Talk at IPAM $\boxed{\text{Ilia Binder}}$

1. On the history:

'86 ~~Kadanoff~~ Halsey, Jensen, Kadanoff, Procaccia, Shraiman

'84 Mandelbrot

'83 Khanin, Sinai

'49 Eggleston

The idea: general.

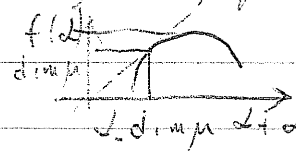
μ - prob. measure on metric X, $J = \operatorname{supp}\mu$.

$\leq \sim \bigsqcup J_\alpha$,

$J_\alpha = \{x : \cancel{\text{loc} \dim_\mu x = \alpha} \quad \mu(B(x,\delta)) \sim \delta^\alpha\}$.

$f(\alpha) = \dim J_\alpha$.

General properties:



1. $f(\alpha) \leq \alpha$
2. $\dim \mu = \inf\{\dim K : \mu(K) = 1\}$

   $f(\dim \mu) = \dim \mu$.
3. $\alpha_{min} \leq \alpha \leq \alpha_{max}$
4. $\sup f(\alpha) = \dim J$.

How to make it rigorous:

- Fine: $f(\alpha) = \lim_{\eta \to 0} H\dim\{x \in J : \exists \delta_j \downarrow 0 : \delta_j^{\alpha+\eta} \leq \mu(B(x_j, \delta_j)) \leq \delta_j^{\alpha-\eta}\}$

- Coarse: $f(\alpha) = \lim_{\eta \to 0} \varlimsup_{\delta \to 0} \frac{\log N(\delta, \alpha, \eta)}{\log \frac{1}{\delta}}$, $N(\delta, \alpha, \eta)$ - max number in disjoint balls, $\delta^{\alpha+\eta} \leq \mu(B(x,\delta)) \leq \delta^{\alpha-\eta}$

We talk about special object: planar harmonic measure.

For simplicity $K \subset \mathbb{C}$ - compact, $\Omega = \hat{\mathbb{C}} \setminus K \ni \infty$,

Geom. meaning $\measuredangle\theta$ $\quad \alpha = \frac{\pi}{\theta}$

Beurling inequality for s.c.: $\alpha \geq \frac{1}{2}$ $(\theta \leq 2\pi)$.

Makarov's thm. $\dim \omega = 1$, ω-a.e. $\alpha = 1$.

Additional structure: local rotation speed.

Fix $z \in \partial J$, small $R > 0$, $z_0 \in J$: $|z - z_0| = R$.

Rotation of J near x at distance δ:

$$\rho(x, \delta) = \inf_{y \notin J, |x-y| = \delta} \exp(\arg(y - x)).$$

The branch is such that $-\pi < \arg(z_0 - z) < \pi$.

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$\rho(x,\delta) \sim \delta^{\lambda\theta}$, λ - rotation speed.

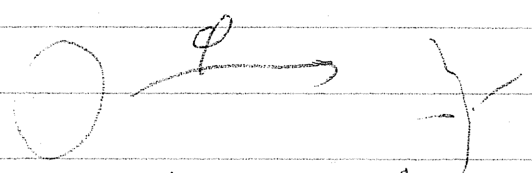
Generalized Beurling: $\alpha \geq 1+\lambda^2$.

Generalized Makarov: $\lambda = 0$ ω-a.e.

Can be generalized to non connected - need to consider rotation along Green lines: G - log potential, normalized to be 0 on J. ω - prob. that random green line hits J.

Can define $f(\alpha,\lambda) = \mathrm{Hdim}\{x \in J \mid \omega(B(x,\delta)) \sim \delta^{\alpha},\ \rho(x,\delta) \sim \delta^{\lambda}\}$.

How to study it: connected sets.



$\varphi: \mathbb{D}_- \to \hat{\mathbb{C}} \setminus J$ - conformal map, $\varphi(\infty) = \infty$.

$\omega = \varphi(\text{length})$. Green lines = images of radii.

$|\xi| = 1$, $r - 1 = \omega(B(\varphi(\xi),\delta))$

$$|\varphi'(r\xi)| \simeq \frac{\delta}{\omega(B(\varphi(\xi),\delta))}, \quad |\varphi'(r\xi)^{i}| \simeq \rho(\varphi(\xi),\delta)$$

If $|\xi| = 1$, $\log \varphi'(r\xi) \approx (a+ib)\log(r-1)$, then

$$\alpha(\varphi(\xi)) = \frac{1}{1-a}, \quad \lambda(\varphi(\xi)) = \frac{b}{1-a}.$$

To make it rigorous:

$$\beta(z) = \overline{\lim_{r \to 1+}} \frac{\log \int_{rS^1} |\varphi'(\xi)^z|\, d|\xi|}{\log(r-1)}$$

$z = t + i\tau$, t - loc. dim, τ - rot. speed.

Legendre-type relation:

$$f(\alpha,\lambda) \leq \inf(\alpha\beta(t+i\tau) + (1-\alpha)t - \lambda\tau + \alpha)$$

$$\beta(t+i\tau) \geq \sup_{\alpha,\lambda}\left(\frac{f(\alpha,\lambda) - (1-\alpha)t + \lambda\tau - \alpha}{\alpha}\right).$$

First, discuss universal bounds.

$F(\alpha,\lambda) = \sup_J f(\alpha,\lambda)$

$F_c(\alpha,\lambda) = \sup_{J\text{-connected}} f(\alpha,\lambda)$.

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$B(z) = \sup \beta(z)$

Legendre-type transform - equality.

Many named conjecture:

$$B(z) = \begin{cases} |z|^2/4, & |z| \le 2 \\ |z|-1, & |z| > 2 \end{cases}$$

Known: $B(z) = |z|-1$, $|z| \geq C \geq 2$.

$$F_c(\alpha, \lambda) = \begin{cases} -\infty, & \alpha < \frac{1+\lambda^2}{2} \\ 2 - \frac{1+\lambda^2}{\alpha}, & \alpha \geq \frac{1+\lambda^2}{2} \end{cases}$$

Includes: — Brennan $t = -2$.

— Makarov thm: $B(t) \le Ct^2$

— Carleson-Jones $B(1) = \frac{1}{4}$.

Coeffs growth for class Σ:

$e_n \le n^{\gamma - 1 + \varepsilon}$, $\gamma = B(1)$.

General domain:

$$F(\alpha, 0) = \begin{cases} \alpha, & \alpha \le 1 \\ F_c(\alpha, 0), & \alpha \geq 1 \end{cases}$$

Explain α.

Other λ - convex hull.

If conj. true

$$F(\alpha, \lambda) = \begin{cases} \alpha/(1+\lambda^2); & \alpha \le 1+\lambda^2 \\ F_c(\alpha, \lambda); & \alpha \geq 1+\lambda^2 \end{cases}$$

How to prove: Fractal Approximation, then - proven for Julia sets.

Becomes a problem in complex dynamics - not yet solved.

Now, let us switch to SLE.

What should be the result? Next page. (B - Duplantier)

What is known? β. (B - Duplantier, Belyaev-Smirnov)

Not optimal! What is optimal? Meaning?

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For SLE_κ: $\kappa \le 4$.

$$\boxed{\begin{gathered} B = 1 + \frac{\kappa}{8} + \frac{2}{\kappa}. \\ \bar{f}(\alpha, \lambda) = B + \alpha \quad B \quad \frac{\alpha^2}{2\alpha - 1 - \lambda^2} \\ (1+\lambda^2) + \left(\frac{\alpha}{1+\lambda^2}, 0\right) - B\lambda^2 \end{gathered}}$$

Negative dim - explain.

$$\bar{\beta}(t + i\tilde{t}) = B + \sqrt{(B-t)^2 + \tilde{t}^2} - (2B)^{\frac{1}{2}} \left(B - t + \sqrt{(B-t)^2 + \tilde{t}^2}\right.$$