# The rotor-router model and Diaconis-Fulton addition

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# Diaconis-Fulton Addition

- Finite sets $A, B \subset \mathbb{Z}^d$.

- $A \cap B = \{x_1, \ldots, x_k\}$.

- To form $A+B$, let $C_0 = A \cup B$ and
$$C_j = C_{j-1} \cup \{y_j\}$$
where $y_j$ is the endpoint of a random walk started at $x_j$ and stopped on exiting $C_{j-1}$.

- Define $A+B = C_k$.

- Abeilan property: the law of $A+B$ does not depend on the ordering of $x_1, \ldots, x_k$.

# Internal DLA

- $A_1 = \{o\}$, $A_n = A_{n-1} + \{o\}$.

- Lawler, Bramson and Griffeath (1992) proved that the limiting shape is a ball.

- More precisely, for any $\varepsilon > 0$, with probability one we have
$$B_{r(1-\varepsilon)} \subset A_{\lfloor \omega_d r^d \rfloor} \subset B_{r(1+\varepsilon)}$$
for all sufficiently large $r$.

- Here $B_r = \{x \in \mathbb{Z}^d \ : \ |x| < r\}$, and $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$.

# Divisible Sandpile

- Start with mass $2$ on $A \cap B$, mass $1$ on $A \cup B - A \cap B$.
- Each site keeps mass $1$, sends excess mass equally to each neighbor.
- As $t \to \infty$, get a limiting region $A \oplus B$ of sites with mass $1$.
  - Sites in $\partial(A \oplus B)$ have fractional mass.
  - Sites outside have zero mass.

# Odometer Function

- $u(x) =$ total mass emitted from $x$.

- Discrete Laplacian:

$$\begin{aligned}
\Delta u(x) &= \frac{1}{2d} \sum_{y \sim x} u(y) - u(x) \\
&= \text{mass received} - \text{mass emitted} \\
&= \begin{cases} -1 & x \in A \cap B \\ 0 & x \in A \cup B - A \cap B \\ 1 & x \in A \oplus B - A \cup B. \end{cases}
\end{aligned}$$

# Least Superharmonic Majorant

- Let

$$\gamma(x) = -|x|^2 - \sum_{y \in A} g(x,y) - \sum_{y \in B} g(x,y),$$

  where $g$ is the Green's function for SRW in $\mathbb{Z}^d$, $d \geq 3$.

  - In dimension two, we use the negative of the potential kernel in place of $g$.

- $s(x) = \inf\{\phi(x) \mid \phi \text{ superharmonic}, \phi \geq \gamma\}$.

- Claim $u = s - \gamma$.

# Proof of the claim

- Let $m(x) =$ amount of mass present at $x$ in the final state. Then

$$\begin{aligned} \Delta u &= m - 1_A - 1_B \\ &\leq 1 - 1_A - 1_B. \end{aligned}$$

- Since

$$\Delta\gamma = 1_A + 1_B - 1$$

  the sum $u+\gamma$ is superharmonic, so $u+\gamma \geq s$.

- Reverse inequality: $s-\gamma-u$ is superharmonic on $A \oplus B$ and is $\geq 0$ outside $A \oplus B$, hence $\geq 0$ inside as well.

# Scaling Limit

- $A, B \subset \mathbb{R}^d$ bounded open sets such that $\partial A, \partial B$ have measure zero

- Let
$$D = A \cup B \cup \{s > \gamma\}$$
where
$$\gamma(x) = -|x|^2 - \int_A g(x,y)dy - \int_B g(x,y)dy$$
and $s$ is the least superharmonic majorant of $\gamma$.

- Odometer: $u = s - \gamma$.

# Scaling Limit for the Divisible Sanpile

- Write $A^{::} = A \cap \delta_n \mathbb{Z}^d$.

- **Theorem** (Levine-P.) Let $D_n = A^{::} \oplus B^{::}$, and let $u_n$ be the corresponding odometer function. Then

$$\delta_n^2 u_n \to u \text{ uniformly}$$

and for any $\varepsilon > 0$

$$D_\varepsilon^{::} \subset D_n \subset D^{\varepsilon::}$$

for all sufficiently large $n$, where $D_\varepsilon, D^\varepsilon$ are the inner and outer $\varepsilon$-neighborhoods of $D$.

# Scaling Limit for Diaconis-Fulton Addition

- **Theorem** (Levine-P.) Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A, \partial B$ having measure zero. For all $\varepsilon > 0$, with probability one

  $$D_\varepsilon^{::} \subset A^{::} + B^{::} \subset D^{\varepsilon ::}$$

  for all sufficiently large $n$, where

  - $D = A \cup B \cup \{s > \gamma\}$;
  - $\gamma(x) = -|x|^2 - \int_A g(x,y)dy - \int_B g(x,y)dy$;
  - $s$ is the least superharmonic majorant of $\gamma$;
  - $D_\varepsilon, D^\varepsilon$ are the inner and outer $\varepsilon$-neighborhoods of $D$;
  - $A^{::} = A \cap \delta_n \mathbb{Z}^d$.

# Lower Bound

- Inspired by the Lawler-Bramson-Griffeath argument for internal DLA.

- Let each random walk continue on after the particle has aggregated to the cluster.

- Fix $z \in D^{::}_{\varepsilon}$, and let

  – $M =$ number of walks that visit $z$ before exiting $D^{::}$.

  – $L =$ number of walks that visit $z$ before exiting $D^{::}$ but after aggregating to the cluster.

- $\mathbb{P}(z \notin A^{::} + B^{::}) = \mathbb{P}(L = M)$.

# Independent Indicators

- $\tilde{L} =$ number of walks that visit $z$ before exiting $D^{::}$ if one walk starts at each point $y \in (D - A \cup B)^{::}$.

- Since $\tilde{L} \geq L$ we have

$$\begin{aligned}\mathbb{P}(L = M) &\leq \mathbb{P}(\tilde{L} \geq M) \\ &\leq \mathbb{P}(\tilde{L} \geq a) + \mathbb{P}(M \leq a).\end{aligned}$$

- Strategy: show $\mathbb{E}\tilde{L} < \mathbb{E}M$ and use concentration of measure.

# Dirichlet Problem

- Let

$$
\begin{aligned}
f(z) &= \mathbb{E}M - \mathbb{E}L \\
&= \sum_{y \in (D - A \cup B)^{::}} g_n(y,z) - \sum_{y \in (A \cap B)^{::}} g_n(y,z),
\end{aligned}
$$

  where $g_n$ is the Green's function for SRW stopped on exiting $D^{::}$.

- Then

$$
\begin{aligned}
\Delta f &= 1 - 1_{A^{::}} - 1_{B^{::}}, && \text{on } D^{::} \\
f &= 0, && \text{on } \partial D^{::}.
\end{aligned}
$$

- The divisible sandpile odometer satisfies

$$
\begin{aligned}
\Delta u_n &= 1 - 1_{A^{::}} - 1_{B^{::}}, && \text{on } D_n \\
u_n &= 0, && \text{on } \partial D_n.
\end{aligned}
$$

# Concentration of Measure

- Using the fact that $D_n \to D$, $u_n \to u$, and the positivity of $u$, can show that

$$f > c_\varepsilon \delta_n^{-2} \qquad \text{on } D_\varepsilon^{::}.$$

- $\tilde{L}$ and $M$ are sums of $\delta_n^{-d}|D|$ independent indicators whose means are of order $\delta_n^{d-2}$. So $\mathbb{E}\tilde{L}$, $\mathbb{E}M$ are of order $\delta_n^{-2}$.

- Large deviations:

$$\mathbb{P}\,(|\tilde{L} - \mathbb{E}\tilde{L}| > \lambda\, \mathbb{E}\, \tilde{L}) < 2e^{-c_\lambda \mathbb{E}\tilde{L}};$$
$$\mathbb{P}\,(|M - \mathbb{E}M| > \lambda\, \mathbb{E}\, M) < 2e^{-c_\lambda \mathbb{E}M}.$$

# Finishing Up

- Conclude that $\mathbb{P}(\tilde{L} \geq M) < 4e^{-c'_\varepsilon \delta_n^{-2}}$.

- Summing over $z \in D_\varepsilon^{::}$ and over $n$, by Borel-Cantelli only finitely many of the events $\{z \notin A^{::} + B^{::}\}$ occur, a.s.

- Hence $D_\varepsilon^{::} \subset A^{::} + B^{::}$ for sufficiently large $n$.

# The Rotor-Router Model

- Deterministic analogue of random walk.

- Invented by Jim Propp.

- Each site $x \in \mathbb{Z}^2$ has a **rotor** pointing North, South, East or West.
  (Start all rotors pointing North, say.)

- A particle starts at the origin. At each site it comes to, it

  1. Turns the rotor clockwise by 90 degrees;

  2. Takes a step in direction of the rotor.

# Rotor-Router Aggregation

- Sequence of lattice regions

$$A_1 = \{o\}$$

$$A_n = A_{n-1} \cup \{x_n\},$$

  where

  - $x_n \in \mathbb{Z}^2$ is the site at which rotor walk first leaves the region $A_{n-1}$.

- Makes sense in $\mathbb{Z}^d$ for any $d$.

# Related Models

- In **internal DLA**, particles perform random walk instead of rotor-router walk.
  - Lawler, Bramson and Griffeath (1992) proved that the asymptotic shape is a ball.

- **Sandpiles**, or **chip-firing**: When 4 or more grains of sand accumulate at a site, it *topples*, sending one grain to each neighbor.
  - Limiting shape might not be a ball (?)

# The Abelian Property

- Choices of which sites to topple in what order don't affect the final sandpile shape.

- Choices of which particles to route in what order don't affect the final shape generated.

- **Equivalent models**:

  - Start with $n$ particles at the origin

  - If there are $m$ particles at a site, send $\lfloor m/4 \rfloor$ to each neighbor.

  - **Sandpile**: Leave the extra particles where they are.

  - **Rotor**: Send extra particles according to the usual rotor rule.

# Main Result

**Theorem** (Levine-P.) Let $A_n$ be the region of $n$ particles formed by rotor-router aggregation in $\mathbb{Z}^d$. Then

$$B_{r-c\log r} \subset A_n \subset B_{r(1+c'r^{-1/d}\log r)}$$

where

- $B_\rho$ is the ball of radius $\rho$ centered at the origin.
- $r = (n/\omega_d)^{1/d}$.
- $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$.
- $c, c'$ depend only on $d$.

**Corollary**: Inradius/Outradius $\to 1$ as $n \to \infty$.

# Divisible Sandpile

- Start with mass $s$ at the origin.

- Each site keeps mass $1$, sends excess mass equally to each neighbor.

- As $t \to \infty$, get a limiting region $A_s$ of mass $1$, fractional mass on $\partial A_s$, and zero outside.

- **Theorem**: There are constants $c$ and $c'$ depending only on $d$, such that

$$B_{r-c} \subset A_s \subset B_{r+c'}$$

where $r = (s/\omega_d)^{1/d}$.

# Odometer Function

- $f(x) =$ total mass emitted from $x$.
- Discrete Laplacian:

$$\begin{aligned} \Delta f(x) &= \frac{1}{2d}\sum_{y\sim x} f(y) - f(x) \\ &= \text{mass received} - \text{mass emitted} \\ &= 1 \end{aligned}$$

except at the origin and on the boundary.

# Theoretical Solution

- Boundary value problem:

$$\begin{aligned} &\Delta f = 1 && \text{on } A_s - \{o\} \\ &\Delta f(o) = 1 - s && \\ &f = 0 && \text{on } \partial A_s. \end{aligned}$$

- Idea: Compare $f$ to the function

$$\gamma(x) = |x|^2 - sa(x).$$

where $a$ is the potential kernel

$$a(x) = \lim_{n\to\infty} (G_n(o) - G_n(x))$$

and $G_n(x)$ is the expected number of visits to $x$ by SRW before time $n$.

- $a(x)$ is harmonic off $o$, and $\Delta a(o) = 1$.

- $\Delta |x|^2 = 1$

# Taylor expansion

- Standard estimate:
$$a(x) = \frac{2}{\pi}\log|x| + k + O(|x|^{-2})$$
gives
$$\gamma(x) = |x|^2 - \frac{2s}{\pi}\log|x| + ks + O(s|x|^{-2}).$$

- Get a constant $K = K(s)$ such that
  - If $r \leq |x| < r+1$, then $\gamma(x) = K + O(1)$.
  - $\gamma(x) \geq K + (r - |x|)^2 + O\left(\frac{r^2}{|x|^2}\right)$.

# Inner Radius

- $f - \gamma$ is superharmonic in $B_r$
- $f - \gamma \geq -K + O(1)$ on the boundary, hence on all of $B_r$.
- $\gamma$ grows quadratically as we move away from the boundary
- $\therefore f > 0$ on $B_{r-c}$.

# Outer Radius

- $f-\gamma$ is harmonic in $A_s$

- $f-\gamma \leq -K+O(1)$ on the boundary, hence on all of $A_s$.

- If $x \in A_s$ with $r \leq |x| < r+1$, then $f(x) \leq c'$.

- **Lemma:** If $y \in A_n - \{o\}$ there exists $z \sim y$ with $f(z) \geq f(y)+1$.

  – **Proof**. For some neighbor $z$,

$$f(z) \geq \frac{1}{4}\sum_{u\sim y} f(u) = f(y)+1.$$

- $\therefore A_s \subset B_{r+c'}$.

# Adapting the Proof

- Rotor-router odometer:

  $f(x) =$ total number of particles emitted from $x$.

- Instead of $\Delta f = 1$, we only know $-2 \leq \Delta f \leq 4$.

- Repeating the argument only gives

$$B_{cr} \subset A_n \subset B_{c'r}.$$

# Smoothing

- To do better, let

$$g(x) = \frac{1}{4k^2} \sum_{y \in S_k(x)} f(y)$$

where $S_k(x)$ is a box of side length $2k$ centered at $x$.

- Using $\Delta =$ div grad, we get

$$\Delta g(x) = \frac{1}{4k^2} \sum_{(y,z) \in \partial S_k(x)} \frac{f(z) - f(y)}{4}$$
$$= 1 + O\left(\frac{1}{k}\right)$$

if $o \notin S_k(x)$ and all sites in $S_k(x)$ are occupied.

# Fancier Smoothing

- Let $T$ be the first exit time of $B_r$, and

$$g(x) = \mathbb{E}_x f(X_T) - \mathbb{E}_x T + n\,\mathbb{E}\,\#\{j < T | X_j = o\}.$$

- Boundary value problem:

$$\begin{aligned} &\Delta g = 1 && \text{on } A_n \cap B_r - \{o\} \\ &\Delta g(o) = 1 - n && \\ &g = 0 && \text{on } \partial A_n. \end{aligned}$$

- Want to show $f \approx g$.

# Green's Function

- End up getting

$$f(x) \geq g(x) - \sum_{y \in B_r} \sum_{z \sim y} |G_{B_r}(x,y) - G_{B_r}(x,z)|.$$

- Error gets smaller as $x$ approaches the boundary, and we can show $B_{r - C \log r} \subset A_n$.

- But for the outer radius, the error is

$$\sum_{y \in A_n} \sum_{z \sim y} |G_{A_n}(x,y) - G_{A_n}(x,z)|.$$

- Can't control this, so we need another approach.

# Spreading Out

- Spherical shells

$$S_k = \{x \in \mathbb{Z}^d \ : \ k \le |x| < k+1\}.$$

- Lawler, Bramson, and Griffeath (1992): If $j < k$, $x \in S_j$, $y \in S_k$, then

$$\mathbb{P}_x(X_{T_k} = y) \le C/(k-j)^{d-1}.$$

- Want to show the same holds for rotor-router walk, with frequency replacing probability.

# Holroyd-Propp Bound

- recurrent graph $G$
- $Y \subset Z$ sets of vertices
- $s(x)$ particles start at $x$
- Stop walks when they hit $Z$; how many land in $Y$?
- Let $H(x) = \mathbb{P}_x(X_T \in Y)$. Then

  $$|RR(s,Y) - RW(s,Y)| \leq \sum_{u \in G} \sum_{v \sim u} |H(u) - H(v)|$$

  independent of $s$ and the initial rotor positions!

# Outer Radius

- $N_j = \#$ particles that ever reach shell $S_j$.

- If $r < j < k$ with $N_k > N_j/2$, then

$$\frac{CN_j}{(k-j)^{d-1}} \#(S_k \cap A_n) \geq \frac{N_j}{2}$$

hence

$$\sum_{i=j}^{k} \#(S_i \cap A_n) \geq C(k-j)^d.$$

- Since $B_{r-C\log r}$ is fully occupied,

$$k \leq j + C(r^{d-1}\log r)^{1/d}$$

which gives

$$A_n \subset B_{r(1+Cr^{-1/d}(\log r)^{1+1/d})}.$$

# What next?

- Improve the bound on the outer radius.
  - The truth: $R(n) = \max_{k \le n}(\text{outrad}(A_k) - \text{inrad}(A_k))$.

| $n$ | $R(n)$ |
|---|---|
| 10 | 0.822 |
| $10^2$ | 1.588 |
| $10^3$ | 1.637 |
| $10^4$ | 1.683 |
| $10^5$ | 1.724 |
| $10^6$ | 1.741 |

- Does the abelian sandpile have a limiting shape, and if so, is it a ball?

- Identify the limiting shape of the "broken rotor" models.