Gravitational allocation to Poisson points

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joint work with

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- An **allocation** (of Lebesgue measure to Ξ) is a measurable function $\psi : \mathbb{R}^d \to \Xi \cup \{\infty\}$ that satisfies

$$\begin{aligned} \mathrm{Vol}(\psi^{-1}(\infty)) &= 0, \\ \mathrm{Vol}(\psi^{-1}(z)) &= 1, \qquad z \in \Xi, \end{aligned}$$

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  where $\mathrm{Vol}(\,\cdot\,)$ is Lebesgue measure in $\mathbb{R}^d$.
- For $x \in \Xi$, we call $\psi^{-1}(z)$ the **cell allocated to** z.

Examples

(a) (b)

Figure: (a) The two-dimensional stable marriage allocation for a Poisson process (picture due to Alexander E. Holroyd). (b) The gradient flow allocation (picture due to Manjunath Krishnapur).

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- One object of interest: The rate of decay of the tail $\mathbb{P}(X > R)$.

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- Holroyd & Peres showed that if ψ_Z is an invariant allocation rule then $T = T_Z = \psi_Z(0)$ is a non-randomized extra head rule, and conversely, given a non-randomized extra head rule T_Z, the mapping $\psi_Z(x) = x + T_{Z-x}$ is an invariant allocation rule.

Allocation to Poisson points: Existing results

- Talagrand: Randomized allocations for $d \geq 3$. Holroyd & Peres used this to construct an allocation with $\mathbb{P}(\psi_Z(0) > R) \leq C\exp(-cR^d)$, where $\psi_Z(0)$ is the typical allocation distance.

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- In the stable marriage allocation, a.s.
  1. all the cells are bounded and contain their owners,
  2. but not all are connected,
  3. and when Z is a Poisson point process the allocation diameter X satisfies $\mathbb{E}X^d = \infty$.

Picture of a stable marriage allocation

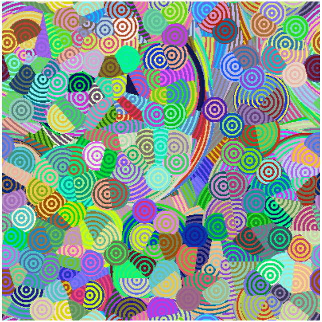

Figure: The 2-diml. stable marriage allocation for a Poisson process

Construction: Each star (point of the process) grows a ball at unit rate and captures all the sites it reaches first, until it is sated (has obtained volume 1).

Nazarov, Tsirelson, Sodin, Volberg

- Allocation to the zeros of the Gaussian Entire Function (GEF)

$$f(z) = \sum_{n=0}^{\infty} \xi_n \frac{z^n}{\sqrt{n!}}, \qquad z \in \mathbb{C},$$

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- Cell of each zero z is defined as the **basin of attraction** of z with respect to the flow induced by the random planar vector field $F(z) = z - \big(\nabla \log |f|\big)(z)$. (See next slide for a picture of the corresponding potential $U(z) = \log |f|(z) - \frac{1}{2}|z|^2$.)

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- Construction due to Nazarov, Sodin & Volberg based on an idea suggested by Tsirelson.

Picture of planar potential

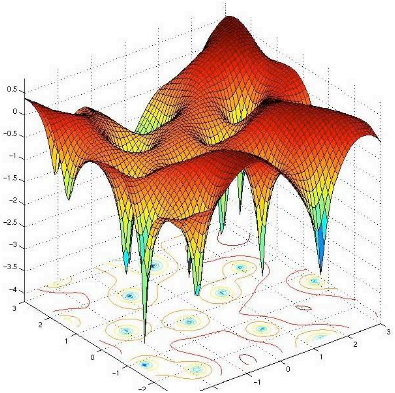

Figure: The random planar potential (courtesy of Manjunath Krishnapur).

Picture of a gradient flow allocation

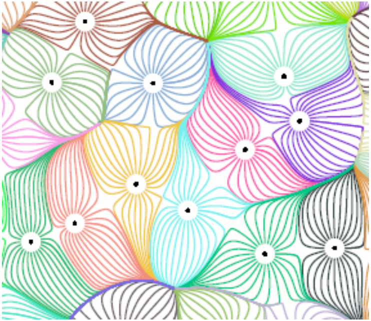

Figure: The gradient flow allocation (courtesy of Manjunath Krishnapur).

Nazarov, Sodin & Volberg showed that the cells are connected, a.s. bounded, and there exist absolute constants $C, c > 0$ such that the allocation diameter X satisfies

$$ce^{-CR(\log R)^{3/2}} \le \mathbb{P}(X \ge R) \le Ce^{-cR(\log R)^{3/2}}, \qquad R > 1.$$

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- Combining, we get $\mathrm{Vol}(B(z)) = \pi$.

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- In fact, a stronger version holds, where we condition on the allocation and choose x uniformly in one cell.

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- First investigated in work of S. Chandrasekhar. Later work by Heath & Shepp.

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- Thus, a.s. convergence of $F(0)$ follows from the Kolmogorov 3-series theorem. By stationarity, same is true for every $F(x)$.
- In our paper we have shown that a.s. the series converges everywhere on $\mathbb{R}^d \setminus \mathcal{Z}$ to give a translation-invariant continuously differentiable random function.

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- Thus, $F_1(0) + \cdots + F_n(0)$ has same law as $n^{(d-1)/d} F(0)$.
- It follows that $F(0)$ is $d/(d-1)$ stable.

A rearrangement identity

Lemma

For each $u, x \in \mathbb{R}^d$, *let*

$$G^{\{u\}}(x) = \sum_{|z_i - u| \uparrow} \frac{z_i - x}{|z_i - x|^d}.$$

Then for any $x, u, v \in \mathbb{R}^d$ *we have* $G^{\{u\}}(x) - G^{\{v\}}(x) = \kappa_d(u - v)$ *a.s., where* κ_d *is the volume of the unit ball in* $\mathbb{R}^d$.

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Proof:

- If $N_{u,x}$ is the number of stars in the ball $B(u, |u - x|)$, then

$$\mathbb{E}\left[G^{\{u\}}(x) \middle| N_{u,x} \right] = N_{u,x} \cdot \frac{u - x}{|u - x|^d}.$$

(Follows from a well-known physics principle.) Thus, $\mathbb{E}(G^{\{u\}}(x) - G^{\{v\}}(x)) = \kappa_d(u - v)$.

Contd. on next slide...

Rearrangement identity contd.

- Now, let $R > 0$ be large, and consider the truncated series

$$G_R^{\{u\}}(x) = \sum_{|z_i - u| < R} \frac{z_i - x}{|z_i - x|^d}.$$

We show that $\mathrm{Var}(G_R^{\{u\}}(x) - G_R^{\{v\}}(x)) \to 0$ as $R \to \infty$. This suffices to complete the proof of the rearrangement identity.

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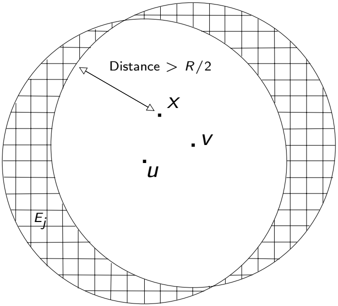
Rearrangement identity contd.



- Since $G_R^{\{u\}}(x) - G_R^{\{v\}}(x)$ is the sum of independent contributions from the E_j's, the variance can be easily bounded.

Gravitational allocation

- Consider now the integral curves $\Gamma(t)$ of the vector field F, that is, solutions of the equation

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- Define the **gravitational allocation rule**

$$\psi_{\mathcal{Z}}(x) = \begin{cases} z & x \in B(z) \text{ for } z \in \mathcal{Z}, \\ \infty & x \notin \bigcup_{z \in \mathcal{Z}} B(z). \end{cases}$$

Main result

Theorem

The mapping $\mathcal{Z} \to \psi_{\mathcal{Z}}$ is an allocation rule of Lebesgue measure to the Poisson point process $\mathcal{Z}$. Almost surely all the cells $\psi^{-1}(z)$ are bounded. The allocation diameter $X = diam(\psi^{-1}(\psi(0)))$ satisfies the following tail bounds: In dimensions 4 and higher, we have

$$\mathbb{P}(X > R) \leq C_1 \exp\left[-c_2 R (\log R)^{\frac{d-2}{d}} \right] \tag{2}$$

for some constants $C_1, c_2 > 0$ (depending on the dimension d) and all positive R. In dimension 3, for any $\alpha > 0$ there exist constants $C_1, c_2 > 0$ (depending on α) such that for all $R > 0$ we have

$$\mathbb{P}(X > R) \leq C_1 \exp\left[-c_2 \frac{R}{(\log R)^{\frac{4}{3}+\alpha}} \right]. \tag{3}$$

Picture of gravitational allocation

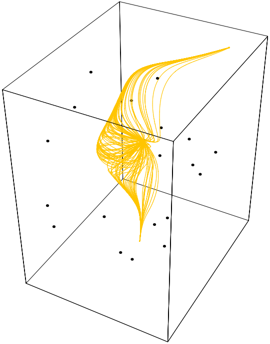

Figure: Simulation of a cell in 3-dimensional gravitational allocation

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- Let E_R denote the event that there exists an integral curve $\Gamma(t)$ connecting $\partial Q(0, R)$ and $\partial Q(0, 2R)$.
- Easy to see: If X is the diameter of the basin containing 0, then for all $R > 0$,
$$\mathbb{P}(X \geq cR) \leq \mathbb{P}(E_R)$$
for some constant c depending only on the dimension.

Sketch contd.

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Sketch contd.

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- Most important observation: Since $F(x) = -\nabla U(x)$, therefore

$$\begin{aligned}\frac{d}{dt}U(\Gamma(t)) &= <\dot{\Gamma}(t), \nabla U(\Gamma(t))> \\ &= <F(\Gamma(t)), -F(\Gamma(t))> = -|F(\Gamma(t))|^2.\end{aligned}$$

In particular, the potential always decreases along a gradient flow.

Sketch contd.

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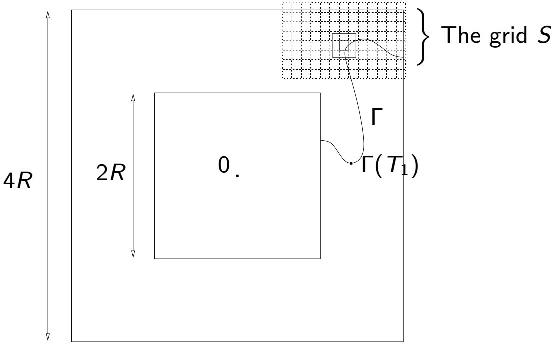
- If $U(x) \leq B$ for all $x \in Q(0, 2R)$ (a high probability event for a right choice of α), then E_R happens only if there is a sequence of R/r *connected* bad cubes.

Schematic illustration



Here T_1 is the time at which the curve Γ enters the phase $U(x) < -B$.

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- Step 2: Get an exponentially small bound on the joint probability of a collection of well-separated cubes being bad (**the percolation step**).
- Step 3: Bound $\mathbb{P}(E_R)$ by summing over all connected sequences of cubes.
- The percolation step turns out to be rather difficult, mainly due to the heavy-tailed nature of the force (unlike the Nazarov-Sodin-Volberg scenario, where the dependence decreases exponentially with distance).

Sketch of the percolation step

The percolation step states, in essence, that the probability of the potential difference being simultaneously small across a large number of well-separated squares is exponentially small in the number of squares. The key component in the proof is the following bound on the joint density of the forces at multiple points.

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The percolation step states, in essence, that the probability of the potential difference being simultaneously small across a large number of well-separated squares is exponentially small in the number of squares. The key component in the proof is the following bound on the joint density of the forces at multiple points.

Lemma

Suppose we have $x_1, \ldots, x_N \in \mathbb{R}^d$ with $|x_i - x_j| > S$ for every $i \neq j$. Fix $\lambda > 0$, and let

$$E = \Big\{ \textit{There is at least one star in } B(x_i, \lambda) \textit{ for every } 1 \leq i \leq N \Big\}.$$

Let $\mathcal{M}$ the σ-algebra generated by the stars in $\left(\cup_{i=1}^N B(x_i, S)\right)^c$. Then there exist constants $c_0, C_1 > 0$ such that if $\lambda < c_0 S(\log N)^{-1/d}$, then conditioned on the event E and on the σ-algebra $\mathcal{M}$, almost surely the joint density of $(F(x_i))_{1 \leq i \leq N}$ exists and is bounded from above by $(C_1 \lambda^{d^2 - d})^N$.

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- The bound is probably suboptimal, but suffices for our purposes.
- Even after this lemma, many other technical hurdles need to be overcome.
- Dimensions 3 and 4 are harder, mainly because the invariant potential is no longer well-defined, and the technicalities are also more difficult.