omega) : |\omega| \leq 1, \omega \in \mathcal{D}^n(U)\}$$

- $T \in \mathcal{D}_n(U)$ is normal if $\mathbf{M}(T) + \mathbf{M}(\partial T) < +\infty$.

Examples of n-current

- Oriented n-dimensional submanifold M of U with $\mathcal{H}^n(M) < +\infty$.

$$[M](\omega) = \int_M \omega = \int_M < \omega(x), \xi(x) > d\mathcal{H}^n(x)$$

for any $\omega \in \mathcal{D}^n(U)$. Note that $\partial[M] = [\partial M]$ and $\mathbf{M}([M]) = \mathcal{H}^n(M)$.

- Differential $m - n$ forms $\phi \in \mathcal{D}^{m-n}(U)$;

$$\phi(\omega) = \int_U \phi \wedge \omega.$$

- Rectifiable currents $\tau(M, \theta, \xi)$

$$\tau(M, \theta, \xi)(\omega) = \int_M < \omega(x), \xi(x) > \theta(x) d\mathcal{H}^n(x)$$

Here: M is a rectifiable n-set, θ is a locally $\mathcal{H}^n$ integrable function and $\xi(x)$ is the orientation of $T_x M$.

Transport paths between Radon measures

Definition. *Given $\mu^+, \mu^- \in P(X)$, a normal 1-current T is called a* *transport path* *from μ^+ to μ^- if there exists a sequence of approximating graphs $\{a_i, b_i, G_i\}$ such that*

$$a_i \rightharpoonup \mu^+, b_i \rightharpoonup \mu^-, G_i \rightharpoonup T$$

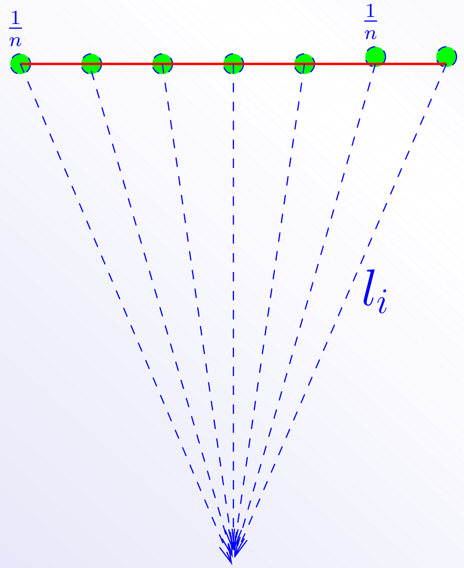
in the sense of distributions.

Note that we automatically have $\partial T = \mu^+ - \mu^-$ as distributions.
For each transport path T, we define

$$\mathbf{M}_\alpha(\mathbf{T}) := \inf_{\{a_i,b_i,G_i\}} \liminf_{\mathbf{i}\to\infty} \mathbf{M}_\alpha(\mathbf{G_i}).$$

Let $Path(\mu^+, \mu^-)$ be the family of all transport paths from μ^+ to μ^-.

Example: How to transport a Lebesgue measure to a Dirac measure?



First attempt:

$$\sum_{i=1}^{n} \left(\frac{1}{n}\right)^{\alpha} l_i$$

$$\approx C \sum_{i=1}^{n} \left(\frac{1}{n}\right)^{\alpha} = Cn^{1-\alpha} \to +\infty.$$

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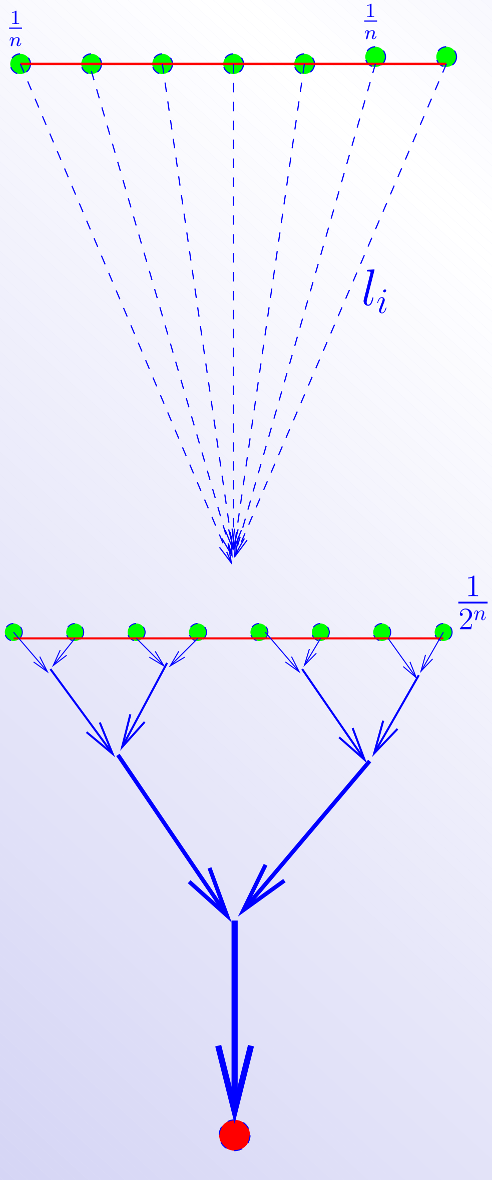

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Second attempt:

$$\sum_{n=1}^{\infty}\sum_{i=1}^{2^n}\left(\frac{1}{2^n}\right)^{\alpha} l_i \approx C\sum_{n=1}^{\infty}\sum_{i=1}^{2^n}\left(\frac{1}{2^n}\right)^{\alpha}\frac{1}{2^n}$$

$$= C\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)^{\alpha} = \frac{C}{1-\frac{1}{2^{\alpha}}}$$

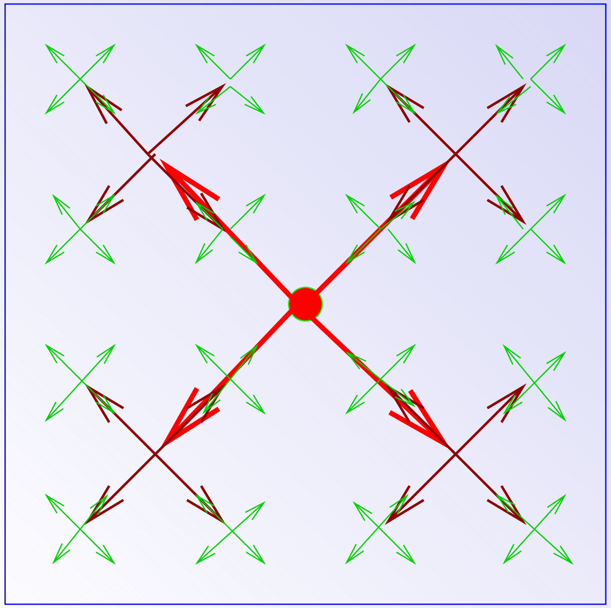
In higher dimension case, if $\alpha > 1 - \frac{1}{m}$, then

$$\sum_{n=1}^{\infty} \sum_{i=1}^{(2^n)^m} \left(\frac{1}{(2^n)^m} \right)^{\alpha} l_i$$

$$\approx C \sum_{n=1}^{\infty} \sum_{i=1}^{(2^n)^m} \left(\frac{1}{(2^n)^m} \right)^{\alpha} \frac{1}{2^n}$$

$$= C \sum_{n=1}^{\infty} \left(\frac{1}{(2^n)^m} \right)^{\alpha} 2^{n(m-1)}$$

$$= C \sum_{n=1}^{\infty} \left(2^{m(1-\alpha)-1} \right)^{n} < +\infty$$



Proposition. [Finite Cost] *(Xia, 2001) Suppose $\alpha > 1 - \frac{1}{m}$. For any $\mu \in P(X)$, there exists a $T \in Path(\mu, \delta_c)$ from μ to a Dirac measure δ_c with* $\mathbf{M}_\alpha(\mathbf{T}) < +\infty$.

Existence theorem (Xia, 2001)

Theorem. *Given* μ^+ *and* $\mu^- \in \mathcal{M}_\Lambda(X), \alpha \in (1 - \frac{1}{m}, 1]$, *there exists an* $\mathbf{M}_\alpha$ *mass minimizer* S *in the family* $Path(\mu^+, \mu^-)$. *Moreover,* $\mathbf{M}_\alpha(\mathbf{S}) < \frac{\mathbf{\Lambda}^\alpha}{\mathbf{2^{1-m(1-\alpha)}-1}} \frac{\sqrt{\mathbf{md}}}{\mathbf{2}}$.

Sketch of the proof:

- Pick $\{a_i, b_i, G_i\}$ with

$$\mathbf{M}_\alpha(\mathbf{G_i}) \searrow \inf\{\mathbf{M}_\alpha(\mathbf{T}) : \mathbf{T} \in \mathbf{Path}(\mu^+, \mu^-)\}$$

- We may assume $\{G_i\}$ has no cycless

$$M(G_i) \leq \mathbf{M}_\alpha(\mathbf{G_i}) < \mathbf{C} \text{ bounded.}$$

- By the compactness of normal currents,

$$G_{i_k} \rightharpoonup T \in Path(\mu^+, \mu^-)$$

- lower semicontinuity of $\mathbf{M}_\alpha$.

A new distance on $P(X)$

Definition. *Given μ^+ and $\mu^- \in P(X)$, define*

$$d_\alpha\left(\mu^+, \mu^-\right) := \min\{\mathbf{M}_\alpha(\mathbf{T}) : \mathbf{T} \in \mathbf{Path}(\mu^+, \mu^-)\}.$$

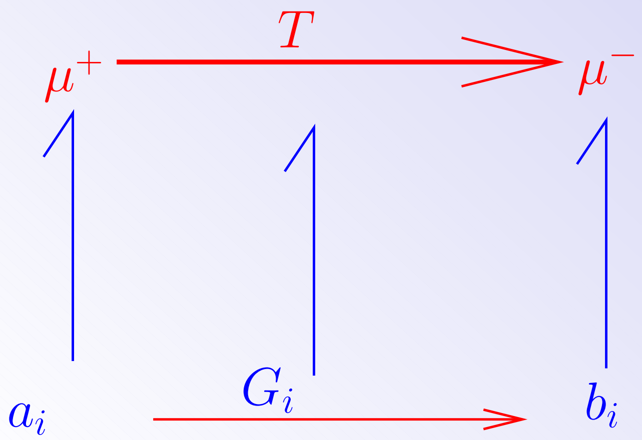
Theorem. *(Xia, 2001) d_α is a distance on $P(X)$.*

Remark: d_α is different from any of the Wassenstein distances.

Theorem. *(Xia, 2001) d_α metrizes the weak * topology of $P(X)$.*

Optimal transport paths

Lemma. *If* $G_i \in Path(a_i, b_i)$ *is an* $\mathbf{M}_\alpha$ *minimizer* , *then* $T \in Path(\mu^+, \mu^-)$ *is also an* $\mathbf{M}_\alpha$ *minimizer in* $Path(\mu^+, \mu^-)$.



Definition. *A transport path* $T \in Path(\mu^+, \mu^-)$ *is called an optimal transport path if there exists a sequence of appximating graphs* $\{a_i, b_i, G_i\}$ *such that each* $G_i \in Path(a_i, b_i)$ *is an* $\mathbf{M}_\alpha$ *minimizer.*

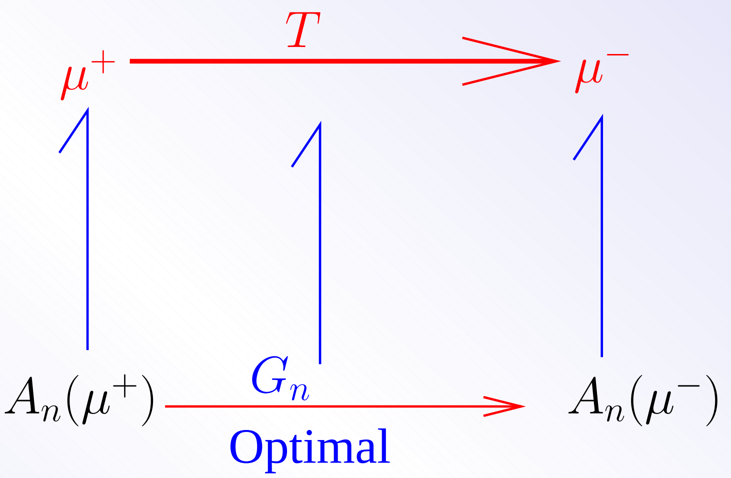
Error estimate

By the lemma, we can pick our favorite approximating atomic measures $\{a_i\}, \{b_i\}$.
We choose "dyadic approximation" $\{A_n(\mu)\}$.

Proposition. *For any* $\mu \in P(X)$,

$$d_\alpha(\mu, A_n(\mu)) \leq C\lambda^n$$

with some constant $C > 0$ *and* $\lambda = 2^{m(1-\alpha)-1} \in (0, 1)$.



Corollary. *If each* G_n *is optimal, then*

$$\mathbf{M}_\alpha(\mathbf{T}) \leq \mathbf{M}_\alpha(\mathbf{G_n}) + \mathbf{2C}\lambda^{\mathbf{n}}$$

Length Space Property

Theorem. *(Xia, 2002)* $(P(X), d_\alpha)$ *is a length space.*

That is, for any $\mu^+, \mu^- \in P(X)$, there exists a continuous map

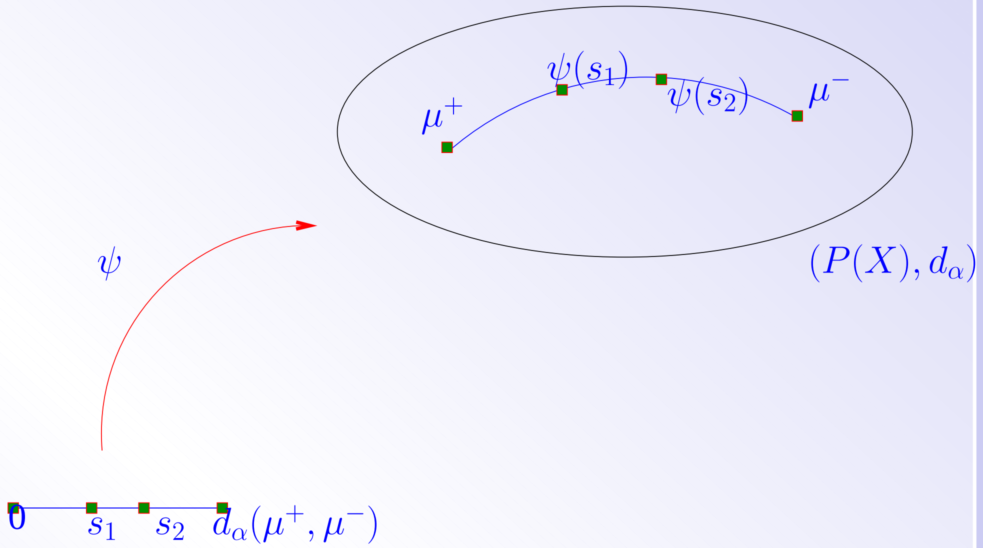
$$\psi : [0, t] \to (P(M), d_\alpha)$$

with $t = d_\alpha(\mu^+, \mu^-)$ such that

$$\psi(0) = \mu^+, \psi(T) = \mu^-$$

and for any $0 \le s_1 < s_2 \le t$,

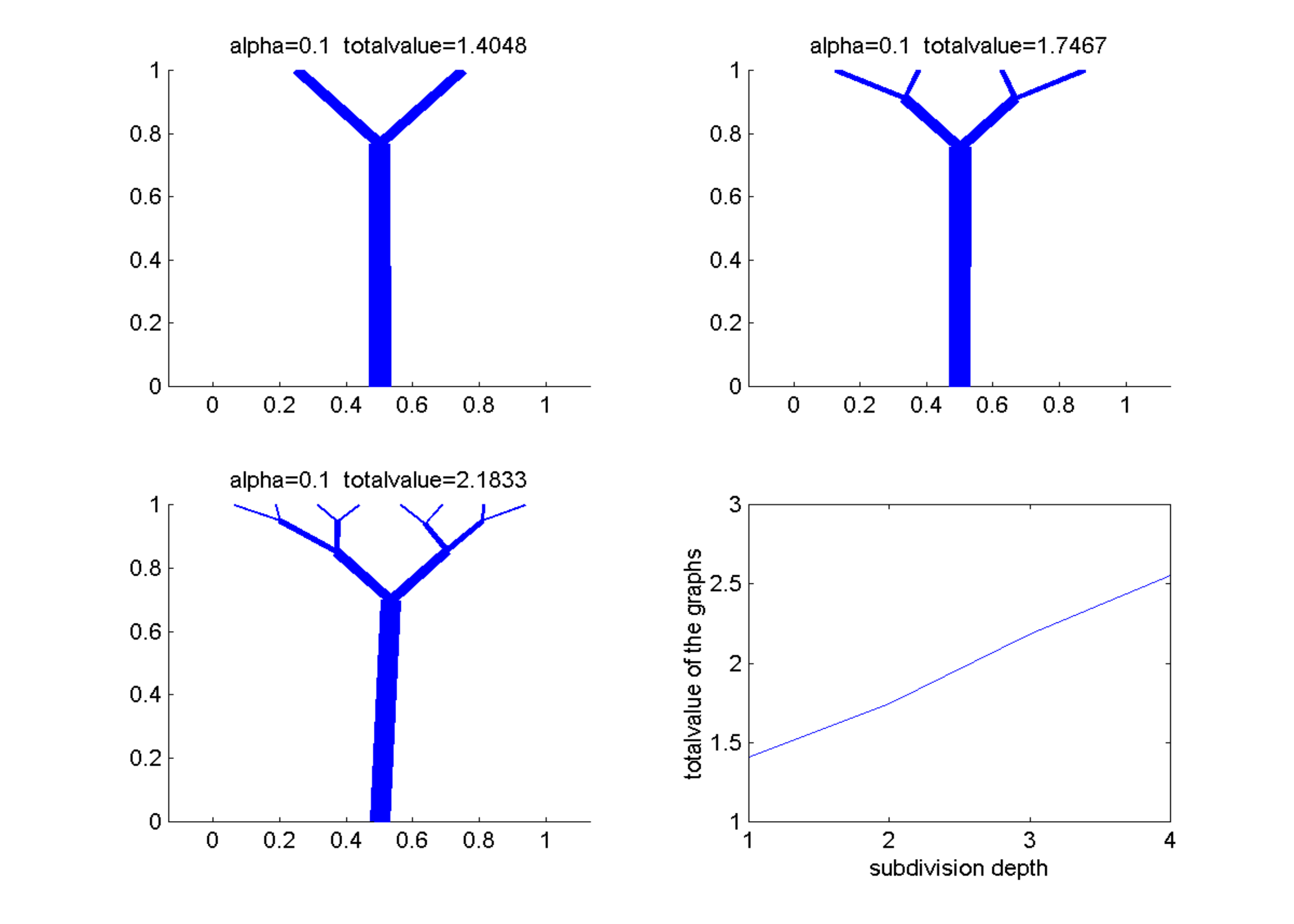
$$d_\alpha(\psi(s_1), \psi(s_2)) = s_2 - s_1.$$



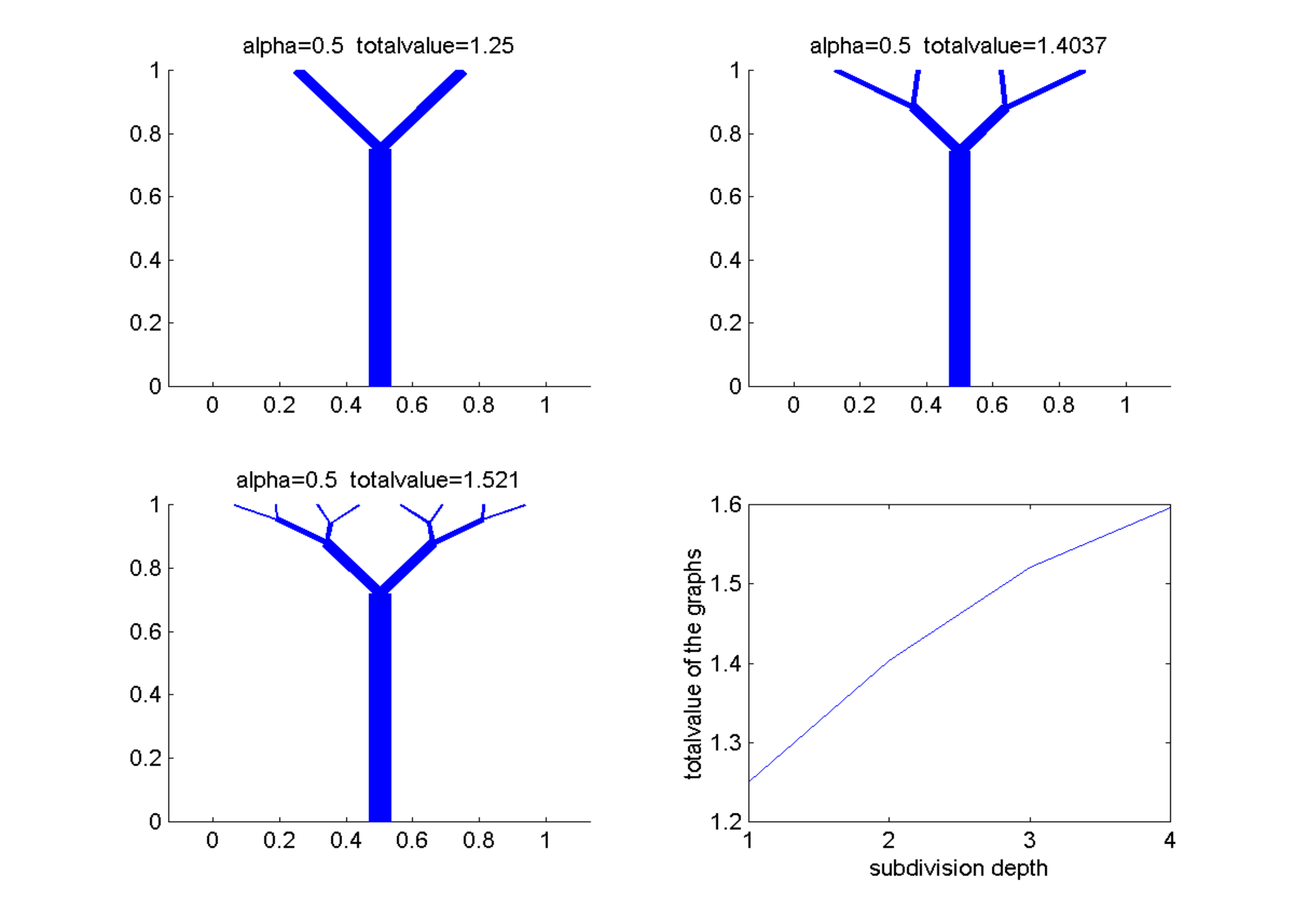
In other words, an optimal transport path between Radon measures plays the role of a geodesic between two points.

Later, we will see that in fact each $\psi(s)$ is purely atomic for any $0 < s < t$.

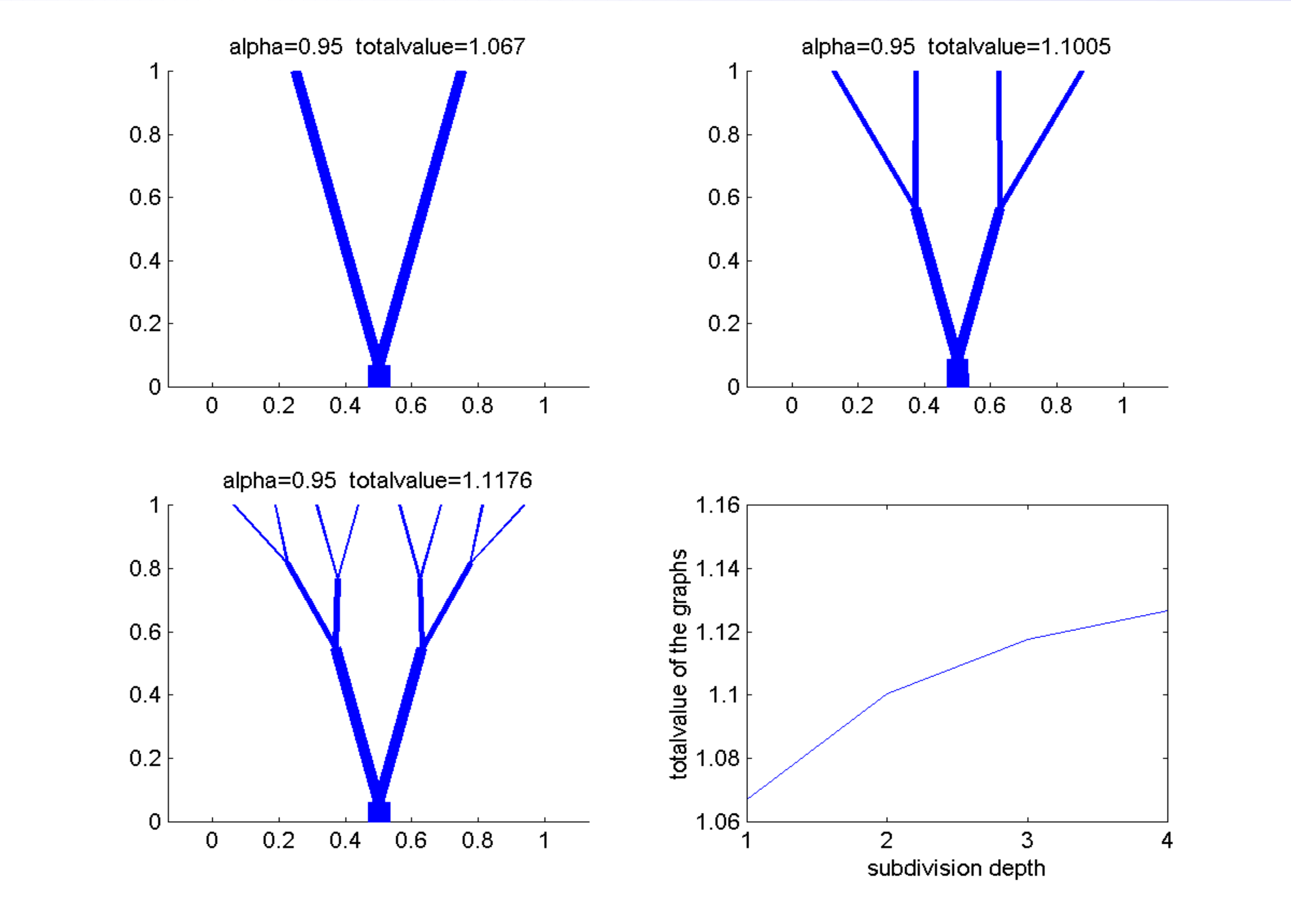
Atomic approximation ($\alpha = 0.1$)



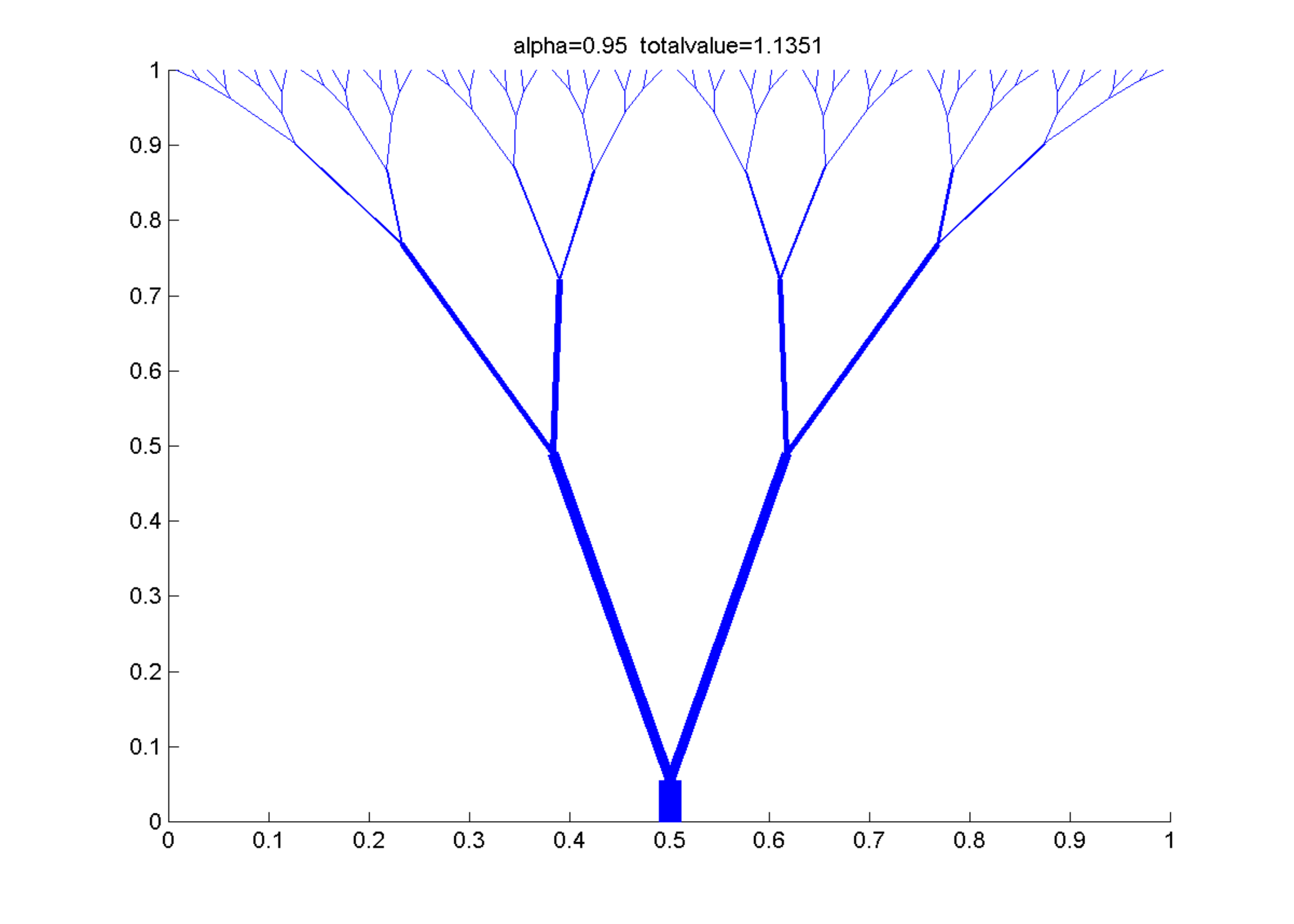
Atomic approximation ($\alpha = 0.5$)



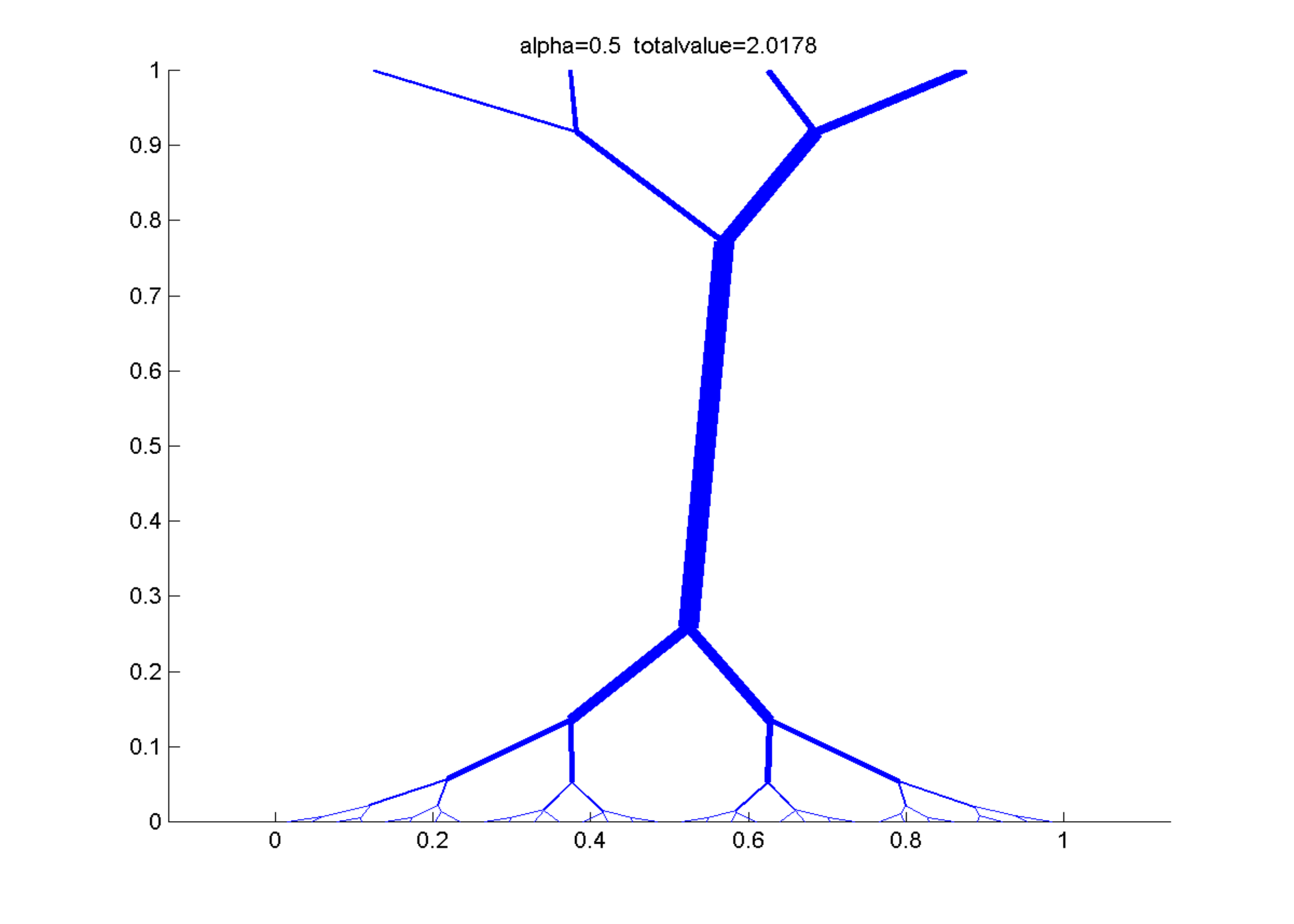
Atomic approximation ($\alpha = 0.95$)



From Lebesgue to Dirac

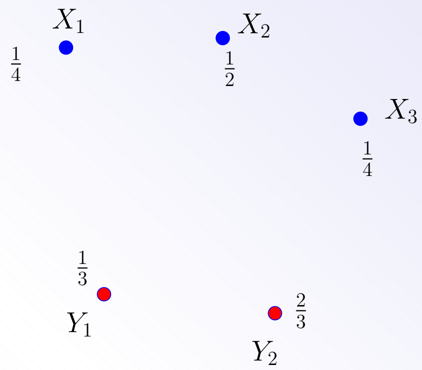


Transporting general measures



Transport Path & Transport Plan

Let a and b be any two atomic measures. For example,



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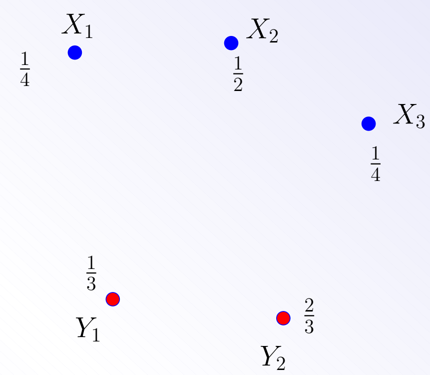


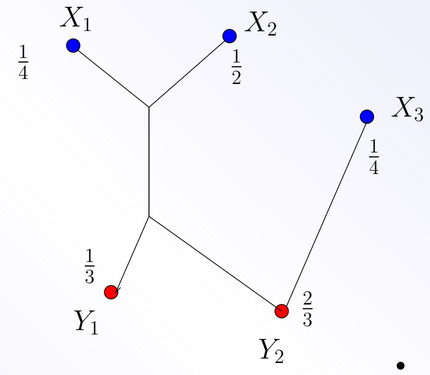
- Each transport plan $\gamma \in Plan(a,b)$ is given by a real valued matrix

$$U = (u_{ij}).$$

e.g.

$$U_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{12} & 0 \\ 0 & \frac{5}{12} & \frac{1}{4} \end{pmatrix} \text{ or } U_2 = \begin{pmatrix} 0 & \frac{1}{12} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{12} & 0 \end{pmatrix}$$

- Each transport path $G \in Path(a,b)$ gives a 1-current valued matrix $g(G) = (g_{ij})$. (no cycles!)



Compatible Pair of Transport Path & Plan

A transport path G and a transport plan γ are said to be compatible if

$$G = \sum u_{ij} \cdot g_{ij}.$$

A compatible pair gives a decomposition of G.

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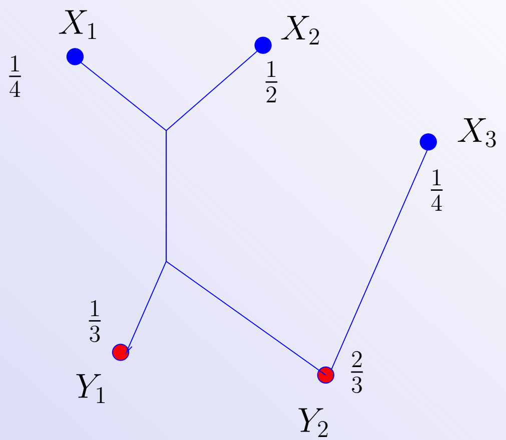

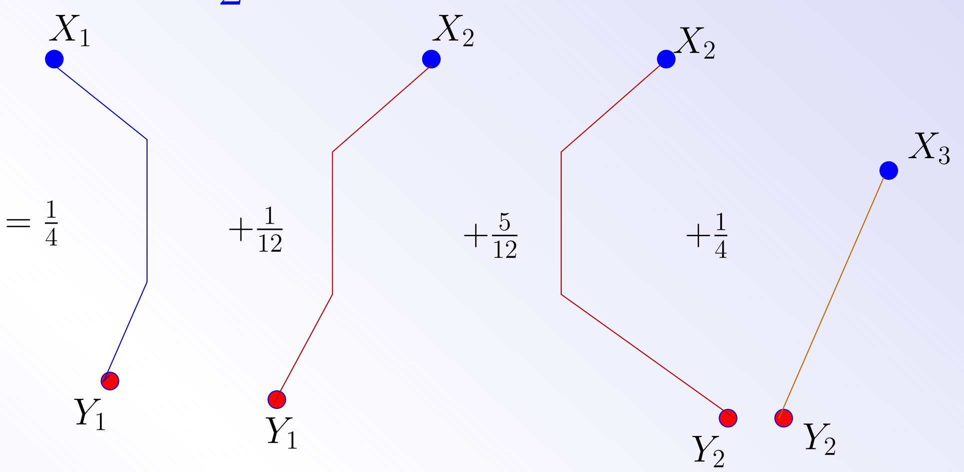

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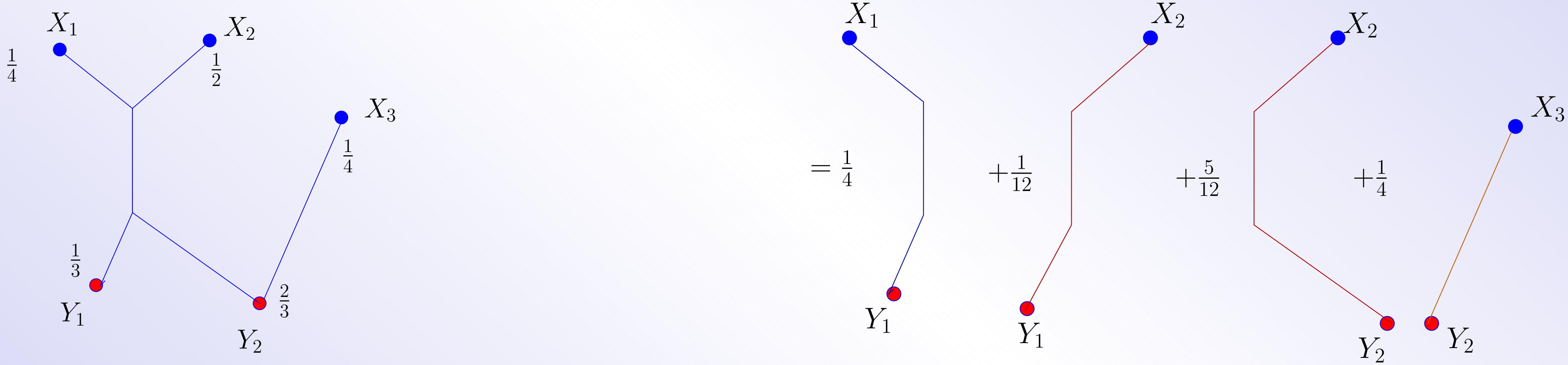


A compatible pair of transport path and transport plan provides the necessary transporting information by its unique matrix representation $\big((u_{ij}), (g_{ij})\big)$.

u_{ij} = amount of mass from x_i to y_j, while g_{ij} = actual transport path.

Some Results (Xia, 2001)

- There exists $G \in Path\,(a,b)$ compatible with all $\gamma \in Plan\,(a,b)$.
- For any $G \in Path\,(a,b)$, there exists a $\gamma \in Plan\,(a,b)$ compatible with G.
- Given a transport plan $\gamma \in Plan\left(\mu^+,\mu^-\right)$, there exists an optimal transport path $T \in Path\left(\mu^+,\mu^-\right)$ with least finite $\mathbf{M}_\alpha$ cost among all compatible pairs (T,γ). (mailing problem)
- Given a transport path $T \in Path\left(\mu^+,\mu^-\right)$, there exists an optimal transport plan $\gamma \in Plan\left(\mu^+,\mu^-\right)$ with least $I\,(\gamma)$ cost among all compatible pairs (T,γ).

How nice is an optimal transport path?

Let $T \in Path(\mu^+, \mu^-)$ be any transport path with $\mathbf{M}_\alpha(\mathbf{T}) < +\infty$, not necessarily optimal.

Theorem. *(rectifiability)(Xia, 2001) T is a real multiplicity 1-rectifiable current $T = \tau(M, \theta, \xi)$ with $\partial T = \mu^+ - \mu^-$. Moreover,*

$$\mathbf{M}_\alpha(\mathbf{T}) = \int_{\mathbf{M}} \theta(\mathbf{x})^\alpha \mathbf{d}\mathcal{H}^{\mathbf{1}}(\mathbf{x})$$

Idea of proof: Follows from the rectifiable slicing theorem.

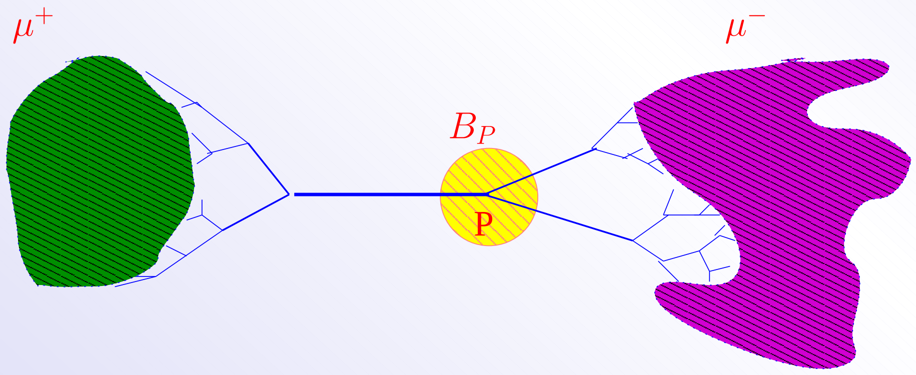
Now, assume that T is optimal. Let us see how nice T is.

Interior regularity: a local finiteness property (Xia, 2002)

For any $p \in spt(T) \setminus spt(\partial T)$, there exists an open ball neighborhood B_p of p such that

$$T \lfloor B_p$$

is a cone at p consisting of finite union of segments with suitable multiplicities. These segments are balanced by a simple balance equation.



How about the boundary ?

Observation: The support of T may not necessarily be 1-dimensional nearby its boundary, which is the difference of the given two measures. This is because the boundary itself may even be dense in the space, as demonstrated by letting the initial measure to be the Lebesgue measure.

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But, how to read this information?

Boundary Regularity

To understand the boundary behavior, a suitable approach is to study the "level sets" of the rectifiable current $T = \tau(M, \theta, \xi)$ instead. For each $\lambda > 0$, let

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Theorem (Xia, 2003): Each level set of an optimal transport path is locally concentrated on a finite union of bilipschitz curves. These curves enjoy some nice properties similar to those satisfied by segments near an interior point.

Key Idea of Proof: Decomposition!

- For any optimal weighted directed graph $G \in Path(a,b)$, if $M^\alpha(a) + M^\alpha(b)$ is bounded above, then we can decompose a, b, G

$$a = a_P + a_R, b = b_P + b_R, G = P + R$$

  so that $P \in Path(a_P, b_P), R \in Path(a_R, b_R)$, the total number of vertices and edges of P are uniformly bounded. The level set G_λ is contained in P. Edges of P are ”nice”.
- Taking the limits to get the decomposition of optimal transport paths.

Advantage: Graphs are much easier to deal with. Just using combinatory.

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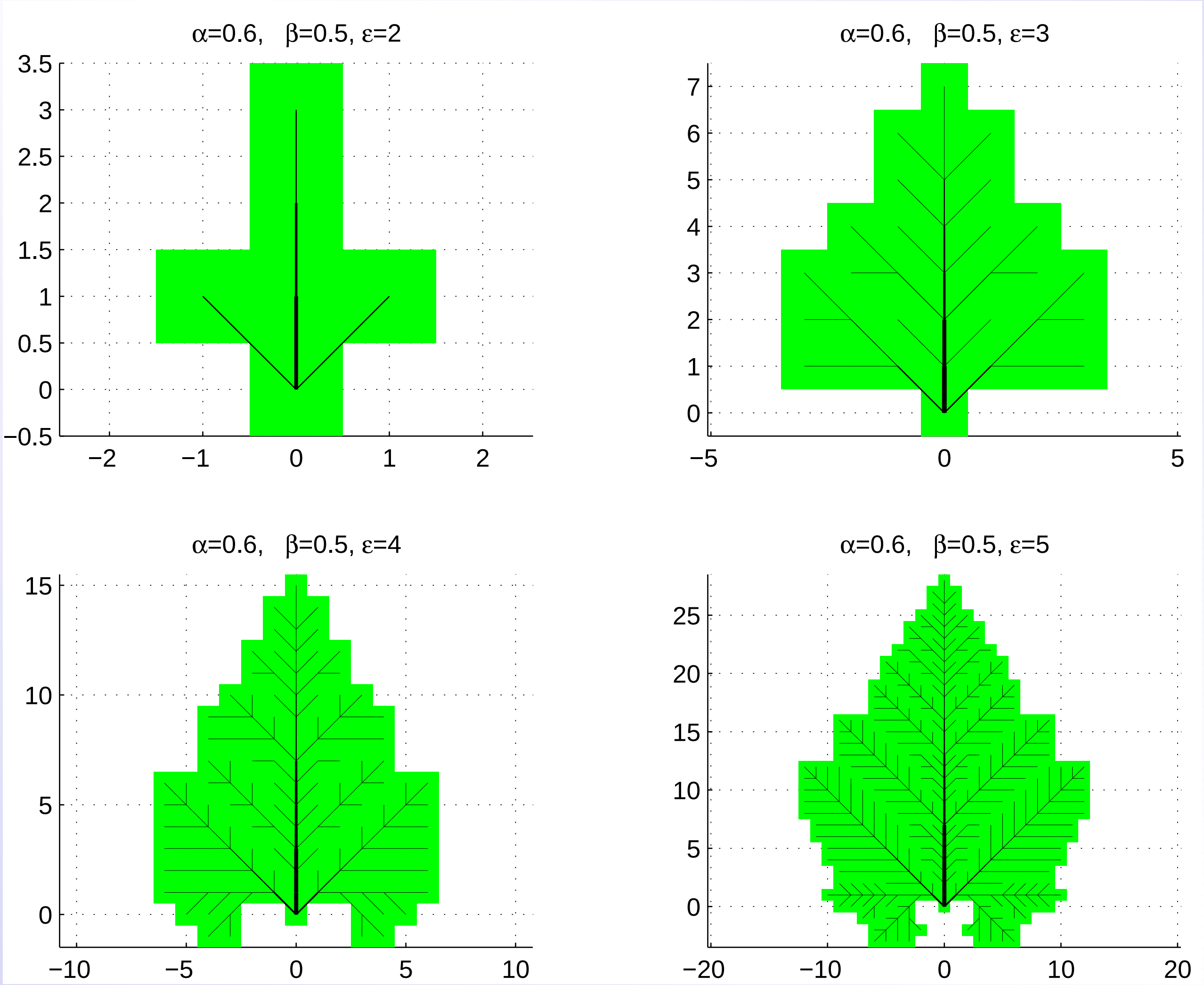
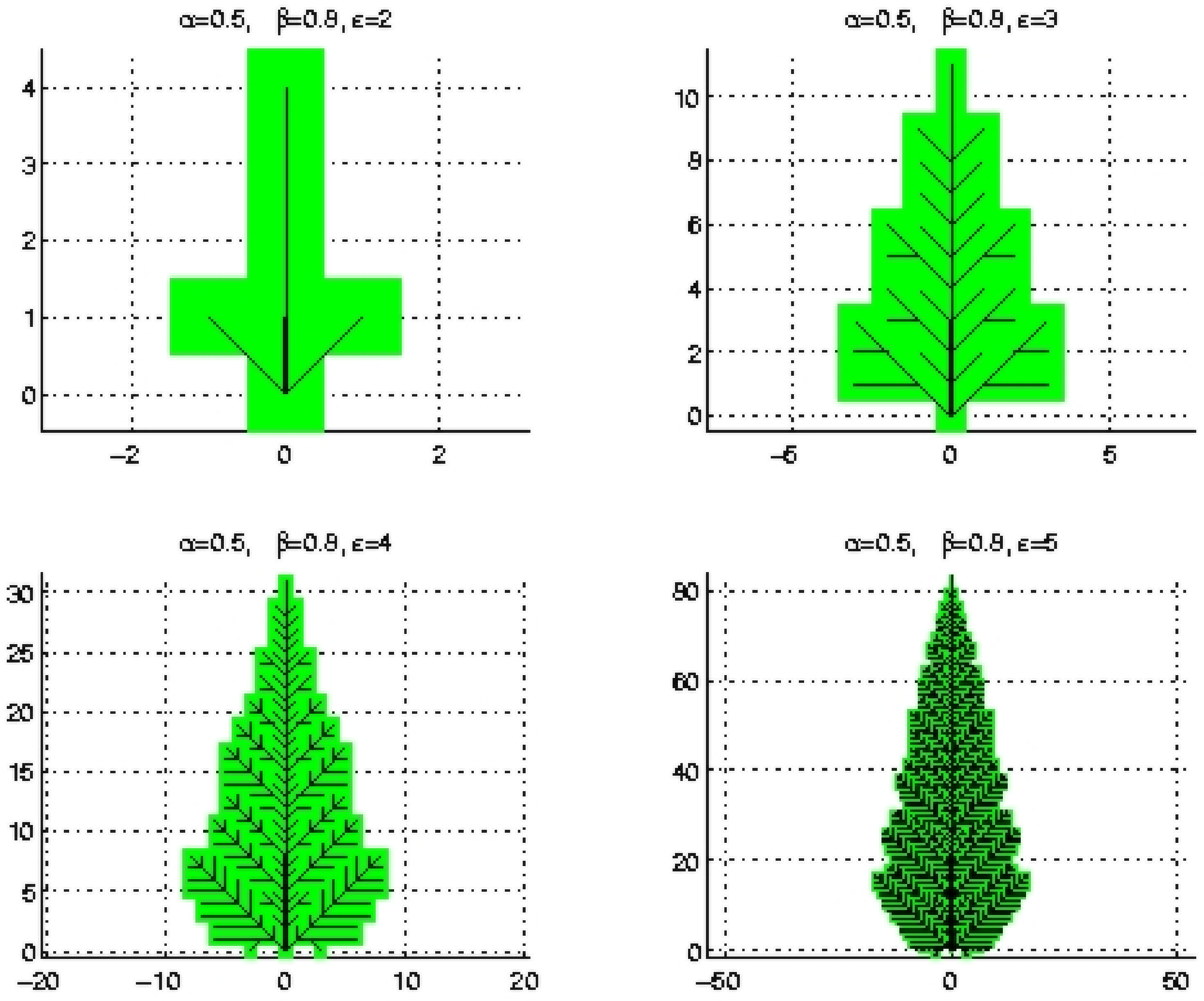
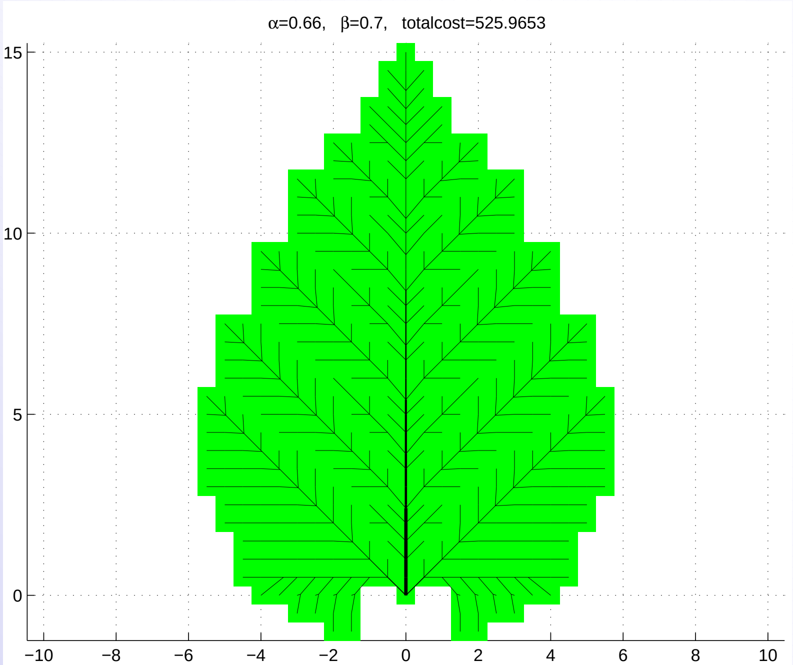
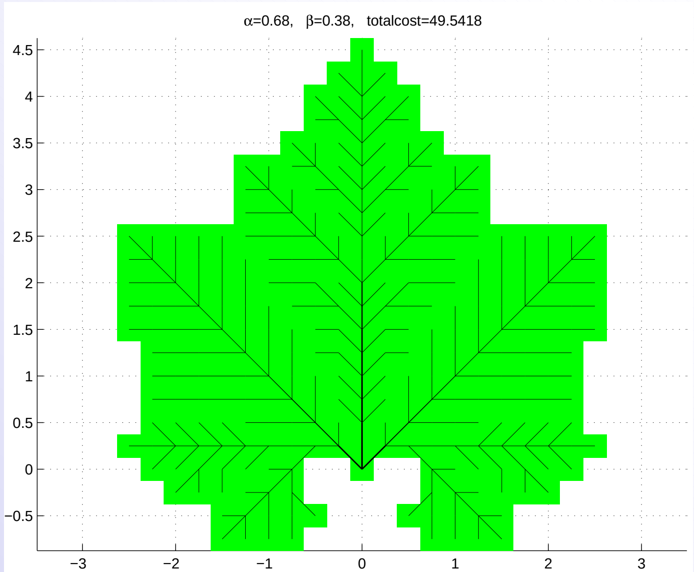
YES!! (Xia, 2004)


α=0.6, β=0.5, ε=2
α=0.6, β=0.5, ε=3
α=0.6, β=0.5, ε=4
α=0.6, β=0.5, ε=5


α=0.5, β=0.8, ε=2
α=0.5, β=0.8, ε=3
α=0.5, β=0.8, ε=4
α=0.5, β=0.8, ε=5


α=0.66,   β=0.7,   totalcost=525.9653
15
10
5
0
−10
−8
−6
−4
−2
0
2
4
6
8
10


α=0.68, β=0.38, totalcost=49.5418
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−3
−2
−1
0
1
2
3

Question: Given a measure μ, ν, for which α, will we have $d_\alpha(\mu, \nu) < +\infty$?

For simplicity, we choose $\nu =$Dirac mass.

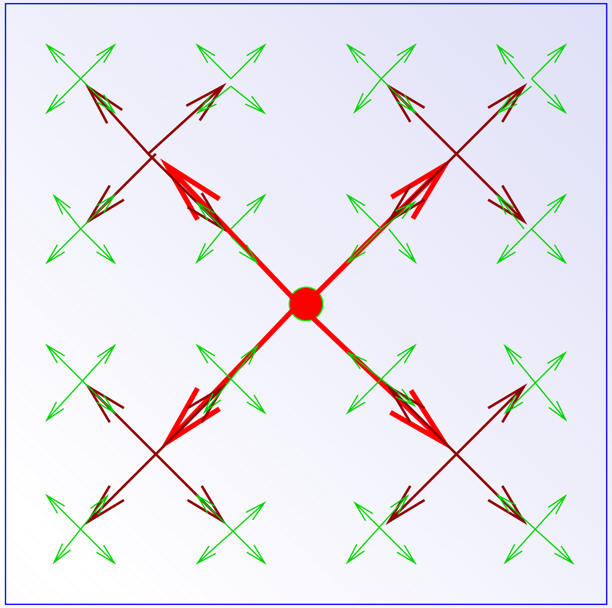
Recall that if $\mu =$Lebesgue measure and $\alpha > 1 - \frac{1}{m}$, then

$$\sum_{n=1}^{\infty} \sum_{i=1}^{(2^n)^m} \left(\frac{1}{(2^n)^m}\right)^\alpha l_i$$

$$\approx C \sum_{n=1}^{\infty} \sum_{i=1}^{(2^n)^m} \left(\frac{1}{(2^n)^m}\right)^\alpha \frac{1}{2^n}$$

$$= C \sum_{n=1}^{\infty} \left(\frac{1}{(2^n)^m}\right)^\alpha 2^{n(m-1)}$$

$$= C \sum_{n=1}^{\infty} \left(2^{m(1-\alpha)-1}\right)^n < +\infty$$



Here, dimension $m = \inf_{\alpha<1}\{\frac{1}{1-\alpha} : d_\alpha(\mu, \delta_0) < +\infty\}$

Example: $\mu =$Cantor set, $\nu =$Dirac mass

$$\sum_{n=1}^{\infty} 2^n \left(\frac{1}{2^n}\right)^{\alpha} \left(\frac{1}{3}\right)^{n} = \sum_{n=1}^{\infty} \left(\frac{2^{1-\alpha}}{3}\right)^{n} < \infty$$

$$\iff \frac{2^{1-\alpha}}{3} < 1$$

$$\iff 2^{1-\alpha} < 3$$

$$\iff \frac{1}{1-\alpha} > \frac{\ln 2}{\ln 3}$$

Here again, dimension of $\mu = \inf_{\alpha<1}\{\frac{1}{1-\alpha} : d_\alpha(\mu, \delta_0) < +\infty\}$.
Note, here α is allowed to be negative.

Example: $\mu =$Fat Cantor set, $\nu =$Dirac mass

Examples: $\mu =$ Fat λ Cantor set (i.e. remove an interval of length λ from the middle of $[0,1]$).

$$\begin{aligned}\sum_{n=1}^{\infty} 2^n \left(\frac{1}{2^n}\right)^{\alpha} \frac{1+\lambda}{4}\left(\frac{1-\lambda}{2}\right)^{n-1} &= \frac{1+\lambda}{2(1-\lambda)} \sum_{n=1}^{\infty} \left(2^{1-\alpha} p\right)^n < \infty \\ &\iff 2^{1-\alpha} p < 1 \\ &\iff 2^{1-\alpha} < \frac{1}{p} \\ &\iff \frac{1}{1-\alpha} > -\frac{\ln 2}{\ln p} = \frac{\ln 2}{\ln 2 - \ln(1-\lambda)}\end{aligned}$$

where $p = \frac{1-\lambda}{2}$.

Again, we have dimension of $\mu = \inf_{\alpha<1}\{\frac{1}{1-\alpha} : d_\alpha(\mu, \delta_0) < +\infty\}$

Example: $\mu =$self-similar set, $\nu =$Dirac mass

Example: $A =$finite union of A_i for $i = 1, \cdots$k. Each A_i is a $\sigma-$rescale of A.

$$\sum_{n=1}^{\infty} k^n \left(\frac{1}{k^n}\right)^{\alpha} \sigma^{n-1} L = \frac{L}{\sigma} \sum_{n=1}^{\infty} \left(k^{1-\alpha}\sigma\right)^n < +\infty$$
$$\iff k^{1-\alpha}\sigma < 1$$
$$\iff \frac{1}{1-\alpha} > -\frac{\ln k}{\ln \sigma}$$

Therefore, $D(\mu) = -\frac{\ln k}{\ln \sigma}$.

Here again, self-similar dimension of $\mu = \inf_{\alpha<1}\{\frac{1}{1-\alpha} : d_\alpha(\mu, \delta_0) < +\infty\}$

Dimensional distance

For any $\mu, \nu \in P(X)$, let

$$D(\mu,\nu) = \inf_{\alpha<1}\{\frac{1}{1-\alpha} : d_\alpha(\mu,\nu) < +\infty\}$$

Proposition. $(P(X), D)$ *is a pseudometric space* .

That is, D is a metric except that $D(\mu,\nu) = 0$ does not imply $\mu = \nu$.
e.g. $D(\delta_x, \delta_y) = 0$ for any $x, y \in X$ because $d_\alpha(\delta_x, \delta_y) = |x - y| < +\infty, \forall \alpha$.

Definition. *For any μ and ν, we say $\mu \simeq \nu$ if $D(\mu,\nu) = 0$. That is, μ and ν are equivalent if and only if $d_\beta(\mu,\nu) < +\infty$ for any β. The equivalent class of μ is denoted by $[\mu]$.*

Lemma. *If $\mu_1 \simeq \mu_2$, then for any ν, $D(\mu_1,\nu) = D(\mu_2,\nu)$.*

Thus, we may define

$$D([\mu],[\nu]) := D(\mu,\nu)$$

Dimensional Distance

Theorem. *(Xia, 2007) D defines a metric on the equivalent classes of probability measures.*

In general, we have

$$d_{Haus}(spt(\mu) \leq D(\mu, \delta_0) \leq d_{box}(spt(\mu)).$$

Thus, when support of μ is nice enough, we get

$$\text{dimension of } spt(\mu) = \text{the distance } D(\mu, \delta_0).$$

As a result, I call D dimensional distance.

Conclusion: Dimension of a set/measure is just the distance from it to a Dirac mass.

Thank You and Enjoy the Nature