Conformal boundary loop models

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Conformal field theory

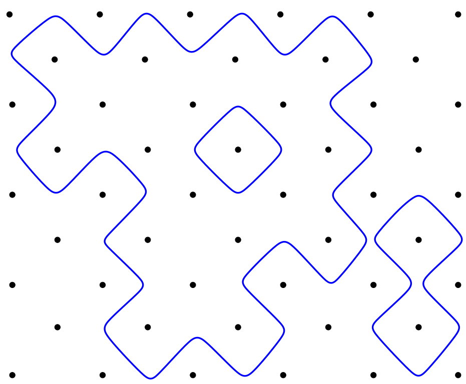
- 2D critical systems are almost always conformally invariant
- Continuum limit of suitable lattice models
- Rational CFT extremely well understood
  - Can solve without knowing a lattice realization
  - When exponents known, one can often identify
  - E.g. unitary minimal models: Ising, Potts,. . .
- Add various perturbations: RG-flow between CFTs
  - E.g. Ising model with 2 and 3-spin interactions

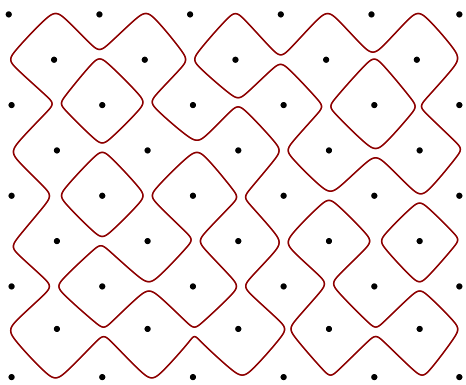
Boundary CFT (J. Cardy)

- Specify local degrees of freedom on boundary
- Conformal boundary condition = RG stable
- Corresponding operator = bc changing operator
- Rational case: bijection bcc $\leftrightarrow$ bulk operators

Non rational CFT: Loop models

- Lattice models of fluctuating loops
- Non-local weight of n per loop
  - Originates from $O(n)$ model of spins $\in \mathbb{R}^n$
- Many relations: Coulomb gas, Temperley-Lieb algebra, XXZ spin chain, SLE, local height probabilities,...
- Variants: fully packed, dense ($\kappa > 4$), dilute ($\kappa < 4$)
- "Obvious" bcs: Dirichlet, Neumann
- Improvements:
  - On annulus, fix L non contractible lines
  - Enhance weight of surface monomers, $K_s > K_b$





Surface transitions for dilute loops

- Ordinary: Dirichlet bcs, $K_\mathrm{s} = K_\mathrm{b}$
  - Bulk orders before surface
  - Just below T_c, magnetization $\to 0$ on surface
- Special: Critically enhanced $K_\mathrm{s}^\mathrm{crit} > K_\mathrm{b}$
  - Bulk and surface order simultaneously
  - Just below T_c, flat magnetization profile
  - Physically, model of critically adsorbed polymers
- Extraordinary: $K_\mathrm{s} = \infty$
  - Teflon effect (ordinary transition with $L \to L + 1$)
- Flow Ordinary $\leftarrow$ Special $\to$ Extraordinary

Critical exponents

- Coulomb gas OK in bulk case (magnetic charge L)
  - Adaptation to boundary case [Cardy, 2006]
- Exact results: integrability of related spin chain
- Fits nicely with CFT setup

$$n = 2\cos\left(\frac{\pi}{p+1}\right) \qquad h(r,s) = \frac{[(p+1)r - ps]^2 - 1}{4p(p+1)}$$

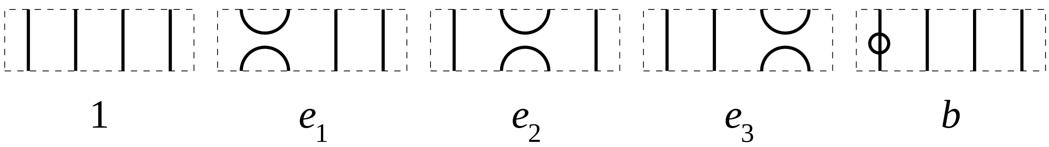
Ordinary $h_{1+L,1}$, Special $h_{1+L,2}$, Extraordinary $h_{2+L,1}$

The JS boundary conditions

- Allows for continuous spectrum of bcc operators
- Give weight $y \neq n$ to loops touching left boundary
  - Looks more non-local than the K_s weights
  - But is algebraically local (in the 1BTL algebra)
  - Like constraining $\mathrm{O}(n)$ to $\mathrm{O}(y)$ on boundary

Relation to one-boundary TL algebra

Graphical representation of generators:



Can be used to prove algebraic relations:

$$\begin{aligned}
e_i e_j &= e_j e_i \text{ for } |i-j| \geq 2 \\
e_i e_{i\pm 1} e_i &= e_i \\
e_i^2 &= n e_i \\
b^2 &= b \\
e_1 b e_1 &= y e_1 \\
e_i b &= b e_i \text{ for } i = 2, 3, \ldots, N-1
\end{aligned}$$

Representation theory for 1BTL

$U_t(SU(2))$ symmetric hamiltonian ($n \equiv 2\cos\gamma \equiv t + t^{-1}$)

$$\mathcal{H}_0 = \sum_{i=1}^{N-1} e_i = \mathcal{H}_{\rm XXZ} - \frac{i}{2}\sin\gamma\,(\sigma_1^z - \sigma_N^z)$$

$$\mathcal{H}_{\rm XXZ} = -\frac{1}{2}\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \cos\gamma\sigma_i^z\sigma_{i+1}^z + \cos\gamma\right)$$

1BTL algebra represented by $\mathcal{H} = \mathcal{H}_0 - ab$ [arbitrary a] setting $y = \sin(\omega+\gamma)/\sin\omega = \sin((r+1)\gamma)/\sin(r\gamma)$

$$b = -\frac{1}{2\sin(\omega+\gamma)}\left(i\cos\omega\sigma_1^z + \sigma_1^x - \sin\omega\right)$$

Spectral equivalence [Nichols-Rittenberg-de Gier, 2004] with

$$\mathcal{H} = \mathcal{H}_{\rm XXZ} - \frac{\sin\gamma}{2}\left[\tan\left(\frac{\omega+\delta}{2}\right)\sigma_1^z + \tan\left(\frac{\omega-\delta}{2}\right)\sigma_N^z + \frac{2\sin\omega}{\cos\omega+\cos\delta}\right], \qquad a = \frac{2\sin\gamma\sin(\omega+\gamma)}{\cos\omega+\cos\delta}$$

implying [Nichols, 2005] for integer r and any a the spectrum generating function

$$K_{r,r+L} = {\rm Tr}\, q^{L_0 - c/24} = \frac{q^{h_{r,r+L}} - q^{h_{r,-r-L}}}{q^{c/24}P(q)}$$

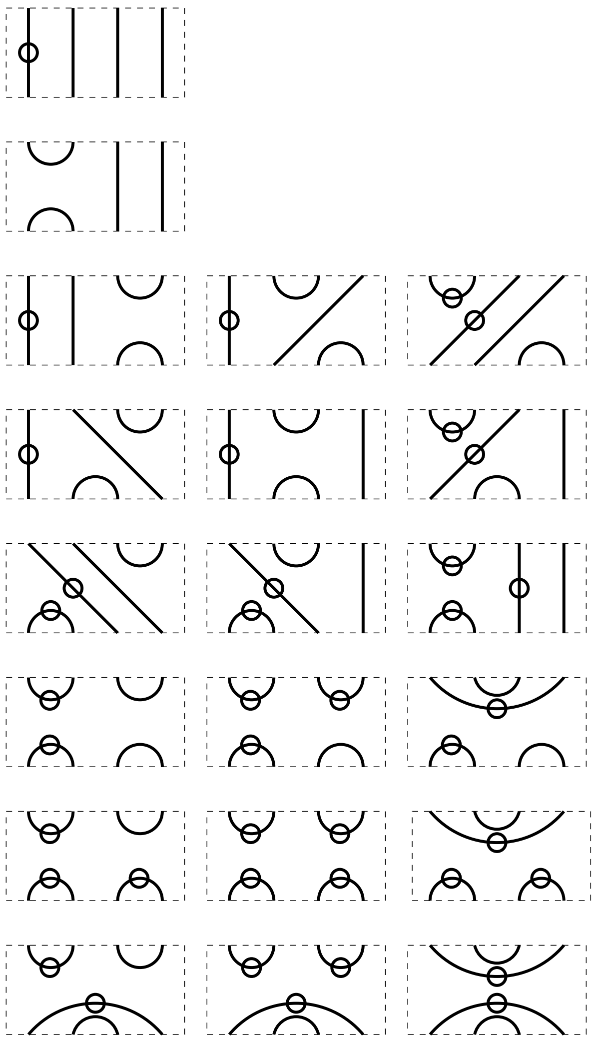
with $P(q) = \prod_{n=1}^{\infty}(1-q^n)$.

Conjecture: This holds true for any real y

Structure of the transfer matrix

$T = b \prod_{i=1}^{\lfloor (N-1)/2 \rfloor} (1 + e_{2i}) \prod_{i=1}^{\lfloor N/2 \rfloor} (1 + e_{2i-1})$

- Consider periodic bc's in the transfer direction
- Need states involving two time-slices
- ℓ bridges, i.e., lines extending back to $t = 0$



Transfer matrix is then block triangular

$$T = \begin{pmatrix} T_{N,N} & 0 & \dots & 0 \\ T_{N-1,N} & T_{N-1,N-1} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ T_{0,N} & T_{0,N-1} & \dots & T_{0,0} \end{pmatrix}$$

and $T_\ell \equiv T_{\ell,\ell}$ are themselves block diagonal (having N_ℓ identical blocks on the diagonal, each of dimension N_ℓ)

- Each eigenvalue of T is an eigenvalue of some T_ℓ
- T_ℓ is the sector with ℓ non contractible lines

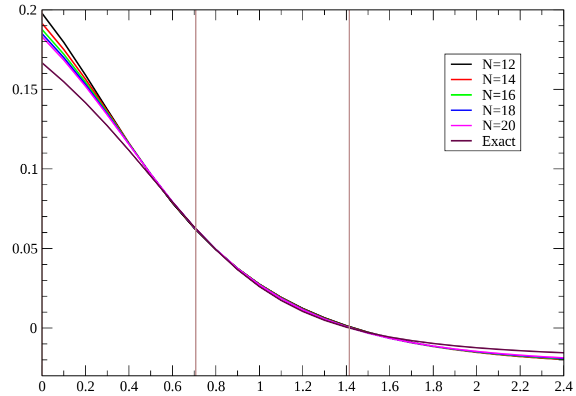
Blobbing of leftmost line also (semi)conserved

- Blobbed (T_ℓ^{b}) and unblobbed (T_ℓ^{u}) sectors

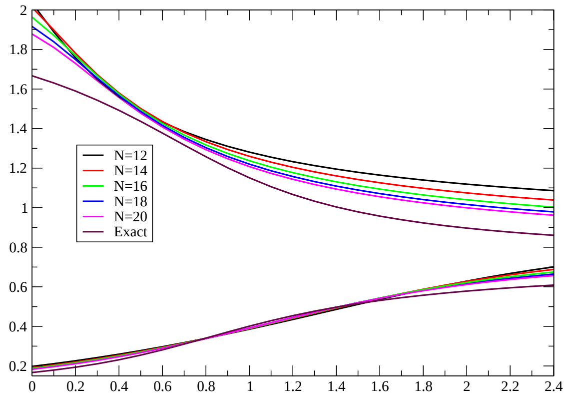
Conformal boundary conditions

- Take any $n = 2\cos\gamma$ and $y = \sin((r+1)\gamma)/\sin(r\gamma)$
- The JS bc's are conformal for any L and both sectors
- Leading exponent $h_{r,r\pm L}$ for $T_\ell^{\rm b}$ (resp. $T_\ell^{\rm u}$)

Check for $L = 0$ and the Ising model:



Check for $L = 2$ and the Ising model:



Refined spectrum generating functions

- Characters of generic irreps of Virasoro algebra:

$$Z_L^{\rm b}(r) = \frac{q^{h_{r,r+L}-c/24}}{P(q)}$$

$$Z_L^{\rm u}(r) = \frac{q^{h_{r,r-L}-c/24}}{P(q)}$$

- No subtraction (no truncation of Bratteli diagram)
- Truncation for r integer (and further for $\gamma = \frac{\pi}{p+1}$)

Checks of the result $h_{r,r\pm L}$

1) Count loops touching a line $P_1 P_2$

- Points P_1 and P_2 on left of annulus
- Potts model with fixed bc's on $P_1 P_2$, free elsewhere
- Potts $\to$ FK clusters $\to$ loops:
  $Z(P_1 P_2) = Q^{V/2} \sum_{\rm loops} Q^{\ell/2} Q^{-\ell(P_1 P_2)}$
- Thus $n = 1/y = Q^{1/2}$, i.e., JS bc's with $r = p - 1$
- Hence bcc operator $\phi_{p-1,p-1} = \phi_{1,2}$
- Matches famous result by [Cardy (1992)]

2) Couple loop model to 2D quantum gravity

- The result $h_{r,r\pm L}$ recovered by [Kostov (2007)]

Eigenvalue amplitudes

Annulus partition function
$Z = \langle v|T^M|u\rangle = \sum_i D_i(n,y)\,[\lambda_i(n,y)]^M$

- Not just a trace (due to non-locality of loops)
- We shall see $D_i = D_{\ell(i)}$ with $\ell = \#$bridges
- Singular behavior of Z when some $D_\ell(n,y) = 0$

D^0_ℓ by combinatorics

- For simplicity, consider 0BTL case ($y = n$)

Reduced TM states for width $2N$ strands and 2ℓ bridges:

$$\#\text{ states} = \#\text{blocks } T^{(i)}_\ell = \dim(T^{(i)}_\ell) \equiv n(N,\ell)$$

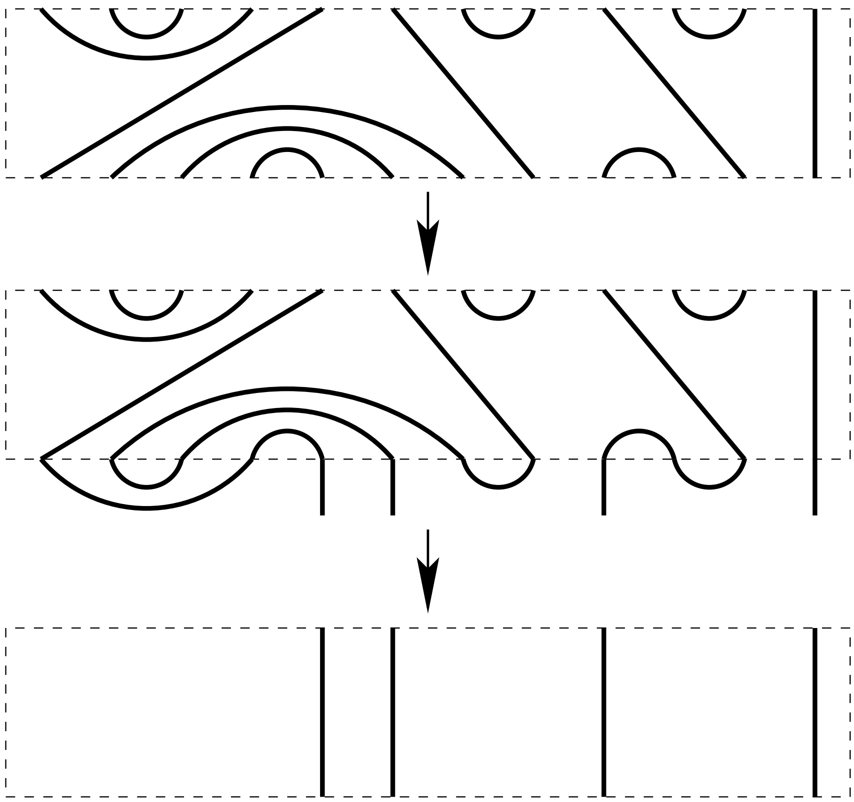
with $n(N,\ell) = \binom{2N}{N-\ell} - \binom{2N}{N-\ell-1}$.

Define character $K_\ell = \mathrm{Tr}(T^{(0)}_\ell)^M$

- Gives generic $K_{r,s}$ character in CFT for $N, M \to \infty$

Let also $Z_j =$ part. sum with $2j$ non contractible lines

- Config with j nc lines occurs $n(j,\ell)$ times in K_ℓ
- Proof: enumerate “compatible states”



So we have shown that

$$K_\ell = \sum_{j=\ell}^{N} n(j,\ell) \frac{Z_j}{n^{2j}}$$

Inverting this gives

$$Z_j = \sum_{\ell=j}^{N} c_j^{(\ell)} K_\ell \quad \text{with} \quad c_j^{(\ell)} = (-1)^{\ell-j} \binom{\ell+j}{\ell-j} n^{2j}$$

And since $Z = \sum_{j=0}^{N} Z_j$ we have

$$D_\ell^0(n) = \sum_{j=0}^{L} c_j^{(\ell)} = U_\ell(n/2)$$

Amplitudes for 1BTL and 2BTL

$$\begin{aligned}
D_\ell^{\rm u} &= U_\ell(n/2) - yU_{\ell-1}(n/2) \\
D_\ell^{\rm b} &= yU_{\ell-1}(n/2) - U_{\ell-2}(n/2) \\
D_\ell^{\rm uu} &= U_\ell(n/2) - (y_l + y_r)U_{\ell-1}(n/2) + y_l y_r U_{\ell-2}(n/2) \\
D_\ell^{\rm ub} &= y_r U_{\ell-1}(n/2) - (1 + y_l y_r)U_{\ell-2}(n/2) + y_l U_{\ell-3}(n/2) \\
D_\ell^{\rm bb} &= y_l y_r U_{\ell-2}(n/2) - (y_l + y_r)U_{\ell-3}(n/2) + U_{\ell-4}(n/2)
\end{aligned}$$

Note $D_\ell^{\rm u} = 0$ or $D_\ell^{\rm b} = 0$ exactly when $y = \frac{\sin((r+1)\gamma)}{\sin(r\gamma)}$ with r integer

- Singular behavior of Z, reduction of Hilbert space, truncation of Bratteli diagram, etc.

More about the amplitudes

Full size of Hilbert space recoved from sumrule

$$\sum_{\ell=0}^{N} n(N,\ell) D_\ell^0 = n^N$$

- Similar results for 1BTL, 2BTL, and dilute versions

Other way to detect singularities: study zeros of Gram matrix

- 0BTL: meander matrix [Di Francesco et al, 1997)]

$$\det[G_{2N}] = \prod_{m=1}^{N} [D_m^0]^{\binom{2N}{N-m} - 2\binom{2N}{N-m-1} + \binom{2N}{N-m-2}}$$

- 1BTL: "one-boundary meander matrix"

$$\det[G_{2N}] = \prod_{m=1}^{N} [D_m^{\rm u} \cdot D_m^{\rm b}]^{\binom{2N}{N-m}}$$

- 2BTL (e.g. with no loop touching both boundaries)

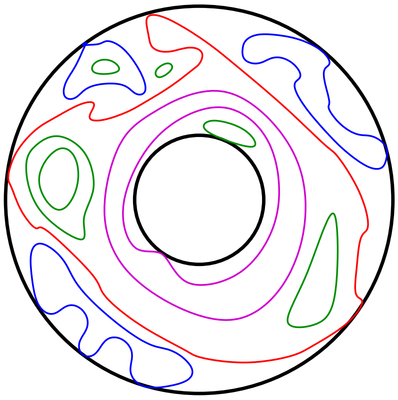
$$\det[G_{2N}] = \prod_{m=1}^{N} [D_m^{\rm uu} \cdot D_m^{\rm b,l} \cdot D_m^{\rm b,r} \cdot D_{m-1}^0]^{\sum_{i=0}^{N-m} \binom{2N}{i}}$$

Other results:

- Various reduction formulae 2BTL → 1BTL → 0BTL
- Alternative quick & dirty "proof" using CFT fusion
- Alternative proof using formal algebraic properties

Partition function identities in 1BTL

- Weights n (resp. y) to contractible loops
- Weights l (resp. m) to non contractible loops



Parameterize: $l = 2\cosh\alpha, \quad m = \frac{\sinh(\alpha+\beta)}{\sinh\beta}$

When N even (set $L = 2j$):

$$Z = q^{-\frac{c}{24}} \left[\sum_{j=0}^{\infty} \frac{\sinh(2j\alpha+\beta)}{\sinh\beta} \frac{q^{h_{r,r+2j}}}{P(q)} - \sum_{j=1}^{\infty} \frac{\sinh(2j\alpha-\beta)}{\sinh\beta} \frac{q^{h_{r,r-2j}}}{P(q)} \right]$$

$$Z = \sum_{j=-[r/2]}^{\infty} \frac{\sinh(2j+r)\alpha}{\sinh r\alpha} K_{r,r+2j}$$

For $\beta = r\alpha$

The subtractions in Z occur in finite size as well

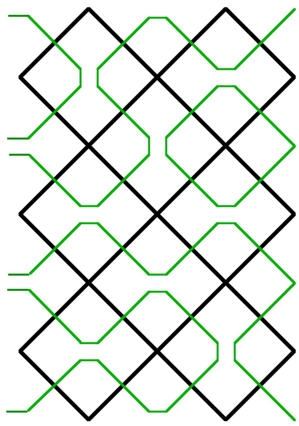
One "geometric" application: $n = l = 0$ and $y = m = 1$

$$Z = \frac{q^{-1/24}}{P(q)} \sum_{j=-\infty}^{\infty} (-1)^j q^{(4j-1)^2/32}$$

Neumann boundary conditions

When $y = 1$ we reinterpret the "blob" as a "fork"

- Weight one to half loops on left boundary



So the Dirichlet-Neumann bcc operator has weight

$$h_{p/2,p/2} = \frac{p^2 - 4}{16p(p+1)}$$

Dilute 1BTL model

- Based on Nienhuis' spin-one model (19 vertex, Izergin-Korepin,...)
- Special "tuned" weight for boundary monomers
- One finds the exponent $h_{r \pm L, r+1}$
- Generalizes special surface transition in O(n) model