On the number of nodal domains of random spherical harmonics

Fedor Nazarov (East Lansing)
M.S. (Tel Aviv)

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Spherical harmonics: traces of homogeneous harmonic poly in $\mathbb{R}^3$ on the unit sphere $\mathbb{S}^2$

$\mathcal{H}_n$: linear space of spherical harmonics of degree n

- $\dim \mathcal{H}_n = 2n + 1$

- $f \in \mathcal{H}_n \iff \Delta^* f = -n(n+1)f$

Δ^*: spherical part of Laplacian in $\mathbb{R}^3$

- $\mathcal{H}_n$ is a Hilbert space with scalar product inherited from $L^2(\mathbb{S}^2)$

Notation: $\| \,.\, \| = \| \,.\, \|_{L^2(\mathbb{S}^2)}$

$Z(f) = \{x \in \mathbb{S}^2 \colon f(x) = 0\}$ zero set of f

Nodal domain connected comp-t of $\mathbb{S}^2 \setminus Z(f)$

$N(f) = \#\{\text{connected comp-ts of } Z(f)\}$
"topological complexity" of $Z(f)$

Courant nodal domains theorem:
for all $f \in \mathcal{H}_n$, $N(f) \le (n+1)^2$

H. Lewy: there exists $f \in \mathcal{H}_n$ s.t. $N(f) \le 3$

Q: What is the "typical" behaviour of $N(f)$?

Gaussian spherical harmonic: $f = \sum_{k=-n}^{n} \xi_k Y_k$

$\{\xi_k\}$ indep Gaussian r.v., $\mathbb{E}\xi_k^2 = \frac{1}{2n+1}$
$\{Y_k\}$ orthonormal basis in $\mathcal{H}_n$

- f doesn't depend on the choice of the o.n.b. $\{Y_k\}$ in $\mathcal{H}_n$

- the distribution of f is rotation invariant

Q: whether $\mathbb{E}N(f) \ge cn^2$? or
$\mathbb{E}N(f) = o(n^2)$ as $n \to \infty$?

Bogomolny and Schmit (2002):

elegant percolation-like model
that ignores correlations
but agrees well with numerics

It suggests that $\mathbb{E}N(f)/n^2 \approx 0.06$ and that the variance of $N(f)$ grows with n like n^2.

THEOREM (Nazarov-S.) There exists a constant $a > 0$ such that, for every $\epsilon > 0$, we have

$$\mathbb{P}\Big\{\Big|\frac{N(f)}{n^2} - a\Big| > \epsilon\Big\} \le C(\epsilon)e^{-c(\epsilon)n}$$

where $c(\epsilon)$ and $C(\epsilon)$ are positive constants depending only on ϵ

Our proof gives $c(\epsilon)$ of (unrealistic) order ϵ^{15}.

The proof of the theorem consists of 3 steps:

I. $\mathbb{E}N(f) \geq cn^2$

II. Exponential concentration of the random variable $N(f)/n^2$ around its median a_n

III. Existence of the limit $\lim_{n\to\infty} a_n = a$

I. $\mathbb{E}N(f) \geq cn^2$

Main trick: existence of a "barrier":

Claim: there exist constants $\rho > 0$ and $c_0 > 0$ s.t. for each $x \in \mathbb{S}^2$, there exists a "barrier function" $b_x \in \mathcal{H}_n$ with the following properties: $\|b_x\| = 1$, $b_x(x) \geq c_0\sqrt{n}$, and $b_x|_{\partial D(x,\rho/n)} \leq -c_0\sqrt{n}$.

If x is the Northern Pole of $\mathbb{S}^2$, then the zonal spherical harmonic $Y(\phi,\theta) = \sqrt{\frac{2n+1}{4\pi}}P_n(\cos\theta)$, $P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}$ is Legendre's polynomial of degree n, gives the barrier. For other x's, just rotate the sphere.

Choose $\gtrsim n^2$ disjoint disks of radius $3\rho/n$, x_j the centers

Ω_j the event

$$\Big\{ f(x_j) \ge C_0 \text{ and } f\Big|_{\partial D(x_j,\rho/n)} \le -C_0 \Big\} \quad (C_0 \gg 1)$$

$$\mathbb{E}N(f) \ge \mathbb{E}\#\Big\{ j\colon \Omega_j \text{ occurs} \Big\} = \sum_j \mathbb{E}1_{\Omega_j} = \sum_j \mathbb{P}(\Omega_j)$$

f_j the Gaussian function built over the orthogonal complement to b_{x_j} in $\mathcal{H}$, $\mathbb{E}f_j^2 = \frac{2n}{2n+1}$. Then $f = \xi_0 b_{x_j} + f_j$, $\mathbb{E}\xi_0^2 = \frac{1}{2n+1}$

Estimating f_j from above, we get $\mathbb{P}(\Omega_j) \ge \kappa > 0$

Then $\mathbb{E}N(f) \ge cn^2 \cdot \kappa = c_1 n^2$

II. Exponential concentration of the random variable $N(f)/n^2$ around its median a_n

Recall (def of the median): $\mathbb{P}(\frac{N(f)}{n^2} < a_n) \leq \frac{1}{2}$, $\mathbb{P}(\frac{N(f)}{n^2} > a_n) \leq \frac{1}{2}$

Recall: $\mathcal{H}_n$ is a large dimensional Euclidean space with Gaussian measure $\mathbb{P}$.

Lemma: For every $\epsilon > 0$, there exists $\rho > 0$ and an exceptional set $\mathcal{E} \subset \mathcal{H}_n$ of probability $\mathbb{P}(\mathcal{E}) \leq C(\epsilon)e^{-c(\epsilon)n}$ such that for all $f \in \mathcal{H}_n \setminus \mathcal{E}$ and for all $g \in \mathcal{H}_n$ satisfying $\|g\| \leq \rho$, we have $|N(f+g) - N(f)| < \epsilon n^2$.

Recall: Borell-Sudakov-Tsirelson Gaussian isoperimetric inequality: Suppose L is a finite dimensional Euclidean space with a Gaussian measure $\mathbb{P}$. For any set $F \subset L$, denote by $F_{+\rho} = \{x \in L \colon \exists y \in F \ s.t. \ ||x - y|| < \rho\}$ its ρ-neighbourhood. Given $t \in (0, 1)$ consider all measurable subsets $F \subset L$ with $\mathbb{P}(F_{+\rho}) \le t$. Among these sets F, affine half-spaces have the largest Gaussian measure.

$\Rightarrow$ P. Levy's concentration on $\mathcal{H}_n$: Let $F \subset \mathcal{H}_n$ be any measurable set of spherical harmonics. Suppose that the set $F_{+\rho}$ satisfies $\mathbb{P}(F_{+\rho}) < \frac{3}{4}$. Then $\mathbb{P}(F) \le 2e^{-c\rho^2 n}$.

Our lemma and Levy's concentration yield exponential concentration of $N(f)/n^2$:

Let $F = \{f \in \mathcal{H}_n \colon N(f) > (a_n + \epsilon)n^2\}$

For $f \in (F \setminus \mathcal{E})_{+\rho}$, we have $N(f)/n^2 > a_n$

$\Rightarrow \mathbb{P}(F \setminus \mathcal{E})_{+\rho} \le \frac{1}{2} \Rightarrow \mathbb{P}(F \setminus \mathcal{E}) \le 2e^{-c\rho^2 n}$

$\Rightarrow \mathbb{P}(F) \le \mathbb{P}(F \setminus \mathcal{E}) + \mathbb{P}(\mathcal{E}) \le C(\epsilon)e^{-c(\epsilon)n}$.

Similarly, we treat the set where $\frac{N(f)}{n^2} < a_n - \epsilon$

Idea of the proof of Lemma:

if the function f has no small critical values, then its nodal portrait is stable under C^1-perturbations

1. Construction of the exceptional set $\mathcal{E}$.

Fix $R \gg 1$ and cover $\mathbb{S}^2$ by approximately $R^{-2}n^2$ spherical disks D_j of radii R/n in such a way that every point is contained in at most 100 disks.

Fix some small $\alpha > 0$.

A disk D_j is stable for a function $f \in \mathcal{H}_n$ if $\min_{3D_j}(|f| + \frac{1}{n}|\nabla f|) > 4\alpha$. Otherwise, the disk D_j is unstable.

Fix a small $\delta > 0$. A function $f \in \mathcal{H}_n$ is exceptional if the number of the unstable disks for this function exceeds δn^2.

2. Estimate of $\mathbb{P}(\mathcal{E})$ uses P. Levy's concentration.

3. For each component of $Z(f)$ choose the disk D_j it intersects.

3a. Discard the components of $Z(f)$ that have diameter $\geq R/n$

Length $Z(f) \leq Cn$, hence the number of such components is $\ll \epsilon n^2$ provided $R \gg 1$

3b. Discard the components of $Z(f)$ that lie in unstable disks

Area of each nodal domain is $\geq c/n^2$, hence unstable disks cannot contain too many components

4. Take a perturbation $g \in \mathcal{H}_n$ with $\|g\| < \rho$, $\rho \ll \alpha$.

By "local Cauchy's estimates", on most of the stable disks $\max_{3D_j}(|g| + \frac{1}{n}|\nabla g|) < \alpha$.

Discard the components of $Z(f)$ that belong to stable disks D_j where $\max_{3D_j}(|g| + \frac{1}{n}|\nabla g|) \geq \alpha$.

5. Recall: the disk D_j is stable for f:

$$\min_{3D_j}(|f| + \tfrac{1}{n}|\nabla f|) > 4\alpha$$

The perturbation g is C^1-small on $3D_j$:

$$\max_{3D_j}(|g| + \tfrac{1}{n}|\nabla g|) < \alpha$$

Therefore, perturbing f by g we cannot change the number of components of $Z(f)$ that "lie deeply" within D_j.

III. Existence of the limit $\lim_{n\to\infty} a_n = a$

Scaling limit: Gaussian plane waves

Denote our Gaussian spherical harmonic f by f_n. Consider covariance function of f (= reproducing kernel in $\mathcal{H}_n$):

$$K_n(x,y) = \mathbb{E} f_n(x) f_n(y)$$
$$= \frac{1}{2n+1} \sum_{k=-n}^{n} Y_k(x) Y_k(y) = P_n(\cos\theta(x,y))$$

P_n Legendre's poly of deg n

$\theta(x,y)$ spherical dist between x and y.

Classical Hilb's asymptotics:

$$\lim_{n\to\infty} K_n(x + \tfrac{u}{n}, x + \tfrac{v}{n}) = J_0(|u-v|)$$

locally uniformly in u and v.

The limiting kernel $J_0(|u-v|)$ is a reproducing kernel in the limiting Hilbert space $\mathcal{H}$ that consists of the Fourier transforms of L^2-densities on the unit circumference:

$$H(u) = \int_{|\lambda|=1} e^{i\lambda \cdot u} h(\lambda)\, dm(\lambda), \quad h(-\lambda) = \overline{h(\lambda)}$$

$\mathcal{H}$ inherits the scalar product from $L^2(\mathbb{S}^1)$.

Note: elements of $\mathcal{H}$ are plane waves
$\Delta H + H = 0$

Let F be a Gaussian function built over $\mathcal{H}$.
Key observation:

$f_n(x + \frac{u}{n})$ can be treated as a "small perturbation" of $F(u)$

This ensures convergence of $\frac{\mathbb{E}N(f_n)}{n^2}$.

The End