Smoothing Rocks by Chipping

P. L. Krapivsky and S.Redner

PRE **75**, 031119 (2007), cond-mat/0611415

NSF grants CHE0532969 (PLK) and DMR0535503 (SR).

Basic question:

inspired by Durian et al., PRL 97, 028001 (2006); PRE 75, 021301 (2007)

What is the shape of rocks as they erode?

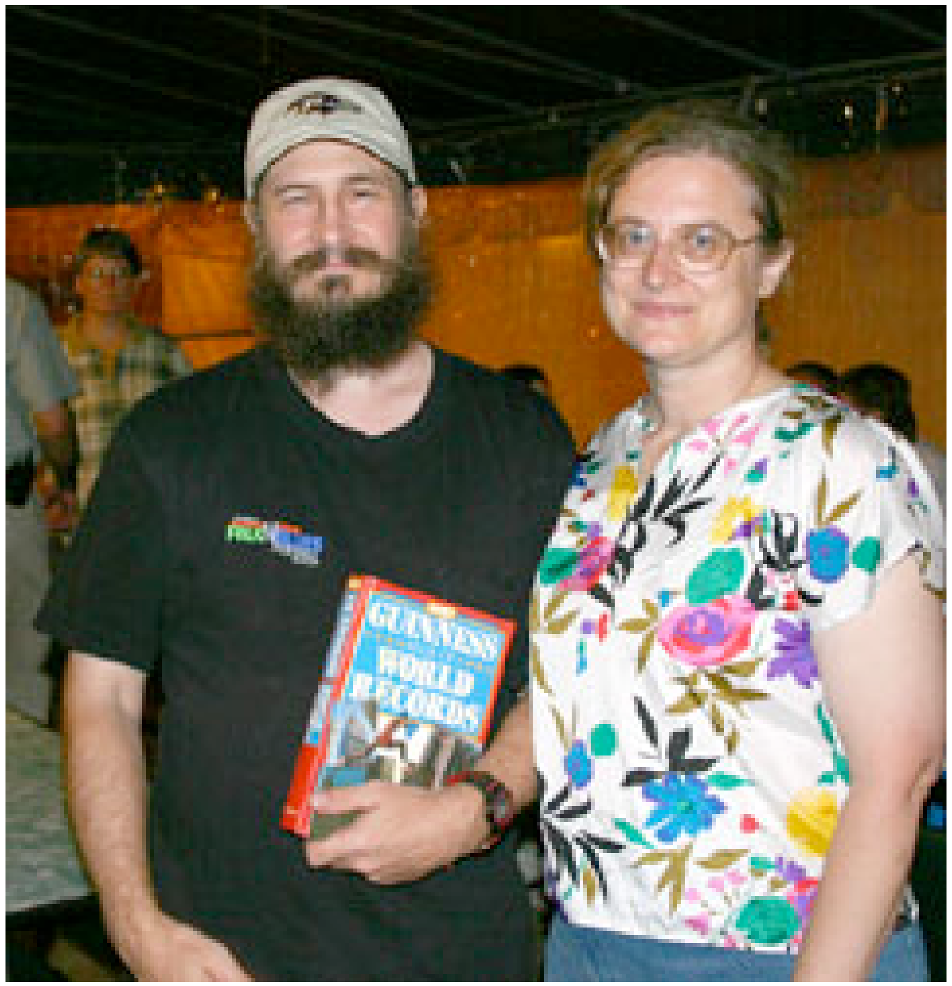
Aristotle:

Rounding by faster erosion at exposed corners.

Main result:

Final shape not round as found by Durian et al.

Motivation: Stone Skipping



Kurt Steiner of Emporium, PA

World record holder (a count of 40)

Warming up for the world-record skip

J. Fluid Mech. (2005), *vol.* 543, *pp.* 137–146.
doi:10.1017/S0022112005006373 Printed in the United Kingdom

Skipping stones

By LIONEL ROSELLINI[1], FABIEN HERSEN[1], CHRISTOPHE CLANET[1] AND LYDÉRIC BOCQUET[2]

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We first report a quantitative experimental study of the collision of a spinning disk with water, from a single to many skips. We then focus on the high spin limit and propose a simple model which enables us to discuss both the physical origin of the bounces and the source of the dissipation which fixes the number of skips.

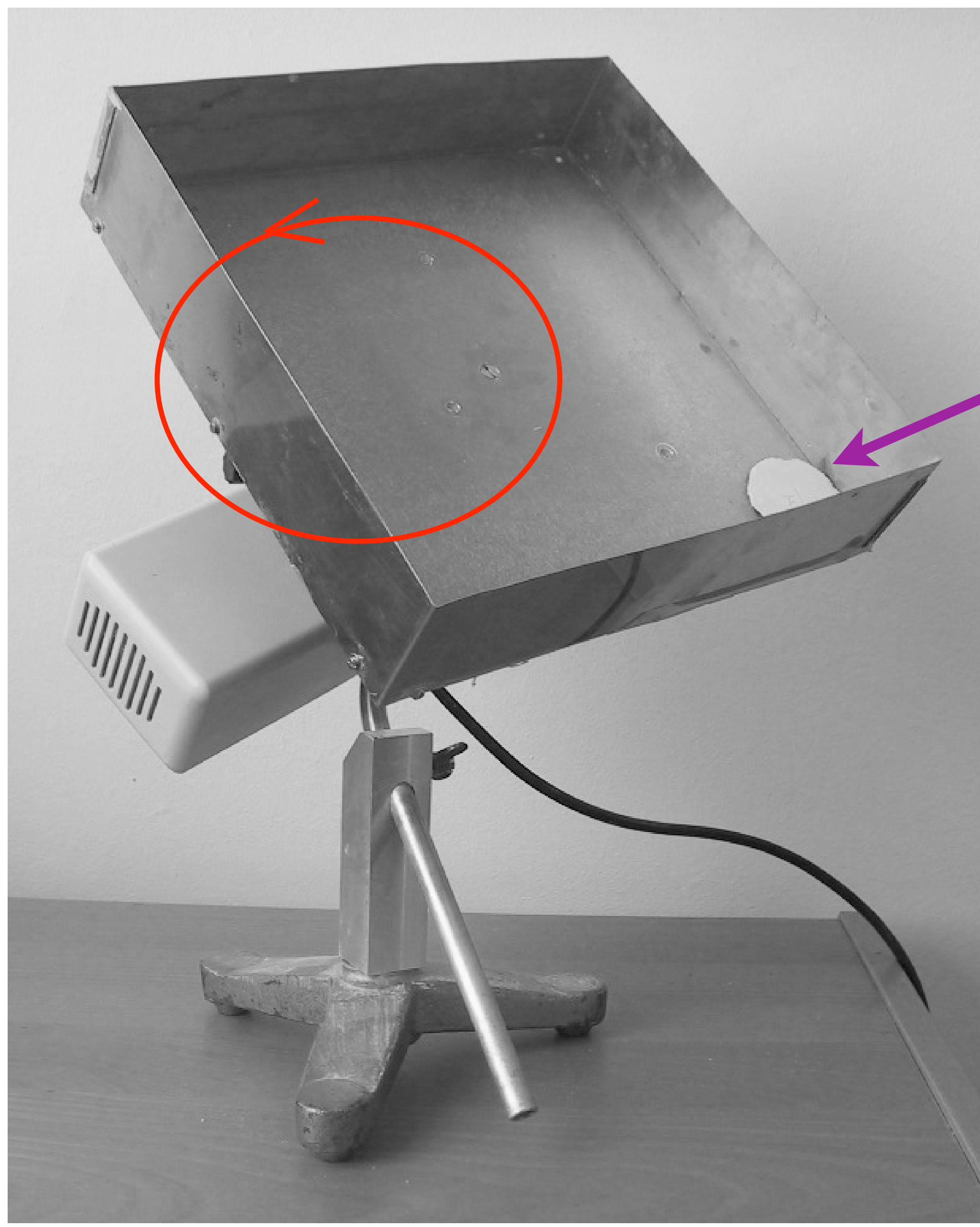
1. Introduction

“One, two, three, four”: this is the number of skips achieved by the stone in figure 1.

The rules of competition for skipping stones have never changed (Thomson 2000): a stone or a shell is thrown over a water surface and the maximum number of bounces distinguishes the winner. Part of the attraction of this game comes from the puzzling questions it raises: How can a stone bounce on water? How many skips can it achieve?

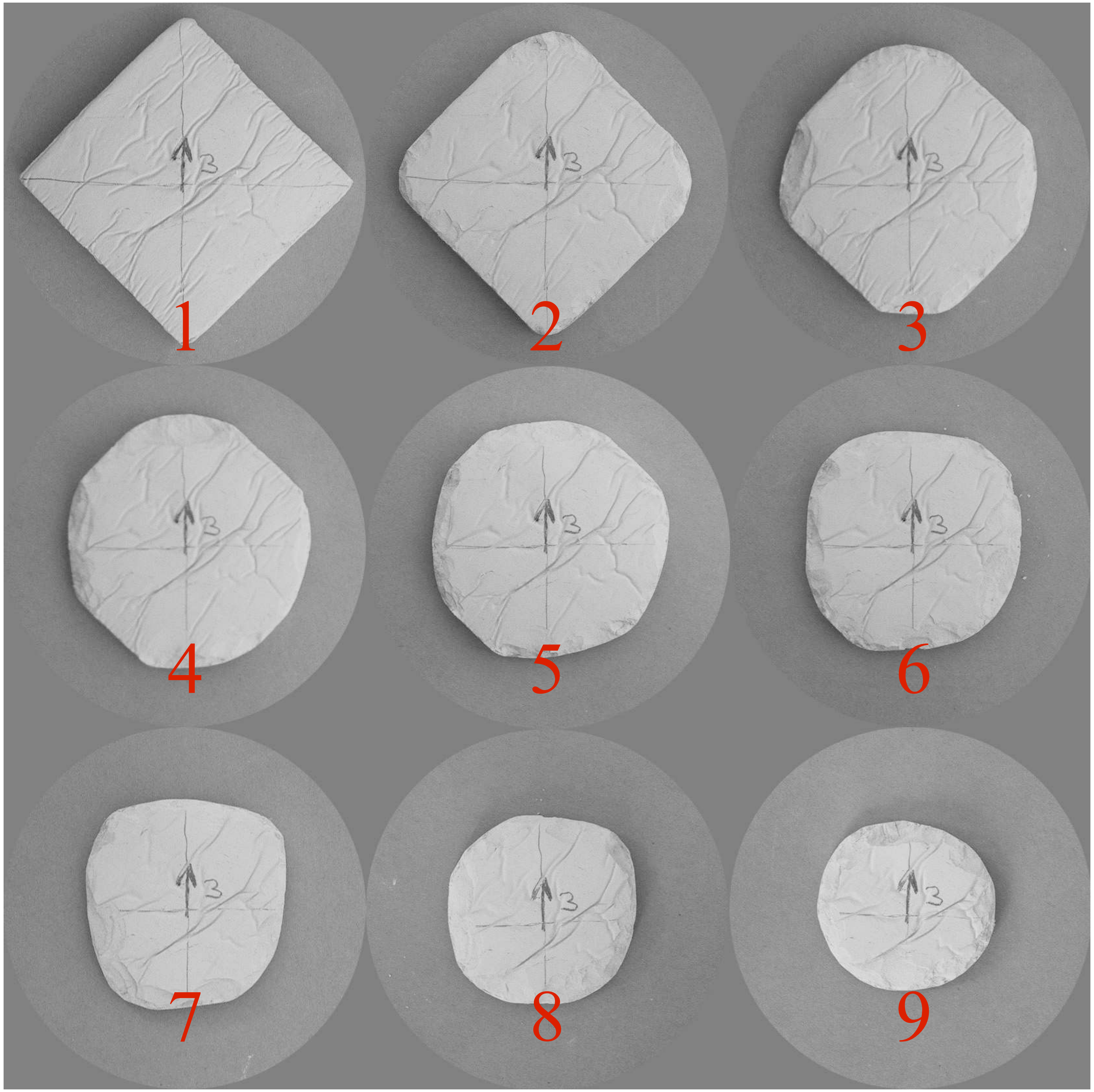
The impact of objects on water has been the object of a large amount of work in the literature (von Kármán 1930; Johnson & Reid 1975; Johnson 1998). Most of these works have focused (mainly due to military applications, e.g. Dambusters) on the impact of spherical and cylindrical objects, and clarified rebound conditions as a function of impact velocity. If R characterizes the size of the object, U its velocity and ρ, ν, σ the fluid properties (respectively density, kinematic viscosity and surface tension) all the above studies are in the limit of large Reynolds number ($Re \equiv UR/\nu \gg 1$) and large Weber number ($\rho U^2 R/\sigma \gg 1$) where inertial effects dominate both viscous and surface forces. Our study belongs to the same domain. However, even if the phenomena at play are similar in the case of stone skipping, the case of a flat (generally spinning) object like a stone is more difficult. In this latter case, a few theoretical analyses have attempted to extract the physical mechanisms (Stong 1968; Crane 1988; Bocquet 2003) and recently, three of us have published the first quantitative experimental results on the first bounce (Clanet, Hersen & Bocquet 2004).

Doug Durian’s Erosion Machine



Evolution of a Square Rock

Durian et al., PRL 97, 028001 (2006);
PRE 75, 021301 (2007)



What should we expect?

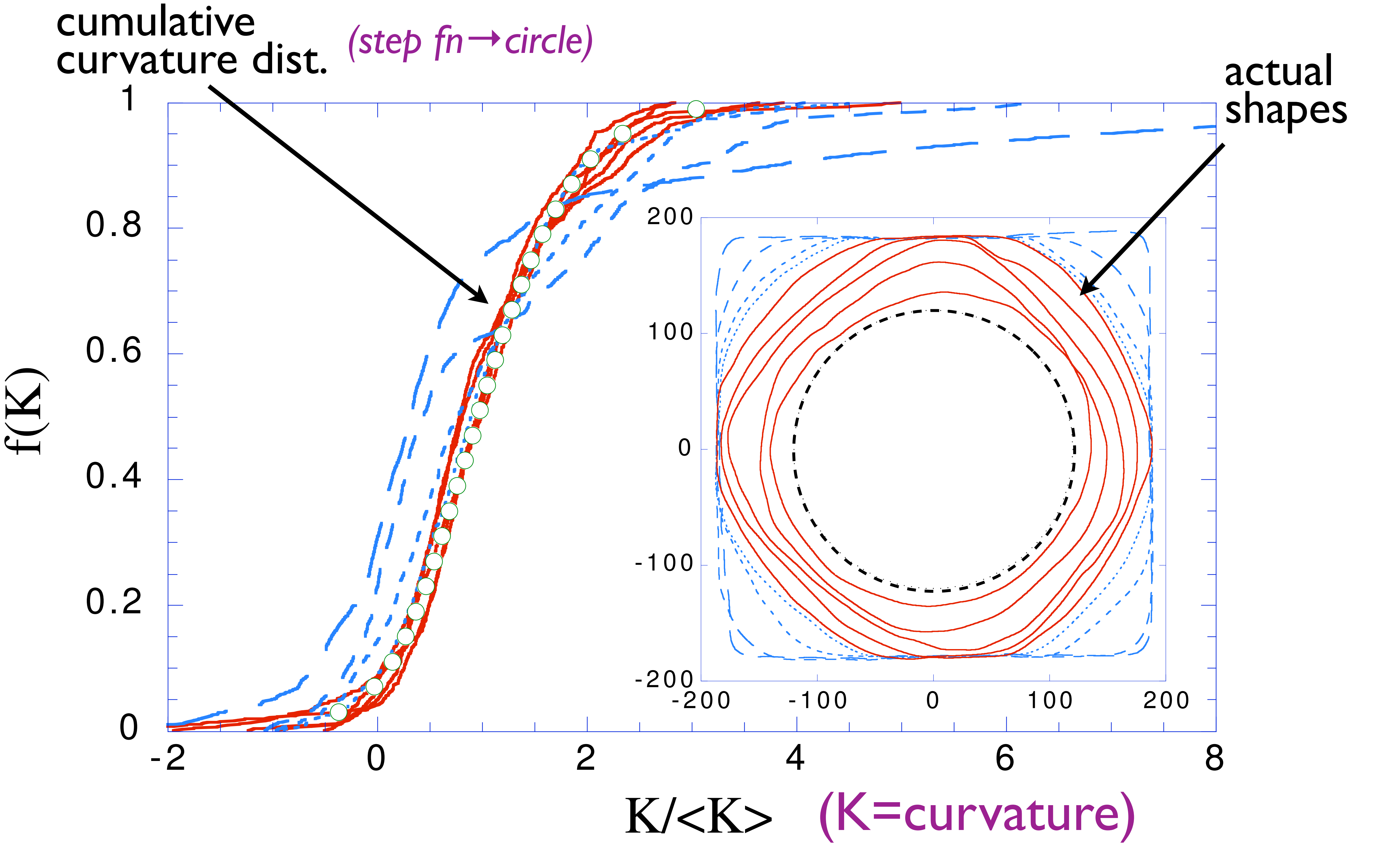
If $v_{\text{interface}} \propto \text{local curvature},$

$\rightarrow$ circular final shape for $d = 2$
(not true for $d > 2$).

Mullins (1956);
many differential geometry publications

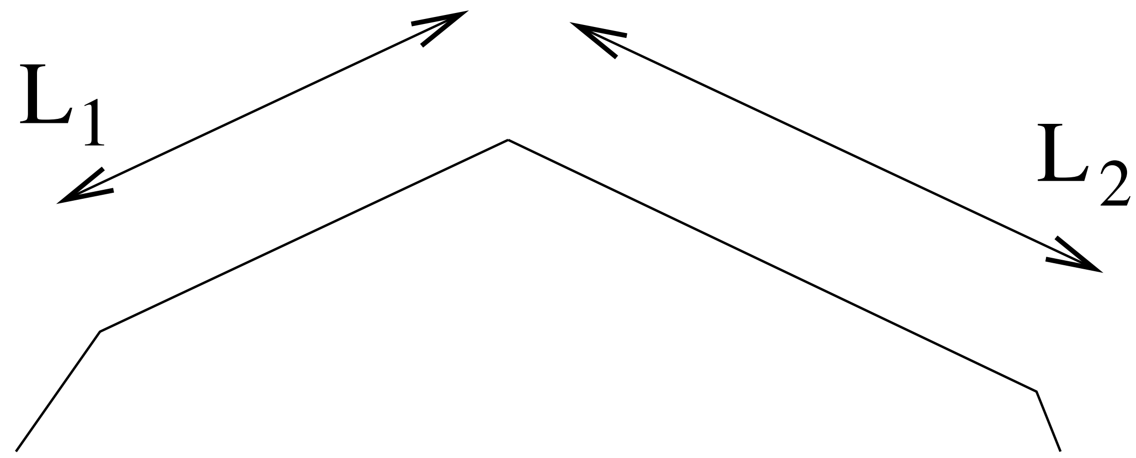
But... Final Shape is ***not*** Circular

Durian et al., Phys. Rev. Lett. 97, 028001 (2006);
Phys. Rev. E 75, 021301 (2007)



Chipping Model

geometry of single event



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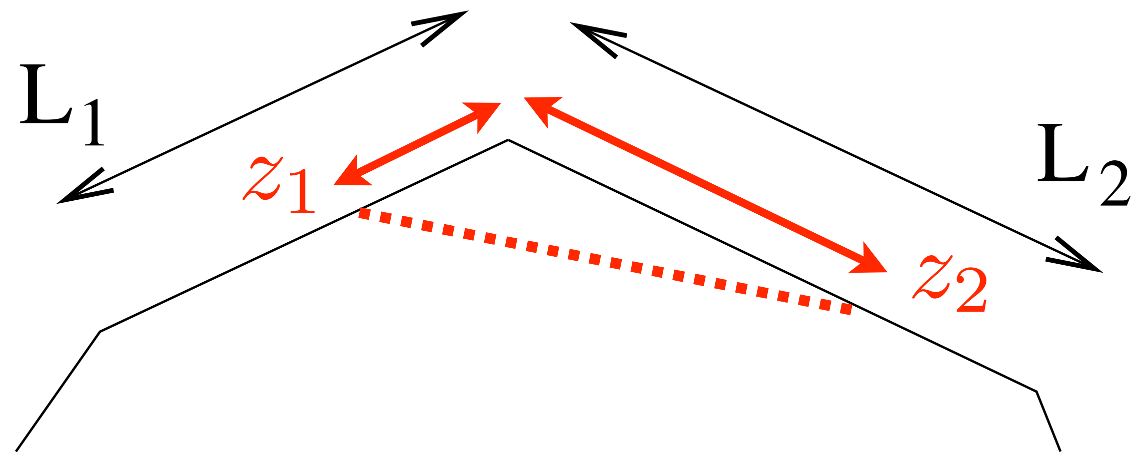

Chipping Model

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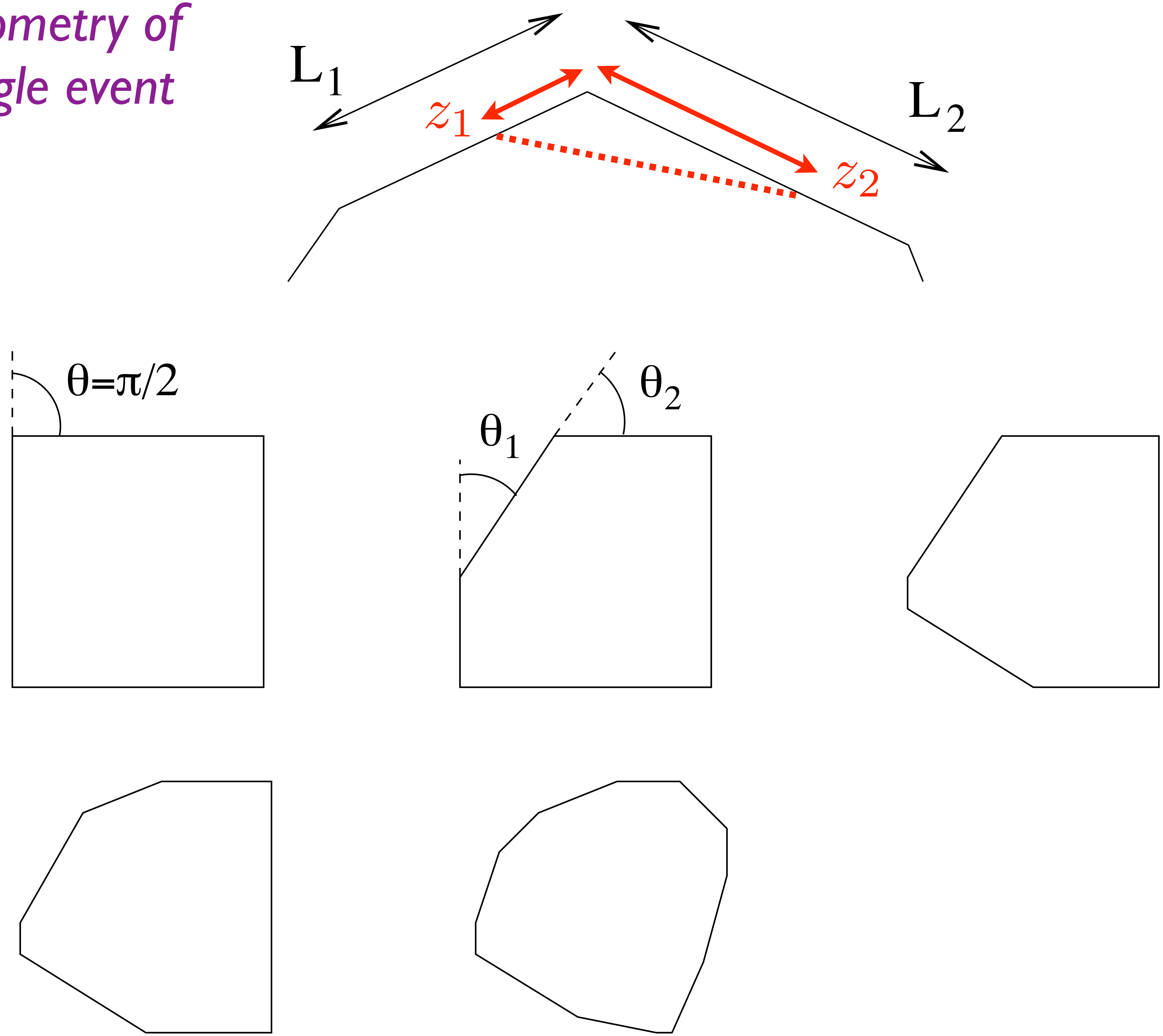
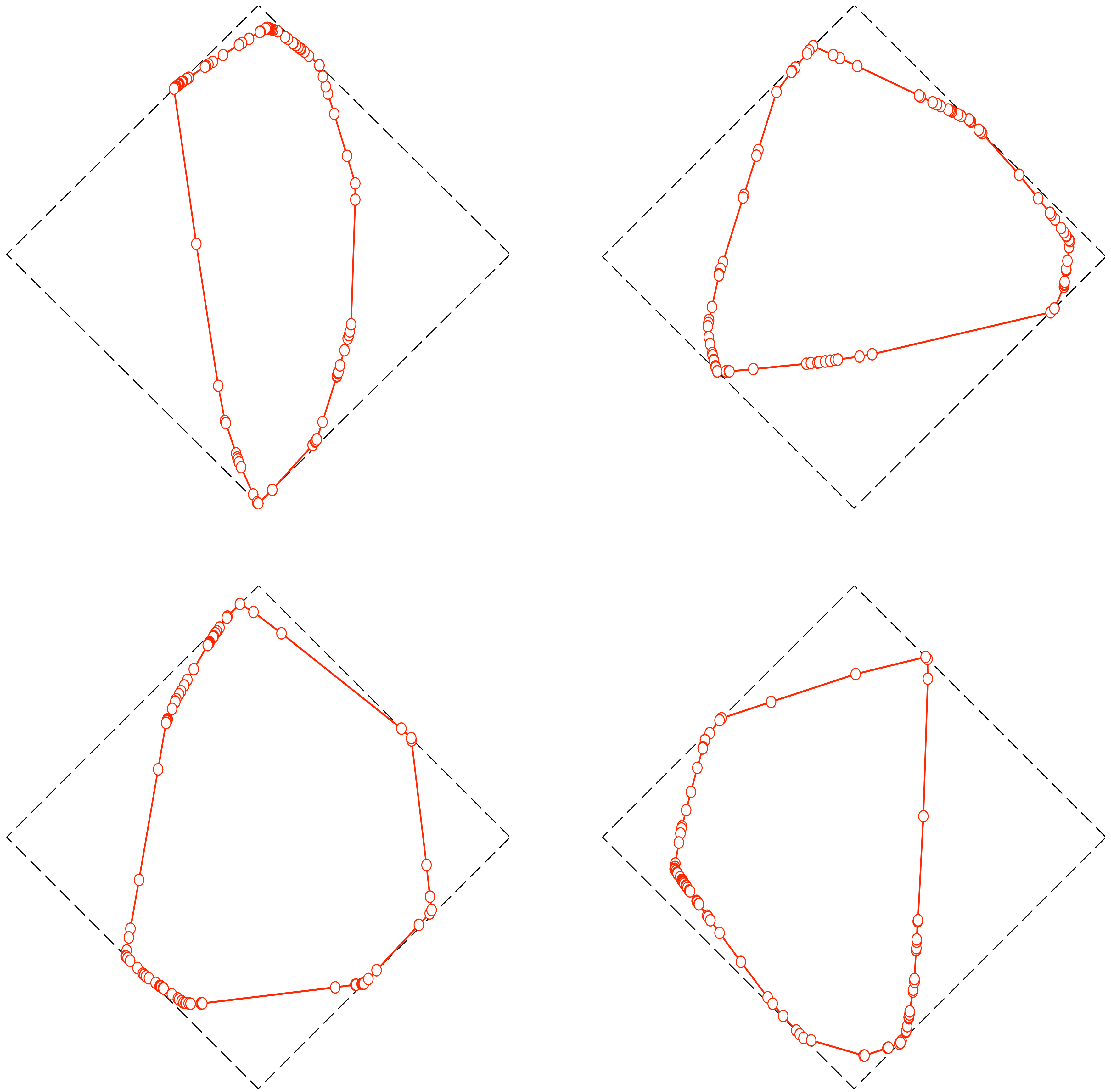


Numerical Realizations (100 corners)



Angle Evolution for Bisection

$n_k \equiv$ # corners with "angle" k

$$\begin{aligned} k &\equiv -\ln_2(2\theta/\pi) \\ &= \text{number of halvings} \end{aligned}$$

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Master equation: *(start with square; t+4 corners at time t)*

$$n_k(t+1) - n_k(t) = -\frac{1}{t+4}\, n_k(t) + \frac{2}{t+4}\, n_{k-1}(t)$$

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lose a k-corner

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Continuum limit:

$$\frac{dn_k}{dt} = -\frac{n_k}{t} + \frac{2}{t}\, n_{k-1}$$

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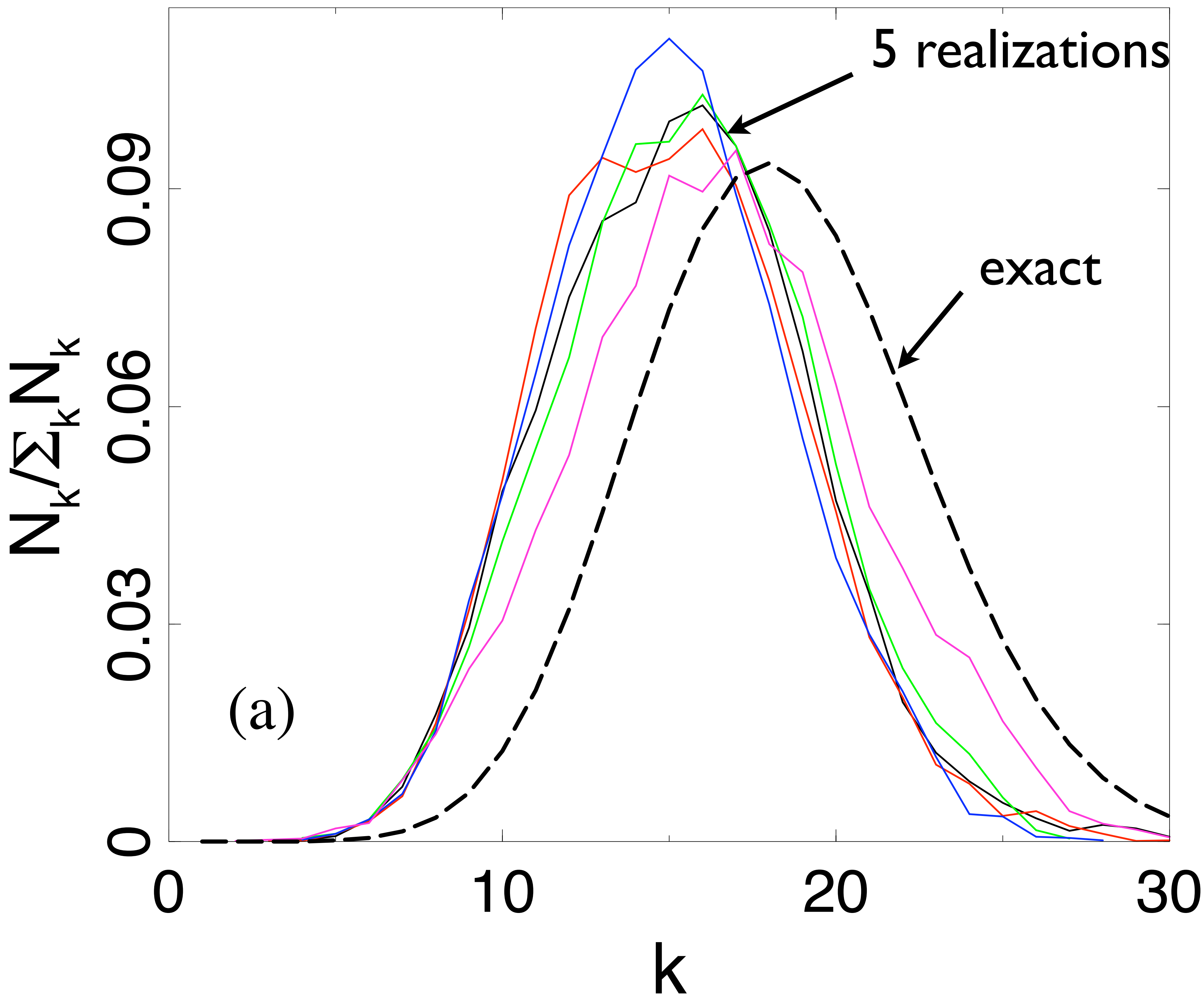
Continuum limit: $$\frac{dn_k}{dt} = -\frac{n_k}{t} + \frac{2}{t} n_{k-1}$$

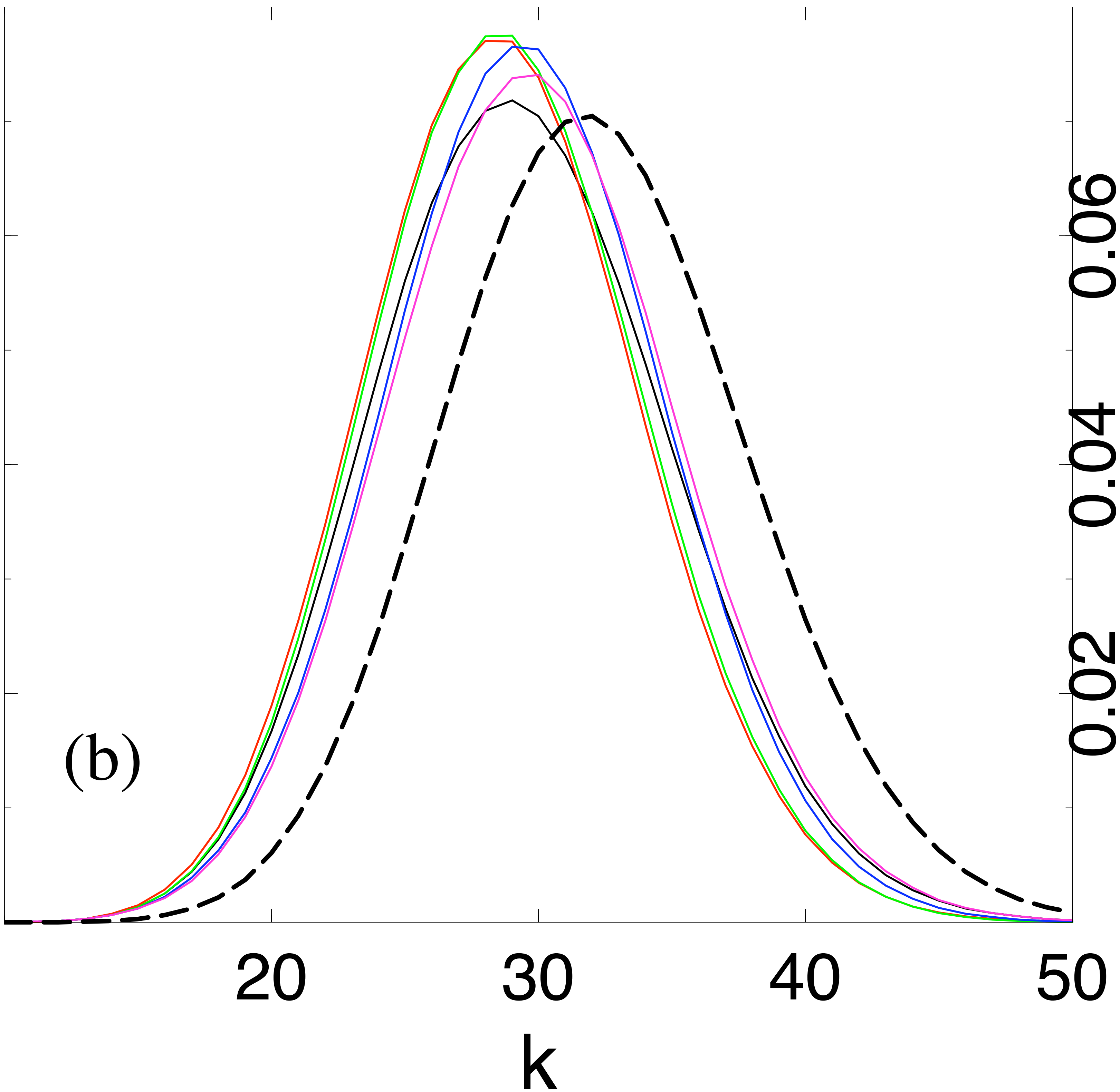
Result: $$n_k(t) = \frac{12}{t} \frac{(2\ln t)^k}{k!}$$

Angle Distribution for Bisection

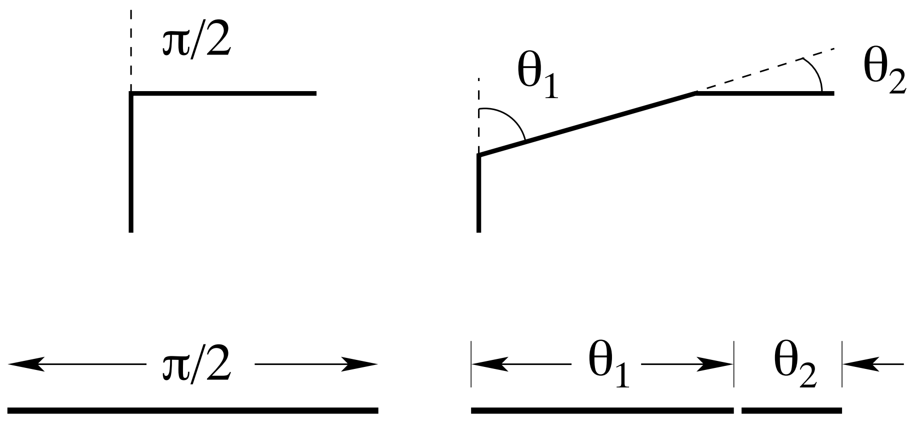
10^4 chipping events



10^7 chipping events

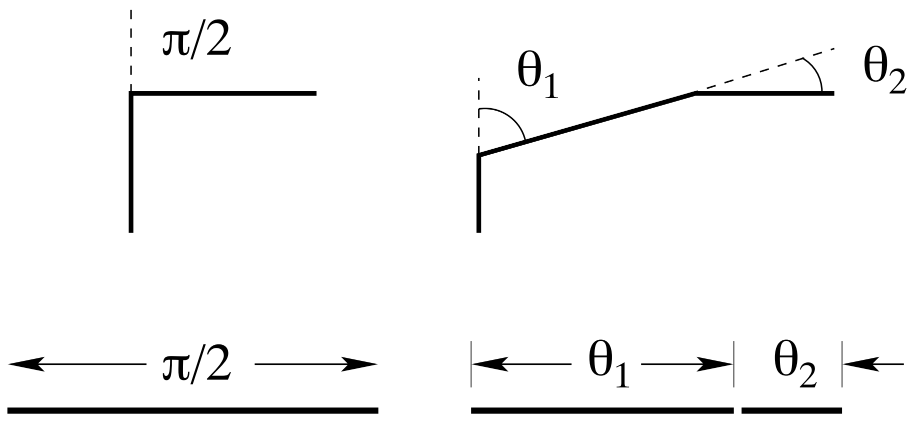


Angle Evolution for General Angles



correspondence with fragmenting a segment

Angle Evolution for General Angles



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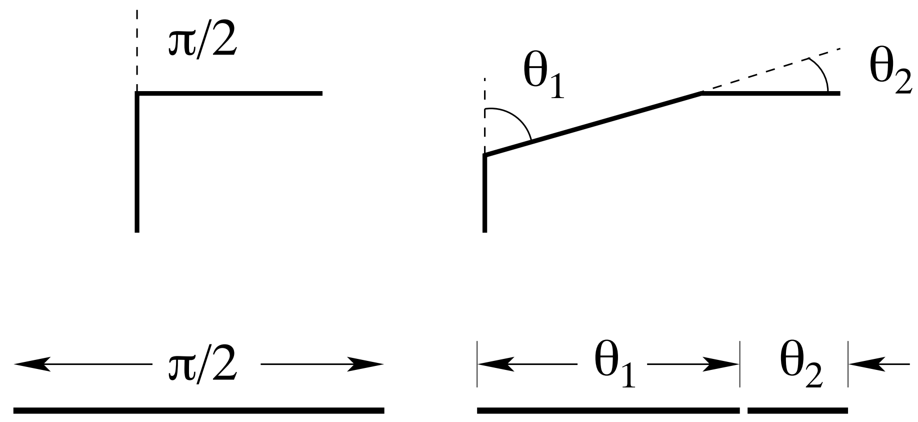
$$c(x,t) = \text{fraction of angles } x = \theta/2\pi$$

Angle Evolution for General Angles



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$$c(x,t) = \text{fraction of angles } x = \theta/2\pi$$

$$\frac{\partial c(x,t)}{\partial t} = -c(x,t) + 2\int_x^1 c(y,t)\,\frac{dy}{y}$$

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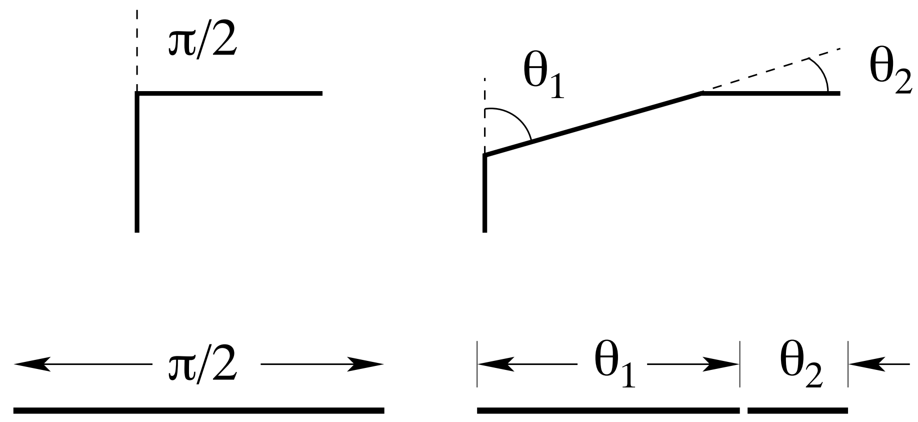

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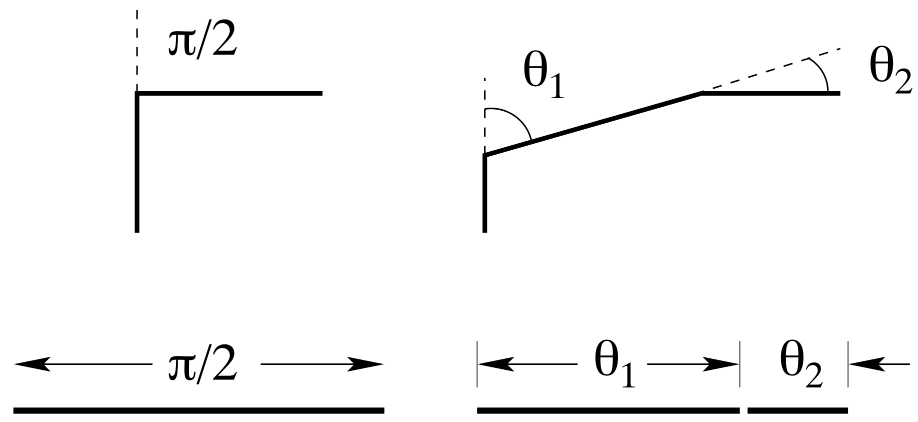

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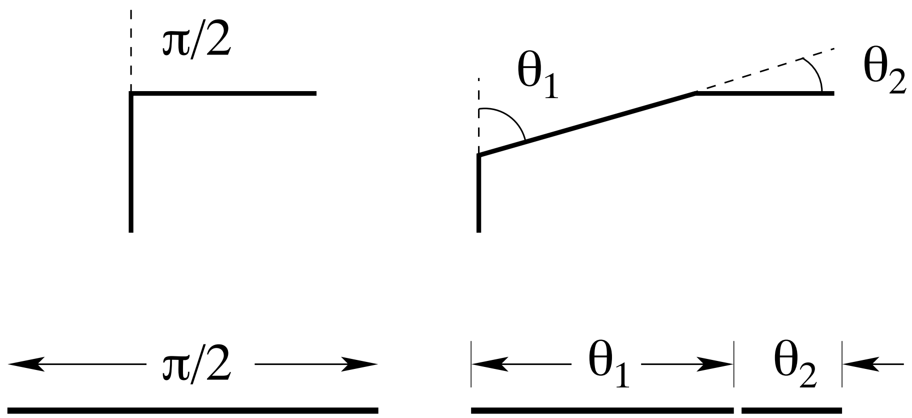
$$\begin{aligned} c(\theta,t) &= \frac{8}{\pi}\sqrt{\frac{2t}{\ln(\pi/2\theta)}}\,e^{-t}I_1\left(\sqrt{8t\ln(\pi/2\theta)}\right) + \frac{8}{\pi}e^{-t}\delta\Big(\theta-\frac{\pi}{2}\Big), \\ &\sim \quad e^{\sqrt{-t\ln\theta}} \end{aligned}$$

Ziff & McGrady (1985); Ziff (1992)

Angle Evolution for General Angles



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$$\sim \quad e^{\sqrt{-t\ln\theta}}$$

Ziff & McGrady (1985); Ziff (1992)

broad distribution of angles

Asymmetry

$$X^2(N) = \frac{1}{N}\sum_{i=1}^{N} x_i^2 \qquad Y^2(N) = \frac{1}{N}\sum_{i=1}^{N} y_i^2$$

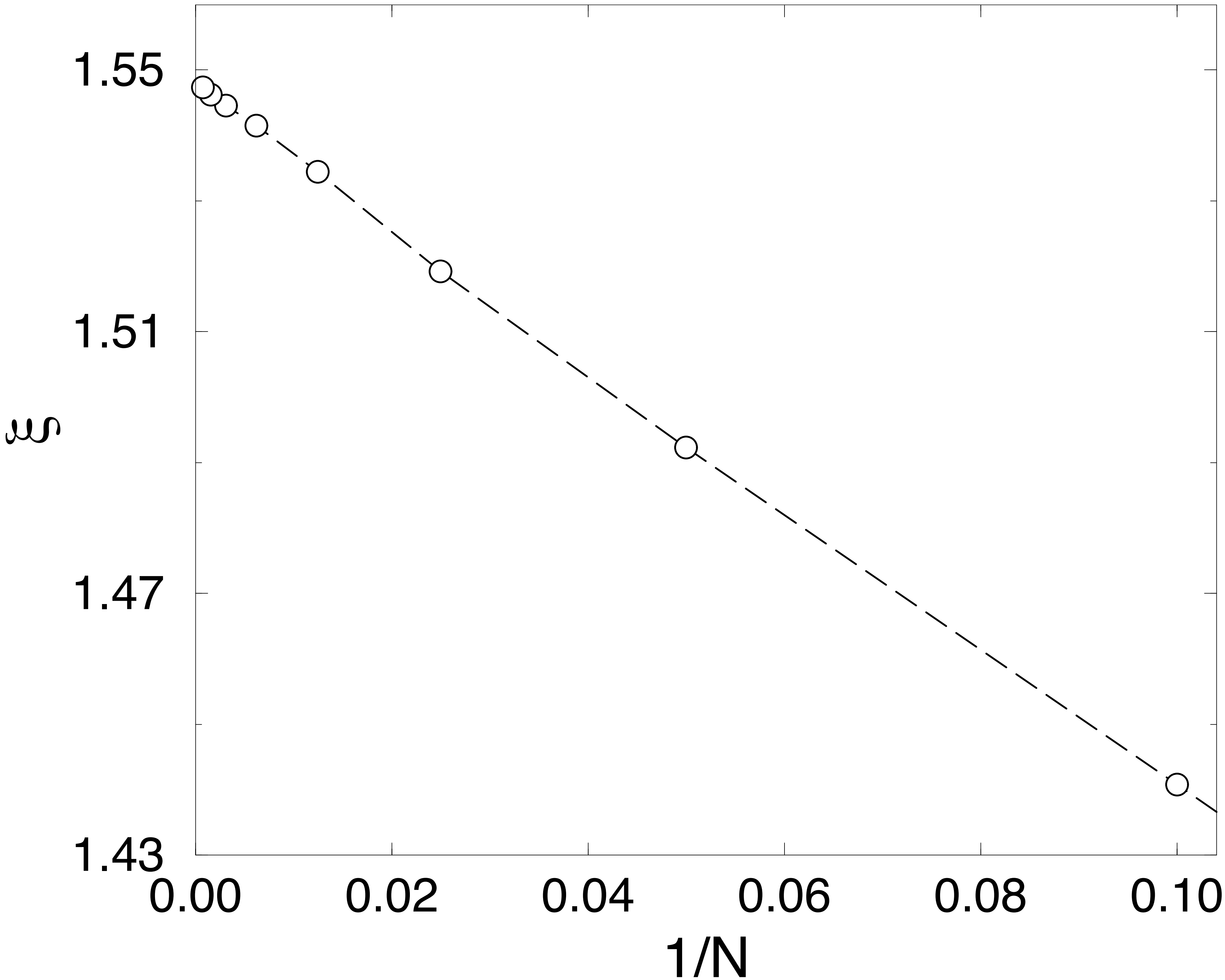
$$\begin{aligned} R_+^2(N) &= \max(X^2(N), Y^2(N)) \\ R_-^2(N) &= \min(X^2(N), Y^2(N)) \end{aligned}$$

for each realization

$$\xi(N) \equiv \sqrt{\langle R_+^2(N)\rangle} / \sqrt{\langle R_-^2(N)\rangle}$$

average over all realizations

Simulation Results



Summary

Eroding rocks are ***not*** round (in d=2)

Large fluctuations between realizations

Robust with respect to extensions

preferentially chip more prominent corners

chip away more than one corner

Smoothing of Interfaces at T=0

P. L. Krapivsky, S.Redner and J.Tailleur
PRE **69**, 026125 (2004)

Basic question:

How interfaces evolve at zero temperature?

Dynamics:

Glauber (mostly); time-dependent GL

Interfaces:

Corner (quadrant); Finger; Square

Dynamics

Below we consider (if not stated otherwise)

1. **Ising spins subject to the zero-temperature Glauber dynamics**
2. **Even coordination number (e.g. a hypercubic lattice)**
3. **Periodic boundary conditions**

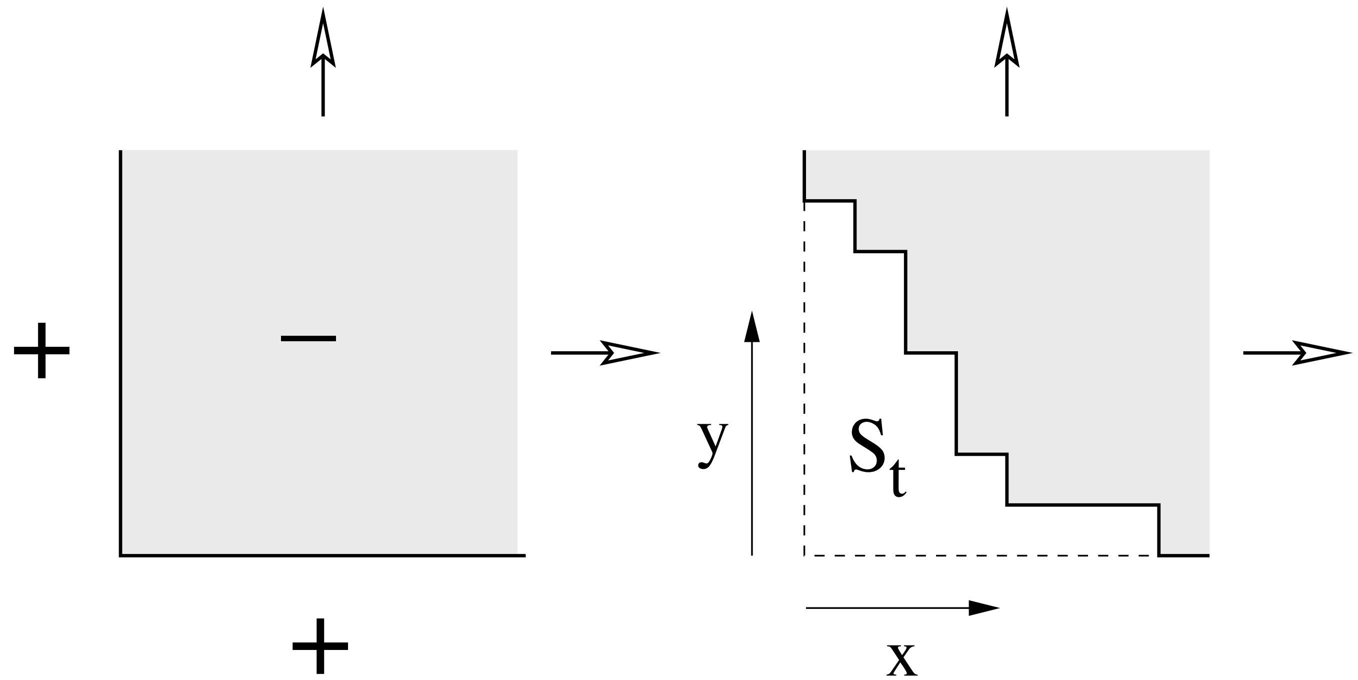
Definition of the zero-temperature Glauber dynamics

Pick up a random spin and compute the energy change ΔE if it were to flip:

1. **If $\Delta E < 0$ flip it**
2. **If $\Delta E > 0$ do not flip it**
3. **If $\Delta E < 0$ flip it with probability 1/2**

An Interface Between Ordered Phases

- **The straight interface is stable**
- **An evolving interface must have corners**
- **The simplest evolving interface is the wedge**



1. **There is always one more spin to flip up than down**
2. **Hence** $\langle S_t \rangle = t$ **and** $x, y \sim \sqrt{t}$
3. **The interface recedes diffusively**

Limiting Shape: Macroscopic Approach

1. **The interface becomes progressively less random as $t \to \infty$**
2. **After $(x, y) \to (x/\sqrt{t}, y/\sqrt{t})$, the interface approaches a deterministic limiting shape.**

We first analyze the limiting shape in the framework of the TDGL equation, or better its reduction to the interface dynamics (Allen and Cahn).

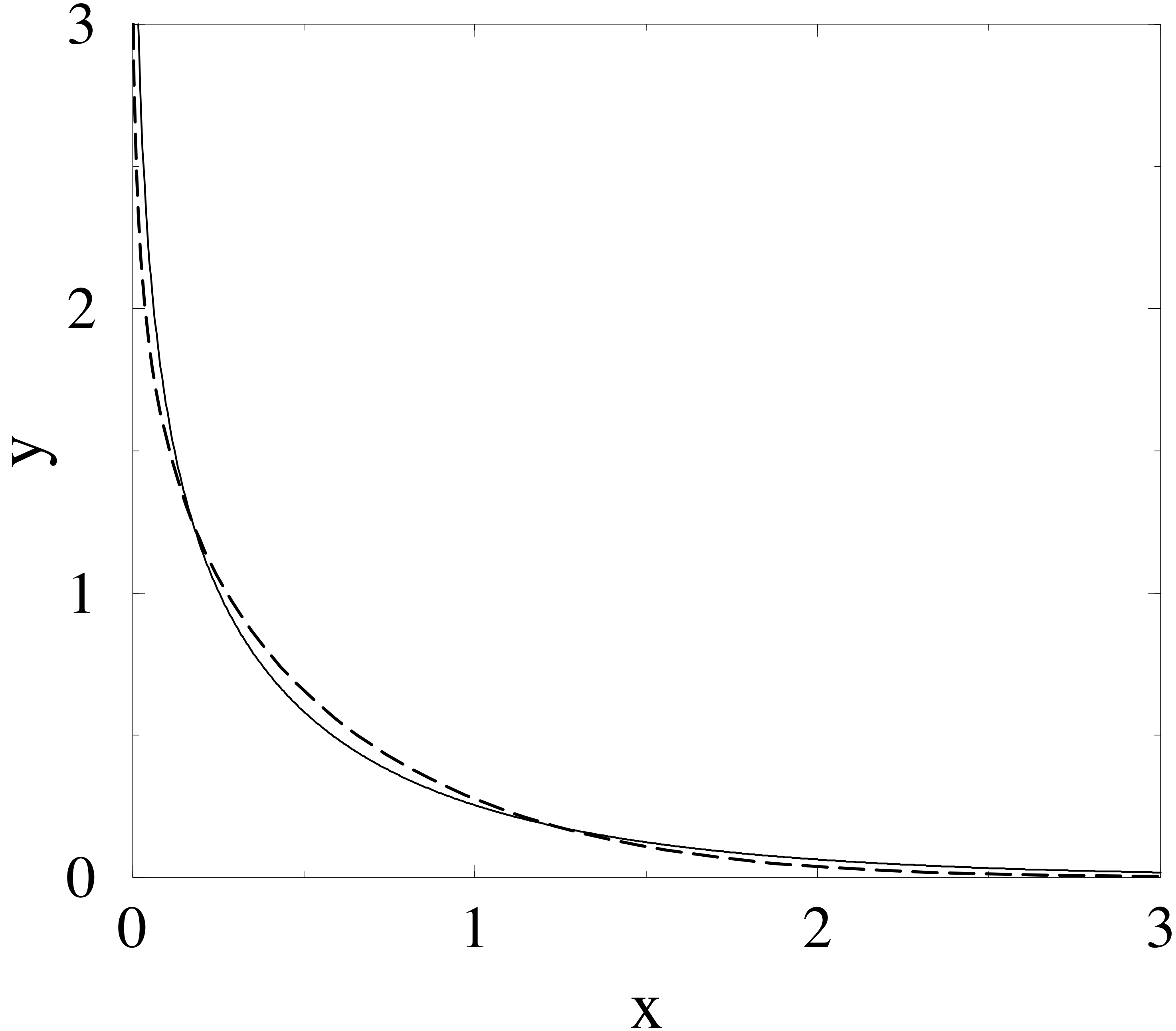
- **The AC asserts that the interface velocity is proportional to the local curvature**

$$y_t = \frac{y_{xx}}{1 + y_x^2}$$

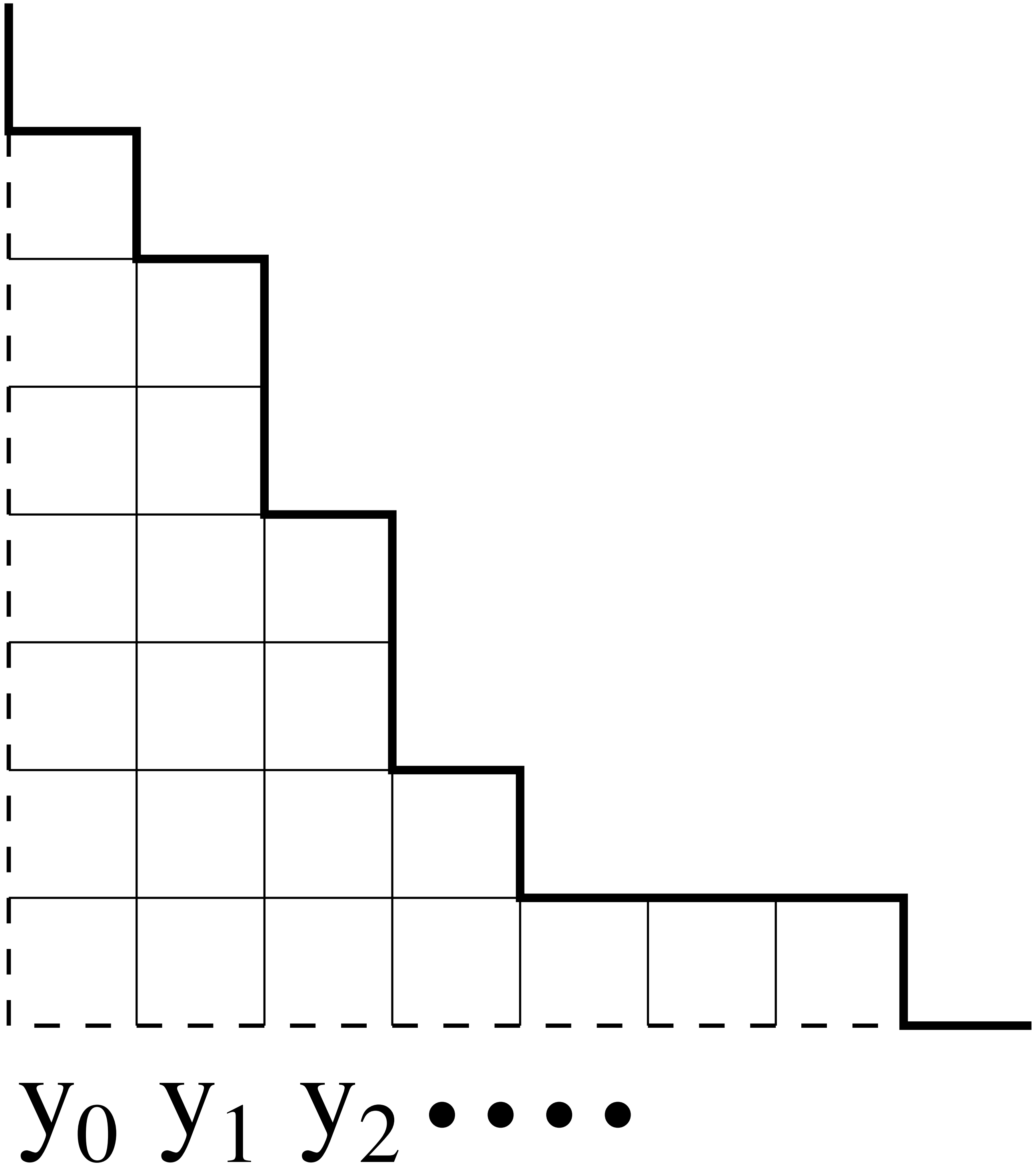
- **The solution must be self-similar on dimensional grounds**

$$y(x,t) = \sqrt{t}\, Y(X), \quad X = x/\sqrt{t}$$

- **The prediction of the TDGL equation (dashed) slightly differs from the exact limiting shape (solid)**



Partitions



This Young diagram is a representation of the partition of S_t. Above is the partitioning of 22 into $\{7, 6, 4, 2, 1, 1, 1\}$.

Limiting Shape: Microscopic Approach

1. **There are $p(t) \sim t^{-1}\exp(2\pi\sqrt{t/6})$ partitions of t (Hardy, Ramanujan)**
2. **Let each occurs with the same weight $1/p(t)$**
3. **Then the limiting shape (Temperley 52)**

$$e^{-\lambda x} + e^{-\lambda y} = 1, \qquad \lambda = \frac{\pi}{\sqrt{6t}}$$

This prediction is hardly distinguishable from the exact limiting shape

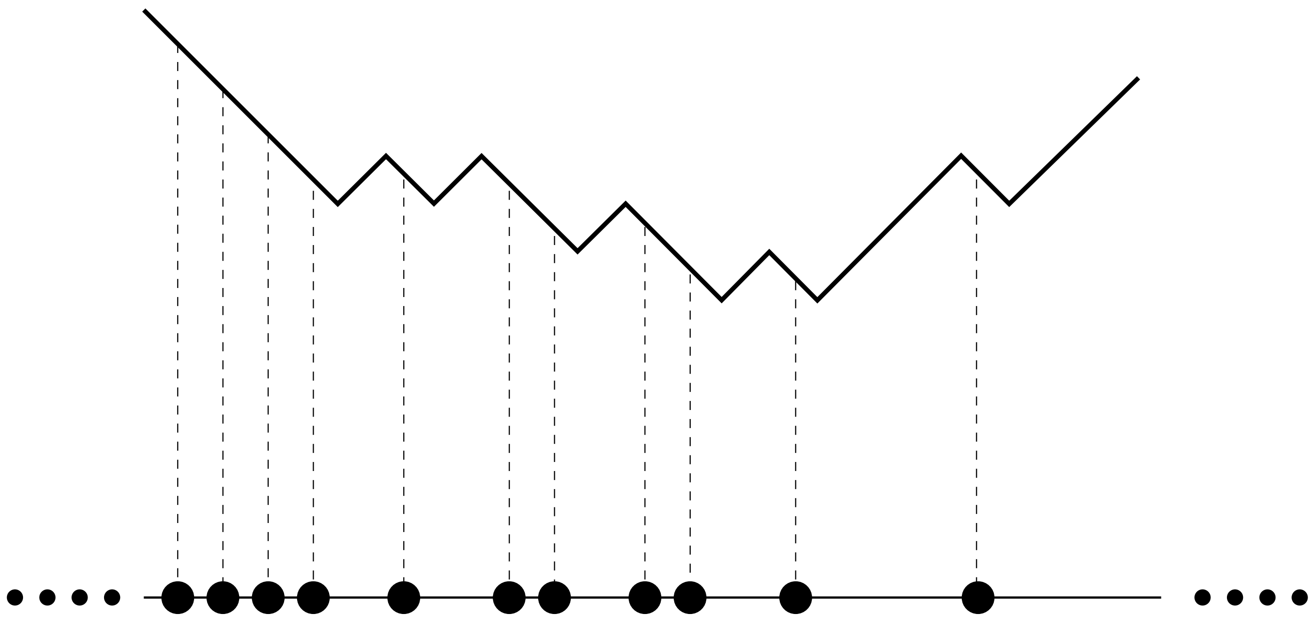
A few more predictions

- **A Langevin equation approach shows that**

$$\langle (S_t - t)^2 \rangle \propto t^{3/2}$$

- **The probability to return to the original (wedge) state is $e^{-\sqrt{t}}$**

Exclusion Processes



The interface dynamics is identical to the symmetric exclusion process (SEP)

1. The interface (rotated by 45°) and the corresponding particle configuration

2. The wedge corresponds to particles in $(-\infty, 0)$ and empty space for $(0, \infty)$

The asymmetric exclusion process (ASEP) is identical to the interface dynamics in the presence of a magnetic field.

Calculation

In the long time 'hydrodynamic' limit

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial z^2} \qquad n(z, t=0) = \begin{cases} 1 & z < 0 \\ 0 & z > 0 \end{cases}$$

The interface (x, y) is determined from

$$y(x,t) = \int_{x-y}^{\infty} dz \, n(z,t)$$

Calculation (magnetic field)

The hydrodynamic description is provided by ‘Burgers’

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial z} n(1-n) = 0$$

$$n(z,t) = \frac{1}{2} \times \begin{cases} 2 & z < -t \\ 1 - z/t & |z| < t \\ 0 & z > t \end{cases}$$

The interface is parabola ! (Rost, 81)

$$\sqrt{x} + \sqrt{y} = \sqrt{t}$$

Continuation (magnetic field)

The $x = y$ diagonal crosses the interface at

$$x = y = \frac{1}{4}t$$

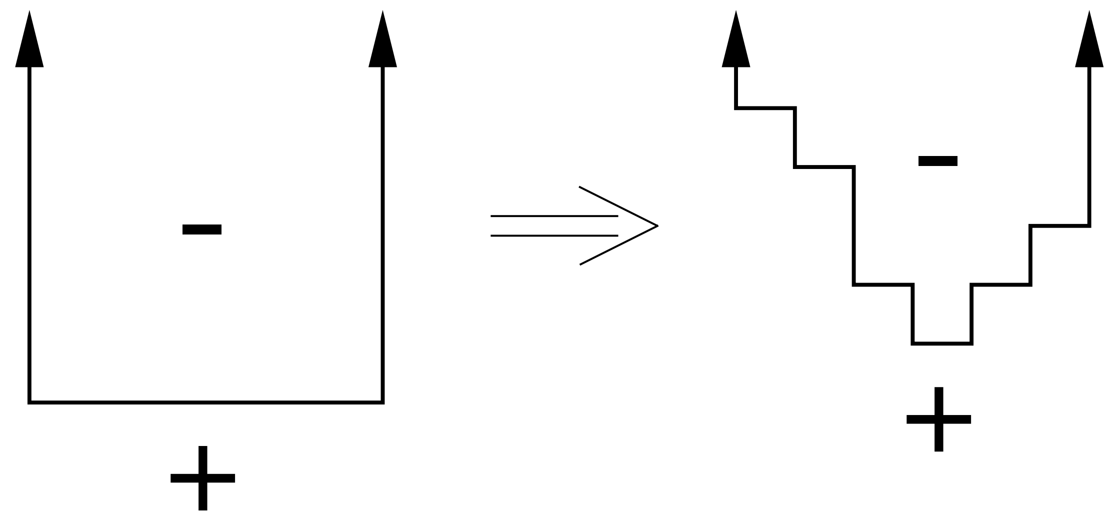
Why? The flux in ASEP $n(1-n)$ is $1/4$ at the origin.

More precisely: The number of particles N_t which have crossed the $(0,1)$ bond up to time t is

$$N_t = \frac{1}{4}t + t^{1/3}\nu$$

(Johansson, Spohn, et at)

Evolution of the Finger



1. **The flip of the lowest spin is an irreversible step that causes the height to advance by one**
2. **The finger recedes with velocity** $v \sim 1/L$
3. **In the co-moving reference frame, the shape of the finger becomes more and more deterministic as** $L \to \infty$
4. **In the AC framework,** $v = \pi/2L$ **and the limiting shape is (Mullins 56, Saffman-Taylor 58)**

$$y = -\frac{2L}{\pi} \ln \left[\cos \left(\frac{\pi x}{2L} \right) \right]$$

Evolution of the finger (exact)

Generally evolution may be addressed in two frameworks:

1. Relying on known surface tension in 2D.
2. Mapping SEP onto kink dynamics

(Karma & Lobkovsky, 2005)

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left[(1+\rho)^{-2}\frac{\partial \rho}{\partial x}\right]$$

Since $$y(x,t) = \int_0^x dx'\, \rho(x',t)$$

Evolution of the finger (exact)

$$y_{xt} = \left[\frac{y_{xx}}{(1+y_x)^2} \right]_x \qquad \text{gives} \qquad v = \frac{y_{xx}}{(1+y_x)^2}$$

Boundary conditions $y(0) = 0$ and $y(L) = \infty$

$$\frac{y}{L} = -\ln\left(1 - \frac{x}{L}\right) - \frac{x}{L}, \quad v = \frac{1}{L}$$

Influence of magnetic field

The governing equation

$$\frac{\partial \rho}{\partial t} = (1+\rho)^{-2}\frac{\partial \rho}{\partial x} + D\frac{\partial}{\partial x}\left[(1+\rho)^{-2}\frac{\partial \rho}{\partial x}\right]$$

In the stationary regime

$$\frac{1}{1+y_x} + D\left(\frac{1}{1+y_x}\right)_x = \mathrm{const}$$

The limiting shape is

$$y = D\left(e^{x/D} - 1 - \frac{x}{D}\right)$$

Evolution of the square

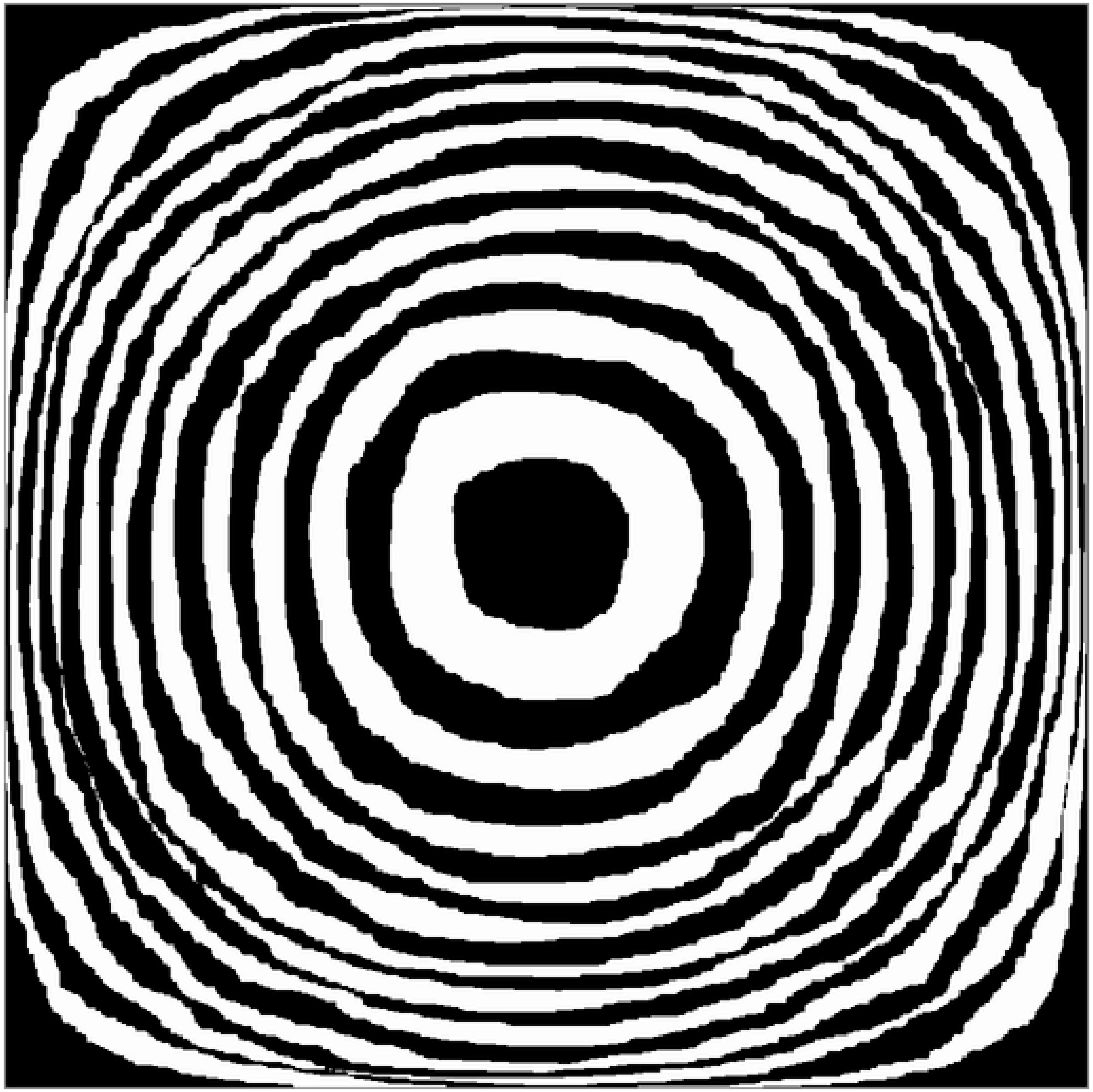
The square becomes circular under TDGL

Hamilton et al (86), Grayson (87)

The cube evolves into a ball under TDGL

Huisken (84)

When the dynamics is T=0 Glauber the square reaches a non-circular limiting shape



R.Cerf and S.Louhichi (2005)

Evolution of a square droplet

The limiting shape in 2D

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\left[(1+\rho)^{-2}\frac{\partial\rho}{\partial x}\right]$$

$$\rho(x,t) = \rho(\xi), \quad \xi = \frac{x}{L}$$

$\rho(0) = 0$ and L is the position where $\rho(L,t) = 1$

$$L\dot{L} = -B, \quad B = \frac{1}{8}\,\rho'(1)$$

$$B\xi\rho' = \left[\frac{\rho'}{(1+\rho)^2}\right]'$$

T=0 Glauber Dynamics

V.Spirin, P. L. Krapivsky, and S.Redner
PRE **63**, 036118 (2001), **65**, 016119 (2002),

Basic question:

Does the system reach a ground state?

Answer: Yes (one dimension)

Sometimes (two dimensions)

Never (three dimensions)

Expectations

Setup

- **Start with $T_i = \infty$ (disordered phase)**
- **Quench to $T_f \leq T_c$ (ordered phase)**

Central Dogma

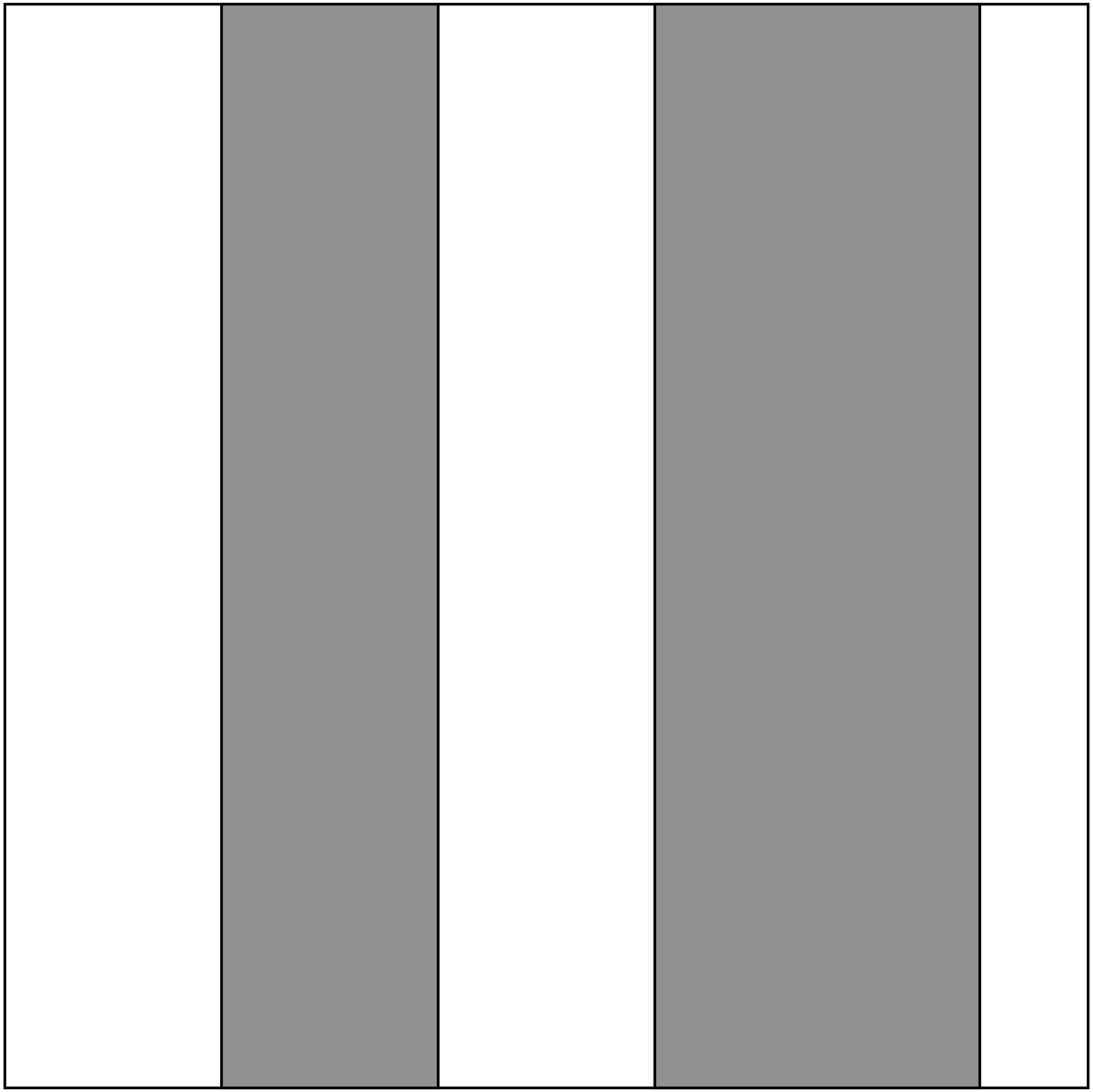
- **Two possible behaviors:**
  1. **All $T_f < T_c$ dynamics are isomorphic to zero-temperature dynamics**
  2. **Critical dynamics ($T = T_c$) is different**
- **Regions of ordered phase grow with time, $L(t) \sim t^z$,**

  $z = 1/2$ **(non-conserved dynamics)**
  $z = 1/3$ **(conserved dynamics)**
- **Ground state is reached in time $\propto N^{2/d}$ for non-conserved (e.g. Glauber or TDGL) dynamics**

Exceptional Behaviors ($T_f = 0$)

- **One dimension**
  1. $T_c = 0$ may cause an atypical behavior
  2. $L(t) \sim \ln t$ (hence $z = 0$) for the TDGL equation
  3. $z = 1/4$ for the XY model with Glauber dynamics
- **Freezing for conserved dynamics**
- **Freezing for odd-coordinated lattices**

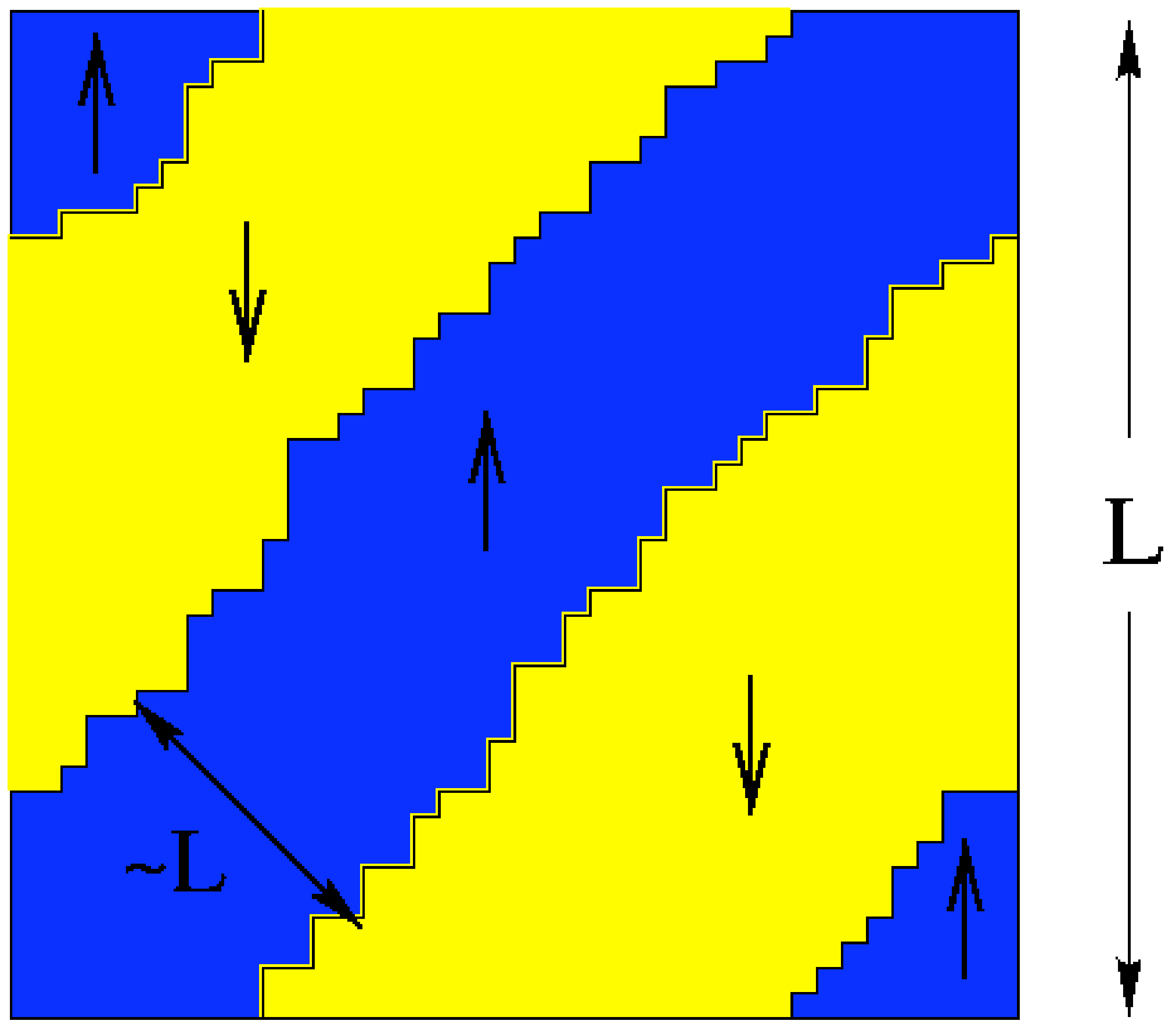
Stripes (two dimensions)



Evolution (two dimensions)

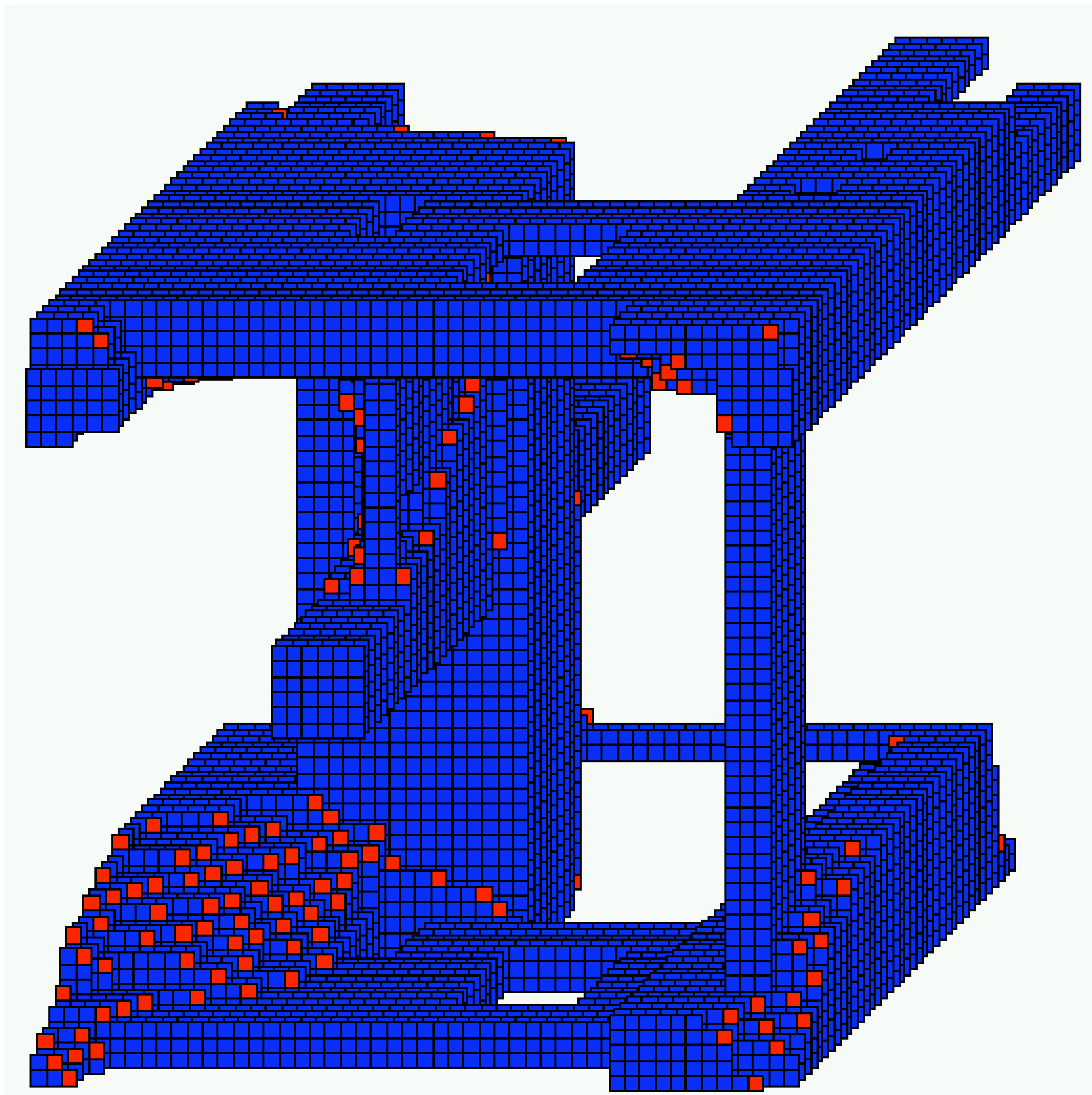
May be slow if the diagonal stripe is formed

Ground states are reached with prob $\approx 2/3$



Three dimensions

Ground state is never reached (in the thermodynamic limit). Blinkers.



Challenges

- **Prove that $P_+(m_0) = \mathrm{sgn}(m_0)$ for $m_0 \neq 0$ and $d \geq 2$**
- **Prove that $P_{\rm stripe} > 0$ in $2D$. Prove (or disprove) the asymptotic $P_k \propto e^{-k^2}$**
- **Count the number of metastable states for $d \geq 3$**
- **Work out the $d \to \infty$ limit**
- **Work out the case of small but positive temperature**
- **Find the limiting shapes in $2D$**
- **Find the limiting shapes (membranes) in three dimensions**
- **The AC dynamics turns a droplet into a shrinking disk, and an initially convex $3D$ droplet into a shrinking ball (Hamilton, Huisken, Grayson). What about Ising-Glauber?**

Thank you !