A Tour of Spin Glasses and Their Geometry

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Support from NSF, ANR

IPAM: Random Shapes - Workshop I
26 March, 2007

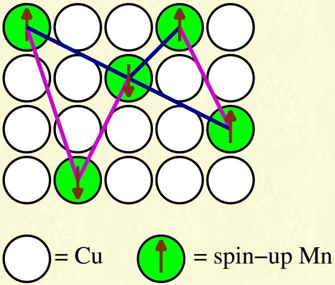
Goals

- **Real** experiment
  - very slow dynamics
  - memory
- Edwards-Anderson spin glass
  - $L \to \infty$?
  - geometric objects?
- **Numerical** experiments
  - evidence for SLE as a **challenge**

An alloy with randomly placed magnetic ions

Competition between

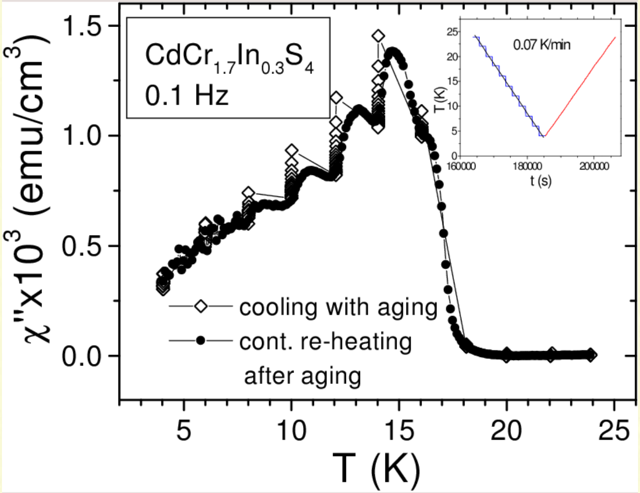
- ferromagnetic and
- anti-ferromagnetic interactions.



Measure response to magnetic field:

1. Slowly cool, then slowly heat.
2. Cool slowly, *pause*, continue cooling, then slowly heat.

A more complicated experiment [Miyashita, Vincent]



* Slow relaxation   * Rejuvenation   * Memory

Edwards-Anderson Hamiltonian - Statics

Energy for Ising spins $s_i = \pm 1$ on a graph with sites i:

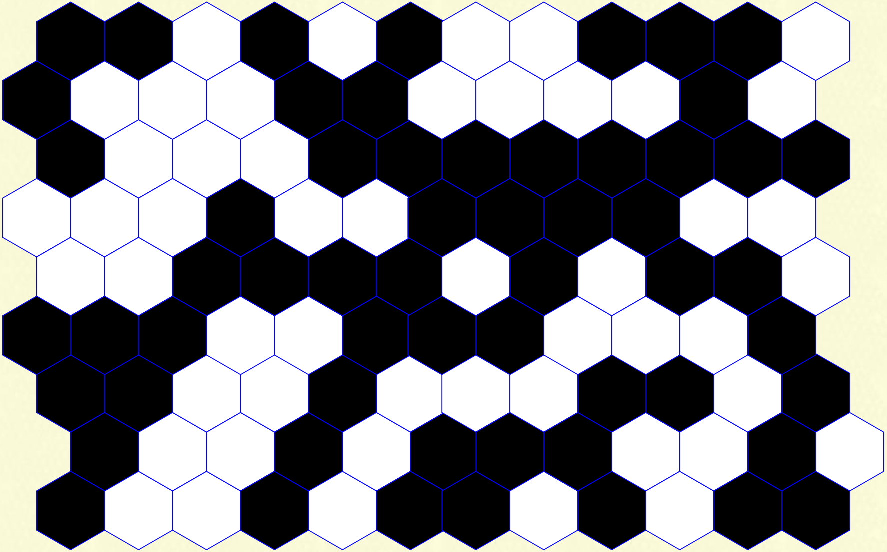
$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j \ ,$$

where the J_{ij} are i.i.d. Gaussian with zero mean.

Temperature T is believed to be **“irrelevant”**:

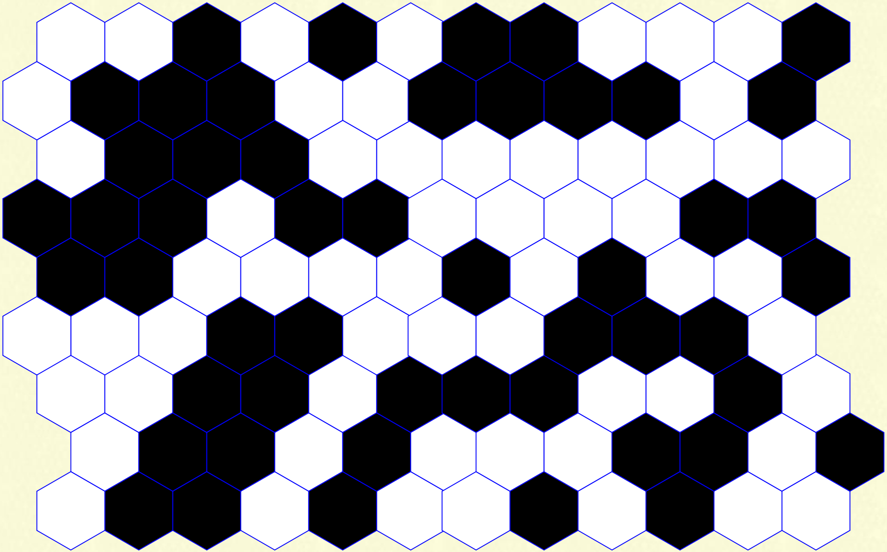
> At small enough T, changing T modifies the effective J_{ij}, but has no other effect on the statics $\Rightarrow$study minima of $\mathcal{H}$.

Given J_{ij}, two ground states (min $\mathcal{H}$)



$\alpha = \{s_i^0\}$

Given J_{ij}, two ground states (min $\mathcal{H}$)



$\beta = \{-s_i^0\}$

Two intertwined questions:

- Only two states as $L \to \infty$?
- What is the effect of boundary conditions?

Ground state description, numerical solution

Single ground state (in finite sample) for F_{ij} bonds dual to edges $\langle ij \rangle$:

$$F_{ij} = J_{ij} s_i s_j \, .$$

If $s_i \to -s_i$ for $i \in B$, then $F_{ij} \to -F_{ij}$ on ∂B and

$$\Delta\mathcal{H} = 2 \sum_{\partial B} F_{ij}$$

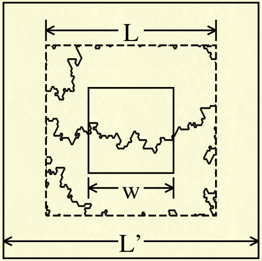
Find ground state numerically on cylinder or open square:

- Use Barahona's mapping of planar ISG to a matching problem (1982).
- The ground state F^0_{ij} has no loops or contractible paths of negative total weight $\sum_\gamma F_{ij}$.

Thermodynamic limit: does it exist?

Numerically:

- **Solve GS** for open system of size L.
- **Grow** to size L' & solve for GS.
- **Compare** the F_{ij} in box of size w.



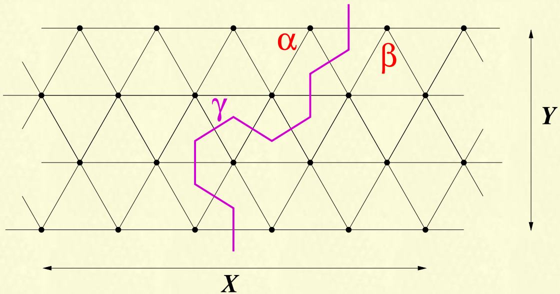
Probability of change in window appears to scale as

$$f(L'/L)\left(\frac{w}{L}\right)^{d-d_f},$$

with $f(x) \to \text{const}$ as $x \to \infty$.

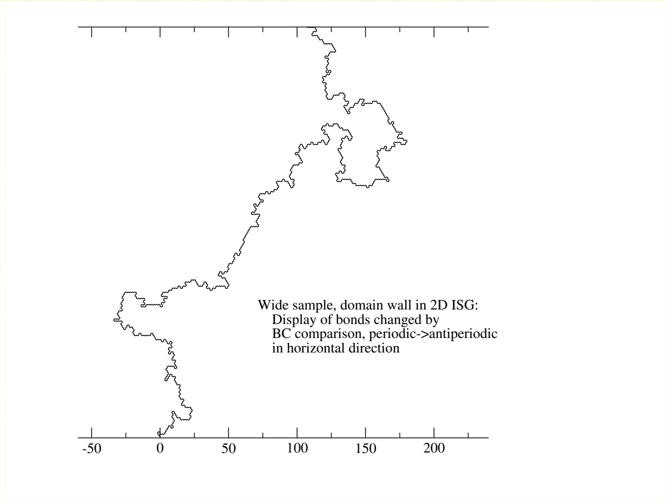
Domain Walls in a cylinder

⇒Changing $J_{ij} \to -J_{ij}$ on a column (periodic→anti-periodic BCs), changes constraints on F_{ij}.
⇒ Introduces a domain wall γ, the shortest path in F^0_{ij} from the bottom to the top.
⇒$E_{\text{DW}} = 2\sum_\gamma F^0_{ij}$.



L rows of W spins; $X = W$, $Y = \frac{\sqrt{3}}{2}L$.

Domain Wall



Study 720^2 or 1024×512 with good statistics ($N > 10^4$, 1% errors).
Domain walls can be **unconstrained** or have a **fixed end**.

Energies and fractals

Domain wall "renormalization group" (Bray, Moore, McMillan) and droplet theory (Fisher, Huse) give scaling hypotheses,

$$|E_{\rm DW}| \sim L^{\theta},\, \theta = -0.28(1)$$

$$|\gamma| \sim L^{d_f},\, d_f = 1.28(1)$$

where estimates are from simulations.

Tests of conformal invariance and domain Markov (SLE)

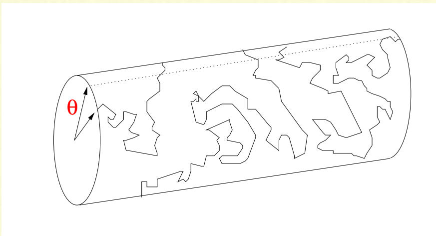
For comparison:

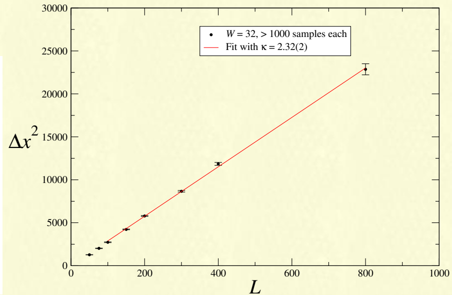
- Loop-erased random walk (LERW): execute a random walk on a lattice, erasing loops as they form.
- [Minimal spanning tree (MST): $d_f = 1.217(3)$, not SLE.]

Numerically, $d_f \approx 1.250$ for LERW, *independent* of whether the boundary conditions are reflecting (not SLE) or absorbing (SLE, $d_f = \frac{5}{4}$).

Conformal Invariance

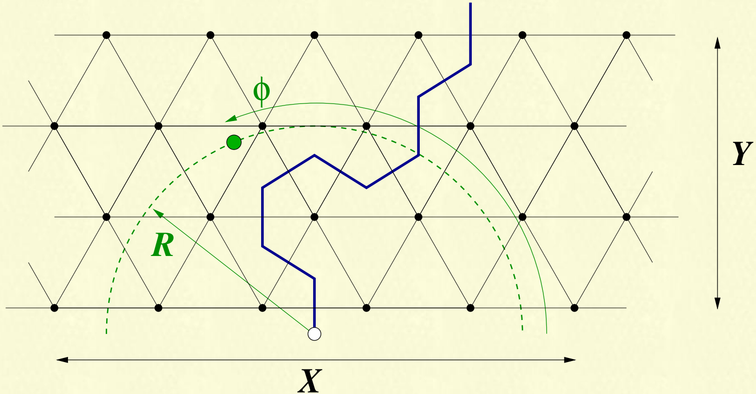
If SLE, $d_f = 1 + \frac{\kappa}{8}$; prediction of winding number on cylinder of circumference 2π: $\langle \theta^2 \rangle = \kappa L$.





Inferred $\kappa_{\text{cyl}} = 2.32 \pm 0.02$ is consistent with $d_f = 1.28 \pm 0.01$ (for both unconstrained & fixed BCs).

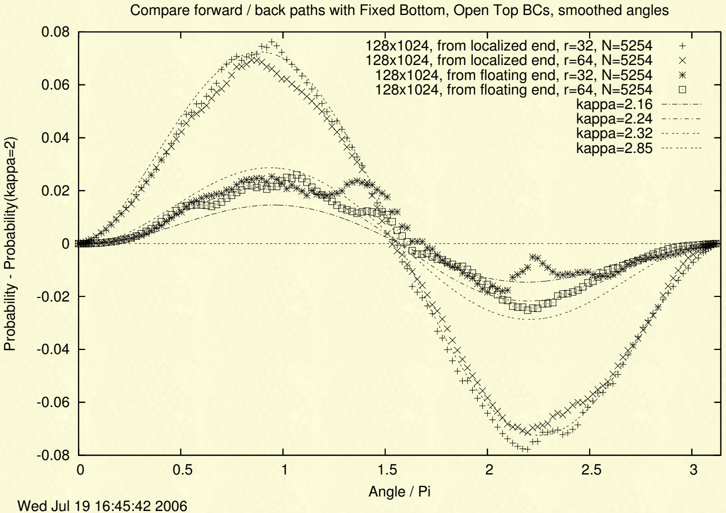
Probability of DW passing to right of a point



Near start, like upper half plane, so use Schramm:

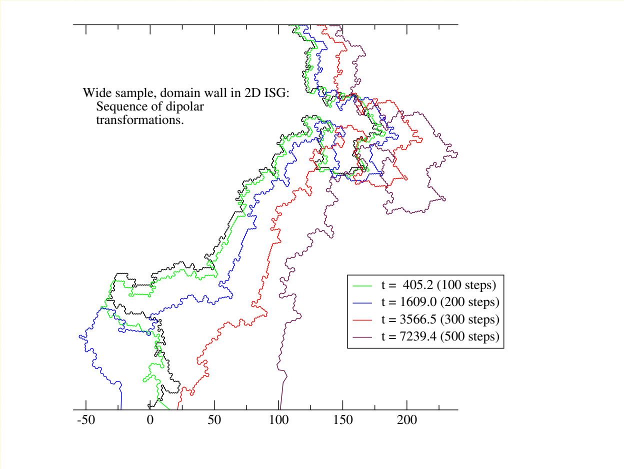
$$P_\kappa(\phi) = \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)} \cot(\phi)\, {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}, \frac{3}{2}; -\cot^2(\phi)\right)$$

Probability of DW passing to the right of a point



Plots show **residuals** $P_\kappa - P_2$, where $P_2(\phi) = \frac{\phi - \sin(2\phi)/2}{\pi}$.

Dipolar maps

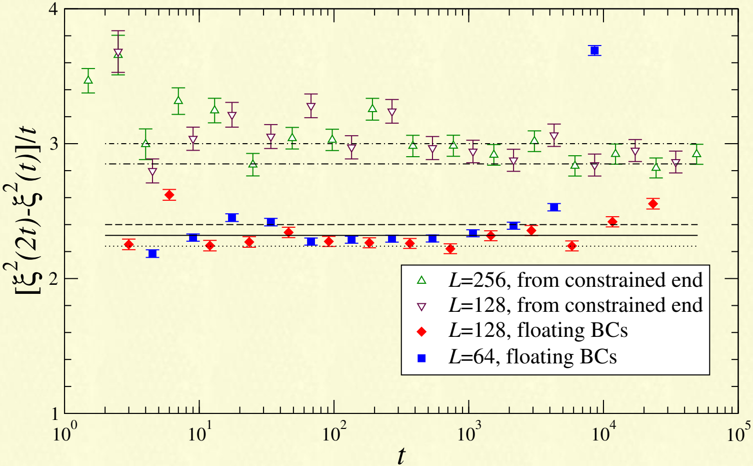


Sequence of maps $g_{t_i}(z)$ for strip height $\pi/2$ to infer $\xi(t_i)$, approximates

$$\frac{dg_t(z)}{dt} = \frac{2}{\tanh[g_t(z) - \xi_t]}\ ;\ g_0(z) = z$$

Dipolar maps

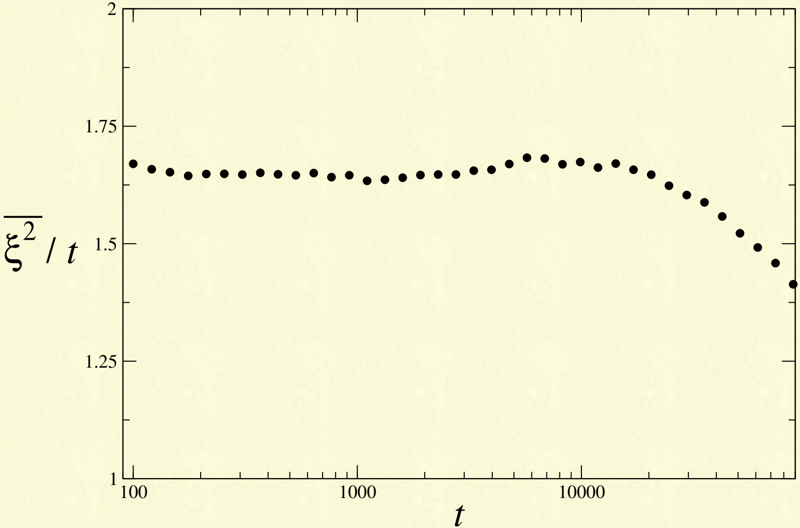
Is $\overline{\xi^2(t)} = \kappa t$, $d_f = 1 + \frac{\kappa}{8}$? YES for unconstrained, NO for fixed end.



Is ξ Gaussian? **“Yes”**. Is ξ correlated? Doesn’t **look** like it.

But $\xi(t)$ is not the best test

MST with $L = 256$, $N = 40000$:



(cf. expected $\kappa \approx 1.74$)

Markov Property

It may be hard to see failure by studying $\xi(t)$.
Another approach is to directly study

$$P[\gamma_2(b,c)|\gamma_1(a,c);a,b,c,\mathbb{D}] \stackrel{?}{=} P'[\gamma_2(b,c)|c;\mathbb{D}\backslash\gamma_1,a]$$

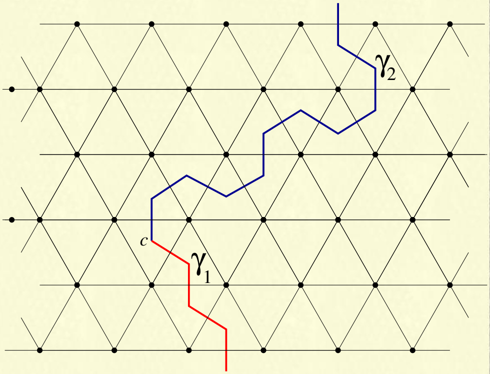
[Note, approximate study by Hartmann, Amoruso]
"Must fail microscopically" - Hastings.

Sample, sample, sample, compute P

Say 4×10^7 samples of 6×6 spins.

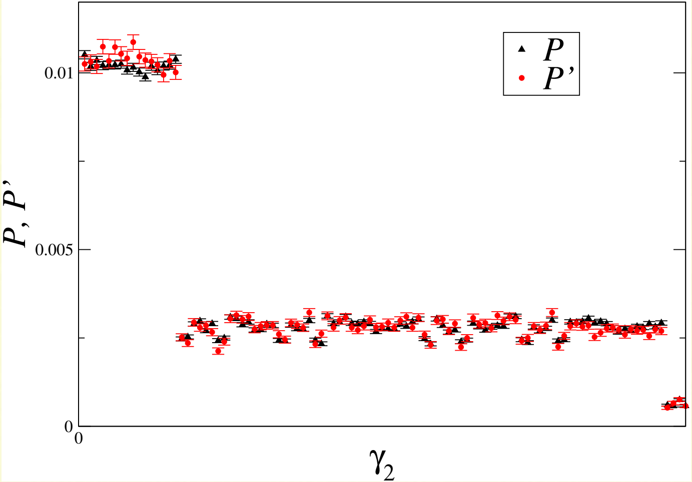
On original strip $\mathbb{D}$, compute DWs; for given start γ_1, compute fraction $P(\gamma_2, \gamma_1)$ that end with γ_2.

Given γ_1, generate J_{ij} and find DW in $\mathbb{D} \backslash \gamma_1$: out of those that start at c, find fraction $P'(\gamma_2, \gamma_1)$ that end in γ_2.



Compare P and P', ISG - lexicographic order for γ_2

2DISG, FF BCs; $\gamma_1 = \mathrm{RLR}$, $L = 6$



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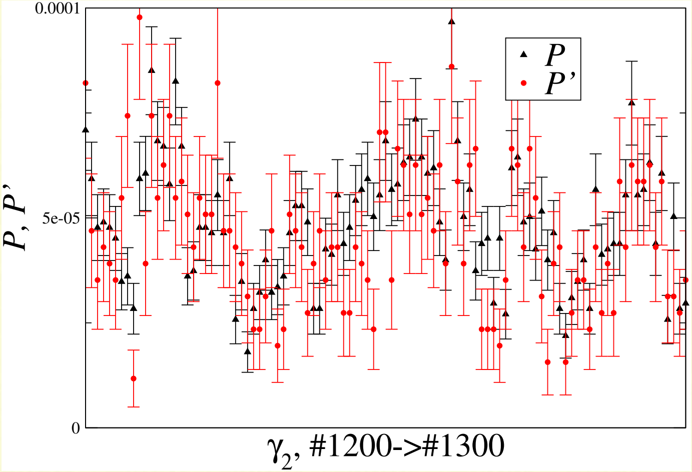
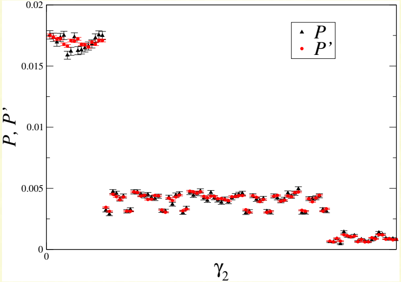
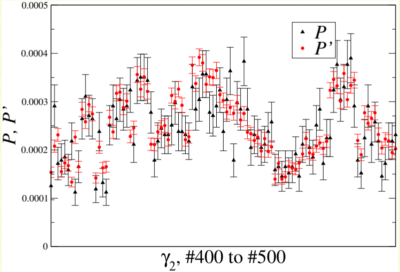


Test Markov, LERW with absorbing boundary conditions

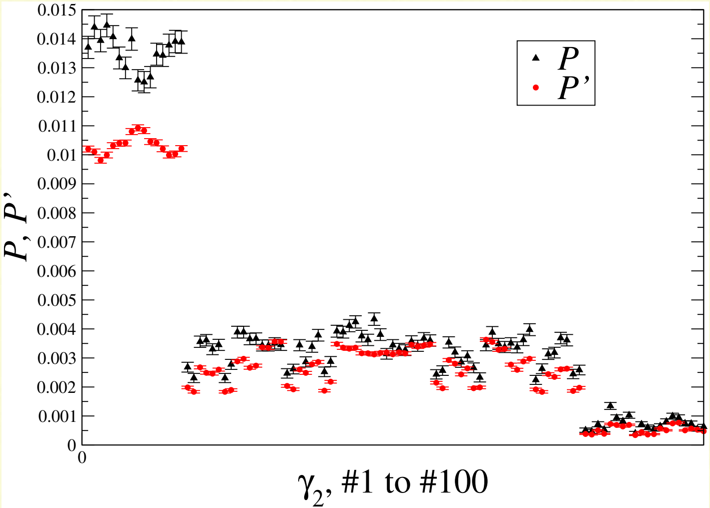
$L = 6;\ \gamma_1 = \mathrm{RLR},\ P = P(\gamma_2|\gamma_1;\mathbb{D}),\ P'(\gamma_2|c;\mathbb{D} - \gamma_1).$



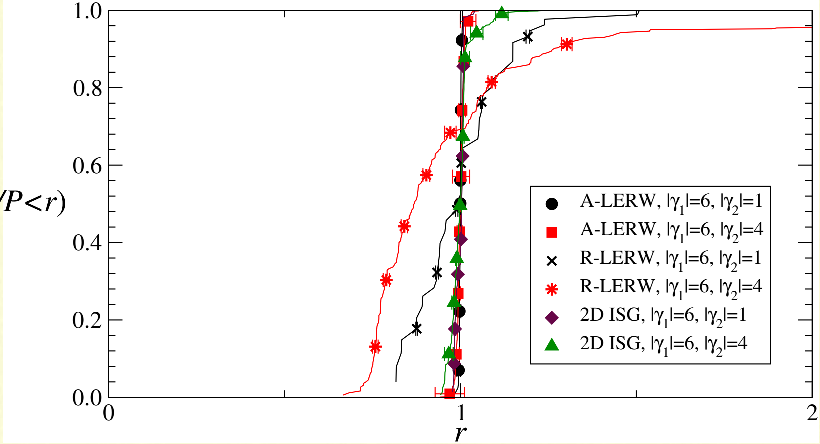


Test Markov, LERW, reflecting boundary conditions

$L = 6; \gamma_1 = \text{RLR}.$



Alternative display



Cumulative probability $p = \sum_{\gamma_1, \gamma_2, P'/P<r} \; P_{\mathbb{D}}(\gamma_1, \gamma_2)$

Highlights & Sequels

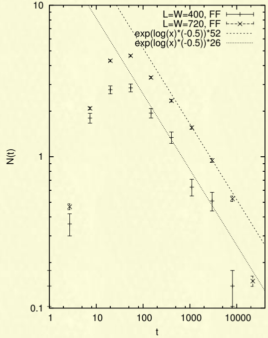
2DISG:

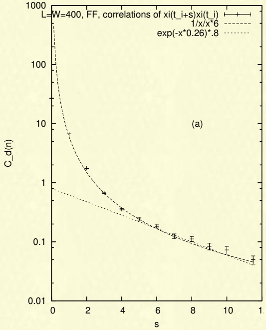
- $\kappa_{\text{eff}} = 2.30(5)$ consistent with $d_f = 1.28(1)$ for unconstrained paths, but not for paths with fixed root.
- Markov property generally "holds" for $L > 4$.

Applied same analysis path for LERW, MST, to study the utility of numerical checks.

- Other curves? (MST, etc.) Conjecture κ?
- What are good tests of domain Markov?
- Understanding of domain Markov? [Hierarchy, reversibility of min path]
- Modifications to SLE for boundary effects?

Correlation Functions





Endpoint distribution on a strip

Compare with formula by Bauer & Bernard, plot location of endpoint relative to start: