Makarov's LIL for SLE

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Outline

Khintchine's LIL

- The law of the iterated logarithm for Brownian motion describes the oscillations of Brownian motion B_t near $t = 0$ and as $t \to \infty$.
- Khintchine's LIL for a Brownian motion B_t:

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1 \quad \text{almost surely.}$$

- The time inversion of a Brownian motion $X_t = tB_{1/t}$ is a Brownian motion.

Bloch Functions

- An analytic function g on $\mathbb{D}$ is called a *Bloch function* if

$$\|g\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)| < \infty.$$

  - This defines a semi-norm.
  - The Bloch functions form a complex Banach space $\mathcal{B}$ with the norm $|g(0)| + \|g\|_{\mathcal{B}}$.

- A connection between Bloch functions and univalent functions:
  - For any conformal map $f : \mathbb{D} \to \mathbb{C}$, $\log f'$ is a Bloch function and $\|\log f'\|_{\mathcal{B}} \leq 6$.
  - Conversely if $\|g\|_{\mathcal{B}} \leq 1$ then $g = \log f'$ for some conformal map f.

Makarov's LIL

- Khintchine's LIL for a Brownian motion B_t:

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 \quad \text{almost surely.}$$

- Makarov's LIL: If $g \in \mathcal{B}$ then, for almost every $\zeta \in \partial\mathbb{D}$,

$$\limsup_{r\to 1} \frac{|g(r\zeta)|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \leq \|g\|_{\mathcal{B}}.$$

- T. Lyons' synthetic proof of Makarov's LIL.

Hausdorff Measure

- L-functions and Measure Functions
  - A real valued function φ on an interval $(0, r_0)$ is called an *L-function* if φ is defined by a finite algebraic combination of exponential functions and logarithmic functions.
  - A *measure function* is a positive, increasing and continuous function.

- Hausdorff Measure
  - For a measure function φ, define

$$\Lambda_\varphi^\delta(E) = \inf \Big\{ \sum \varphi(r_j) : E \subseteq \bigcup B(z_j, r_j), r_j \leq \delta \Big\}.$$

  - Then the limit $\Lambda_\varphi(E) = \lim_{\delta \to 0} \Lambda_\varphi^\delta(E)$ exists in $[0, \infty]$, and $\Lambda_\varphi(E)$ is called the *φ-Hausdorff measure* of E.

Harmonic Measure and Hausdorff Measure

- Setting
  - φ : an L-measure function satisfying $\lim_{r\to 0}\varphi(r)/r=\infty$.
  - f : a conformal mapping from $\mathbb{D}$ onto a s.c. Ω.
  - ω : the harmonic measure for some point $w_0\in\Omega$.

- $\omega \ll \Lambda_\varphi \iff \liminf\limits_{r\to 1}\dfrac{(1-r^2)|f'(r\zeta)|}{\varphi^{-1}(1-r)} > 0$ for a.e. $\zeta\in\partial\mathbb{D}$.

- $\omega \perp \Lambda_\varphi \iff \liminf\limits_{r\to 1}\dfrac{(1-r^2)|f'(r\zeta)|}{\varphi^{-1}(1-r)} = 0$ for a.e. $\zeta\in\partial\mathbb{D}$.

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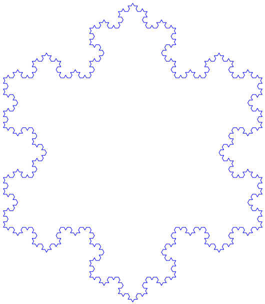
Makarov's LIL

Theorem (Makarov)

- $\dim \operatorname{supp} \omega = 1$.
- *For* $\varphi_C(r) = re^{C\sqrt{\log \frac{1}{r} \log\log\log \frac{1}{r}}} (C > 0)$, Λ_{φ_C} *is the proper measure for the size of* $\operatorname{supp} \omega$.



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Jones' Theorem

Theorem (Jones)

If $\|g\|_{\mathcal{B}} \leq 1$ *and if there exists* β *and* $M < \infty$ *such that for all* $z_0 \in \mathbb{D}$*,*

$$\sup_{\{z:\rho(z,z_0)<M\}} (1-|z|^2)|g'(z)| \geq \beta,$$

then there exists $c = c(\beta, M) > 0$ *such that almost every* $\zeta \in \partial\mathbb{D}$*,*

$$\limsup_{r\to 1} \frac{\operatorname{Re} g(r\zeta)}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \geq c.$$

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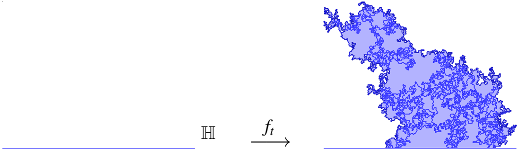
Schramm-Loewner Evolution



- SLE$_\kappa$ map $g_t(z)$: $\partial_t g_t(z) = \dfrac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z.$
- SLE$_\kappa$ backward flow $f_t = g_{-t}$:

$$\partial_t f_t(z) = \frac{-2}{f_t(z) + \sqrt{\kappa} B_t}, \quad f_0(z) = z.$$

- $f_t(z)$ has the same distribution as $z \mapsto g_t^{-1}(z + \sqrt{\kappa} B_t) - \sqrt{\kappa} B_t$.

Main Results

- Conjecture: Let $D = [-1/2, 1/2] \times (0,1)$. For a fixed time t and a sufficiently small $\epsilon > 0$, on the event $E = \{\inf_{z \in D} \operatorname{Im} f_t(z) \geq \epsilon\}$,

$$\limsup_{y \to 0} \frac{\operatorname{Re} \log f_t'(x+iy)}{\sqrt{\log \frac{1}{y} \log \log \log \frac{1}{y}}} = c(\kappa) := \frac{1}{\sqrt{\kappa/8} + \sqrt{2/\kappa}},$$

  for almost all $x \in [-1/2, 1/2]$, almost surely.

- Theorem: An upper bound.

- For given $x \in [-1/2, 1/2] \setminus \{0\}$,

$$\mathbb{E} |\operatorname{Re} \log f_{T_0}'(x+iy)|^2 = \frac{1}{2} c(\kappa)^2 \log \frac{1}{y} + o(\log \frac{1}{y}),$$

  where $T_u = T_u(z) := \inf\{t \geq 0 : \operatorname{Im} f_t(z) \geq e^u\}$.

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Corollary

Suppose ω is a harmonic measure restricted to $\{f_t(x) : x \in [-1/2, 1/2]\}$.

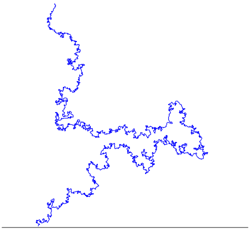
For a sufficiently small $\epsilon > 0$, almost surely, on the event E,

- Theorem:

$$\omega \ll \Lambda_{\varphi_c} \text{ for } c > c(\kappa) = \frac{1}{\sqrt{\kappa/8} + \sqrt{2/\kappa}}.$$

- Conjecture:

$$\omega \perp \Lambda_{\varphi_c} \text{ for } c \leq c(\kappa) = \frac{1}{\sqrt{\kappa/8} + \sqrt{2/\kappa}}.$$



Duplantier's Duality

- From the dimension estimate for the trace and outer boundary of the hull, B. Duplantier conjectured that $\mathrm{SLE}_{\kappa'}$ ($\kappa' = 16/\kappa$) should describe the boundary of the hull of SLE_κ when $\kappa > 4$.
- Note that $c(\kappa') = c(\kappa) = \dfrac{1}{\sqrt{\kappa/8} + \sqrt{2/\kappa}}$.

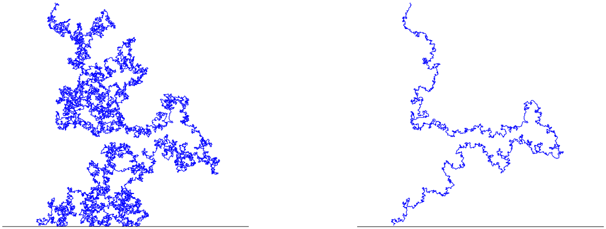


Figure: The boundary of the hull of SLE_6 describes $\mathrm{SLE}_{8/3}$.

$\mathbb{E}|\operatorname{Re}\log f'_{T_0}(x+iy)|^2$

- $\mathbb{E}|\operatorname{Re}\log f'_{T_0}(x+iy)|^2 = \frac{1}{2}c(\kappa)^2\log\frac{1}{y} + o(\log\frac{1}{y}),\ c(\kappa) = \frac{1}{\sqrt{\kappa/8}+\sqrt{2/\kappa}}.$
- $\mathbb{E}|\log f'_{T_0}(x+iy)|^2 = c(\kappa)^2\log\frac{1}{y} + o(\log\frac{1}{y}).$
- Itô calculus on $\log f'_t(z)$ and $\log(f_t(z)+\sqrt{\kappa}B_t)$:

$$d\log f'_t(z) = \frac{2}{\sqrt{\kappa/4}+\sqrt{4/\kappa}}\frac{dB_t}{f_t(z)+\sqrt{\kappa}B_t} - \frac{4}{4+\kappa}d\log(f_t(z)+\sqrt{\kappa}B_t).$$

- $\mathbb{E}\Big[\Big|\int_0^{T_0}\frac{dB_t}{f_t(z)+\sqrt{\kappa}B_t}\Big|^2\Big] = \mathbb{E}\Big[\int_0^{T_0}\frac{dt}{|f_t(z)+\sqrt{\kappa}B_t|^2}\Big]$
- $\mathbb{E}\Big[\Big|\int_0^{T_0}\frac{dB_t}{f_t(z)+\sqrt{\kappa}B_t}\Big|^2\Big] = \mathbb{E}\Big[\frac{1}{2}\log\operatorname{Im}f_t(z)\Big|_0^{T_0}\Big] = \frac{1}{2}\log\frac{1}{y}.$

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$\mathbb{E}|\operatorname{Re}\log f'_{T_0}(x+iy)|^2$

- $\mathbb{E}|\operatorname{Re}\log f'_{T_0}(x+iy)|^2 = \frac{1}{2}c(\kappa)^2\log\frac{1}{y} + o(\log\frac{1}{y}), c(\kappa) = \frac{1}{\sqrt{\kappa/8}+\sqrt{2/\kappa}}.$
- $d\log|f'_t(z)| = \sqrt{2}c(\kappa)\operatorname{Re}\frac{dB_t}{f_t(z)+\sqrt{\kappa}B_t} + \cdots .$
- $dM_t := \operatorname{Re}\frac{dB_t}{f_t(z)+\sqrt{\kappa}B_t}.$
- $d\langle M_{T_u}\rangle_u = \frac{1}{2}\frac{W_u^2}{1+W_u^2}\,du.$
  - $T_u = T_u(z) := \inf\{t \ge 0 : \operatorname{Im} f_t(z) \ge e^u\}.$
  - $f_{T_u}(z) + \sqrt{\kappa}B_{T_u} = X_u + iY_u, W_u := X_u/Y_u.$
  - $dW_u = -2W_u du + \sqrt{\kappa/2}\sqrt{1+W_u^2}d\widetilde{B}_u.$
- $\frac{1}{2}\frac{W_u^2}{1+W_u^2}\,du = \frac{\kappa}{4(\kappa+4)}\,du - \frac{1}{2(\kappa+4)}d\log(1+W_u^2) + \frac{\sqrt{2\kappa}}{2(\kappa+4)}\frac{W_u}{\sqrt{1+W_u^2}}\,dB_u.$

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Distribution of $\operatorname{Re}\log f'_{T_0}(x+iy)$

- (S. Rohde, O. Schramm) $y^p \mathbb{E}\left[(1+X_0^2)^b |f'_{T_0(z)}(z)|^p\right] = (1+x^2/y^2)^b y^q$.
  - $z = x+iy \in \mathbb{H}$ with $y<1$.
  - For each $b \in \mathbb{R}$, $p := 2b + \kappa b(1-b)/2$, $q := 4b + \kappa b(1-2b)/2$.
- $\mathbb{E}|f'_{T_0}(z)|^p \leq C y^{p(1-\frac{1-\sqrt{1-2p/\mu}}{p/\mu})}$.
- (K) $\mathbb{P}\left[\left|\frac{f'_{T_0}(z)}{y}\right| > \lambda\right] \leq C\,\mathrm{erfc}(\sqrt{\frac{\mu}{2}}\frac{\log(\lambda y)}{\sqrt{\log\lambda}})$, $\mu = \kappa/4 + 2 + 4/\kappa$.
- (K) f_t is a.s. h-Hölder ($\kappa \neq 4$) if $h < h(\kappa) = 1 - \frac{1}{\mu} - \sqrt{\frac{1}{\mu^2} + \frac{2}{\mu}}$.

  (Choose $p = -(2+\frac{1}{\mu}) + (\mu+1)\sqrt{\frac{1}{\mu^2}+\frac{2}{\mu}}$ for Chebyshev.)
- (I. Binder) $f(1/h(\kappa)) = 0$ for the multifractal spectrum f of SLE_κ.
- (D. Beliaev) The average integral means spectrum for the SLE.
- (J. Lind) Hölder regularity for the SLE trace.

Distribution of $\mathrm{Re}\log f'_{T_0}(x+iy)$

- (S. Rohde, O. Schramm) $y^p \mathbb{E}\left[(1+X_0^2)^b |f'_{T_0(z)}(z)|^p\right] = (1+x^2/y^2)^b y^q$.
  - $z = x+iy \in \mathbb{H}$ with $y < 1$.
  - For each $b \in \mathbb{R}$, $p := 2b + \kappa b(1-b)/2$, $q := 4b + \kappa b(1-2b)/2$.
- $\mathbb{P}\left[\left|\frac{f'_{T_0}(z)}{y}\right| > \lambda\right] \leq C\,\mathrm{erfc}\left(\sqrt{\frac{\mu}{2}}\frac{\log(\lambda y)}{\sqrt{\log\lambda}}\right)$, $\mu = \kappa/4 + 2 + 4/\kappa$.
- $\mathbb{P}_n = \mathbb{P}\left[\log|f'_{T_0}(x+iy_n)| > \alpha\sqrt{\log\frac{1}{y_n}\log\log\log\frac{1}{y_n}}\right]$.
- For $\log\frac{1}{y_n} = r^n (r>1)$, $\frac{\log\lambda y_n}{\sqrt{\log\lambda}} = \alpha\sqrt{\log(n\log r)}\left(1 + O\left(\sqrt{\frac{\log(n\log r)}{r^n}}\right)\right)$.
- $\mathbb{P}_n \leq C(n\log r)^{-\frac{\mu}{2}\alpha^2}$, and Borel-Cantelli.

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Distribution of $\mathrm{Re}\log f'_{T_0}(x+iy)$

- (S. Rohde, O. Schramm) $y^p\mathbb{E}\left[(1+X_0^2)^b|f'_{T_0(z)}(z)|^p\right] = (1+x^2/y^2)^b y^q.$
  - $z = x+iy \in \mathbb{H}$ with $y<1$.
  - For each $b\in\mathbb{R}$, $p := 2b+\kappa b(1-b)/2$, $q := 4b+\kappa b(1-2b)/2$.
- $\mathbb{E}|f'_{T_0}(z)|^p \leq Cy^{p(1-\frac{1-\sqrt{1-2p/\mu}}{p/\mu})}.$
- $\mathbb{P}_n = \mathbb{P}\left[\log|f'_{T_0}(x+iy_n)| > \alpha\sqrt{\log\frac{1}{y_n}\log\log\log\frac{1}{y_n}}\right].$
- For $\log\dfrac{1}{y_n} = r^n (r>1)$, choose $p = \mu\alpha\sqrt{\log(n\log r)}r^{-n/2}$ for Chebyshev.
- $\mathbb{P} \leq C(n\log r)^{-\frac{\mu}{2}\alpha^2}$, and Borel-Cantelli.

The Linear Dependence Correlation coefficients

- $S_n = \log |f_t'(x + iA^{-n})| \; (A > 1), \; S_n = X_1 + \cdots + X_n.$
- $X_n \approx C \operatorname{Re} y_n \dfrac{f_t''(z_n)}{f_t'(z_n)}.$
- $L_t = L_t(z) := \dfrac{y f_t''(z)}{f_t'(z)} + \dfrac{4}{4+\kappa}\left(\dfrac{y f_t'(z)}{f_t(z) + \sqrt{\kappa} B_t} - \dfrac{y}{z}\right)$
- $L_t = \dfrac{-2}{\sqrt{\kappa/4} + \sqrt{4/\kappa}} \displaystyle\int_0^t \dfrac{y f_s'(z)}{(f_s(z) + \sqrt{\kappa} B_s)^2} \, dB_s.$
- The complex martingale $L_t(z)$ at $t = \infty$ has exponential decay of correlations:

$$\left|\mathbb{E} L_\infty(z_1) \overline{L_\infty(z_2)}\right| = \frac{1}{2(\sqrt{\kappa/4} + \sqrt{4/\kappa})^2} \cosh^{-2} \frac{d_{\mathbb{H}}(z_1, z_2)}{2},$$

where $\cosh d_{\mathbb{H}}(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2 y_1 y_2} = -1 + \frac{|z_1 - \overline{z_2}|^2}{2 y_1 y_2}.$

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The Linear Dependence Correlation coefficients

- $L_t(z) = \dfrac{-2}{\sqrt{\kappa/4}+\sqrt{4/\kappa}} \displaystyle\int_0^t \frac{y f_s'(z)}{(f_s(z)+\sqrt{\kappa}B_s)^2}\, dB_s.$
- $N_t(z) := \displaystyle\int_0^t \frac{y f_s'(z)}{(f_s(z)+\sqrt{\kappa}B_s)^2}\, dB_s.$
- $\mathrm{Cov}(N_\infty(z), N_\infty(w)) = \mathbb{E}\Big[\displaystyle\int_0^\infty \frac{y v f_t'(z)\overline{f_t'(w)}}{(f_t(z)+\sqrt{\kappa}B_t)^2(\overline{f_t(w)}+\sqrt{\kappa}B_t)^2}\, dt\Big].$
- A chordal version of Goluzin's identities:
  - $\dfrac{d}{dt}\dfrac{\partial}{\partial z}\dfrac{\partial}{\partial w}\log\dfrac{f_t(z)-f_t(w)}{z-w} = \dfrac{d}{dt}\dfrac{f_t'(z)f_t'(w)}{(f_t(z)-f_t(w))^2}.$
  - $= \dfrac{\partial}{\partial z}\dfrac{\partial}{\partial w}\dfrac{2}{(f_t(z)+U_t)(f_t(w)+U_t)} = \dfrac{2f_t'(z)f_t'(w)}{(f_t(z)+U_t)^2(f_t(w)+U_t)^2}.$
- $\mathrm{Cov}(N_\infty(z), N_\infty(w)) = \mathbb{E}\Big[\displaystyle\int_0^\infty \frac{1}{2}\frac{d}{dt}\frac{y v f_t'(z)\overline{f_t'(w)}}{(f_t(z)-\overline{f_t(w)})^2}\, dt\Big] = -\frac{1}{2}\frac{yv}{(z-\overline{w})^2}.$

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Mixing Coefficients

- Philosophically, LIL holds for any process for which
  - the Borel-Cantelli lemma,
  - CLT with a reasonably good remainder, and
  - a certain maximal inequality are valid.
- Several strongly mixing coefficients were
  - introduced to measure the dependence between the σ-fields and
  - were used in the context of limit theorems for dependent random variables.
- For a certain class of weakly dependent random variables, their partial sums can be approximated by Brownian motion with probability one.

Mixing Coefficients

- For any two σ-fields $\mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{F}$, define

$$\alpha(\mathcal{A},\mathcal{B}) := \sup_{A\in\mathcal{A},\, B\in\mathcal{B}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|;$$

$$\rho(\mathcal{A},\mathcal{B}) := \sup_{f\in L^2(\mathcal{A}),\, g\in L^2(\mathcal{B})} \frac{|\mathbb{E}[fg] - \mathbb{E}[f]\mathbb{E}[g]|}{\|f\|_2\|g\|_2}.$$

- Suppose $\{X_n\}_{n=-\infty}^{\infty}$ is a sequence of random variables. For $-\infty \le m \le n \le \infty$, define the σ-field $\mathcal{F}_m^n := \sigma(X_k, m \le k \le n)$. Define

$$\alpha(n) := \sup_{j\in\mathbb{Z}} \alpha(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}); \quad \rho(n) := \sup_{j\in\mathbb{Z}} \rho(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}).$$

- $\{X_n\}_{n=-\infty}^{\infty}$ is said to be *strongly mixing* (or α-*mixing*) if $\alpha(n) \to 0$ as $n \to \infty$, ρ-*mixing* if $\rho(n) \to 0$ as $n \to \infty$.

Mixing Conditions and LIL

Theorem (Yoshihara)

Suppose that $\{X_n\}_{n=1}^{\infty}$ *is a (not necessarily strictly stationary) sequence of random variables with* $\mathbb{E}[X_n] = 0$ *and* $\mathbb{E}[X_n^2] < \infty$. *Let* $S_n = \sum_{k=1}^{n} X_k$, $\sigma_n^2 = \mathbb{E}[S_n^2]$, *and* $\sigma_{m,n}^2 = \mathbb{E}[(S_n - S_m)^2]$. *Suppose that* $\{X_n\}_{n=1}^{\infty}$ *is almost surely uniformly bounded and*

$$\frac{\sigma_{n+1}}{\sigma_n} \to 1 \text{ as } n \to \infty, 0 < \liminf_{n\to\infty} \frac{\sigma_n^2}{n} \leq \limsup_{n\to\infty} \frac{\sigma_n^2}{n} < \infty, \sigma_{m,n}^2 = (\sigma_n^2 - \sigma_m^2)(1 + o(1)) \text{ as } n - m \to \infty.$$

Moreover, suppose that $\{X_n\}_{n=1}^{\infty}$ *satisfies a strong mixing condition with*

$$\alpha(n) = O(n^{-1-\epsilon}),$$

for some $\epsilon > 0$. *Then almost surely*

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2\sigma_n^2 \log\log \sigma_n^2}} = 1.$$

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More Computations

- $L_t(z) = \dfrac{-2}{\sqrt{\kappa/4} + \sqrt{4/\kappa}} \displaystyle\int_0^t \frac{y f_s'(z)}{(f_s(z) + \sqrt{\kappa} B_s)^2}\, dB_s.$
- $N_t(z) := \displaystyle\int_0^t \frac{y f_s'(z)}{(f_s(z) + \sqrt{\kappa} B_s)^2}\, dB_s.$
- For two points $z = x + iy$ and $w = u + iv \in \mathbb{H}$,
  - $\mathrm{Cov}(N_\infty(z), N_\infty(w)) = -\dfrac{1}{2}\dfrac{yv}{(z-\overline{w})^2}.$
  - $\mathrm{Cov}(N_\infty^2(z), N_\infty^2(w)) = \dfrac{1}{2}\Big(\dfrac{yv}{(z-\overline{w})^2}\Big)^2.$
- For $z_0 = x_0 + iy_0, z_1 = x_1 + iy_1, w_0 = u_0 + iv_0$, and $w_1 = u_1 + iv_1 \in \mathbb{H}$,

$$\mathrm{Cov}(N_\infty(z_0)N_\infty(z_1), N_\infty(w_0)N_\infty(w_1)) = \frac{1}{4}\Big(\frac{y_0v_0}{(z_0-\overline{w_0})^2}\frac{y_1v_1}{(z_1-\overline{w_1})^2} + \frac{y_0v_1}{(z_0-\overline{w_1})^2}\frac{y_1v_0}{(z_1-\overline{w_0})^2}\Big).$$