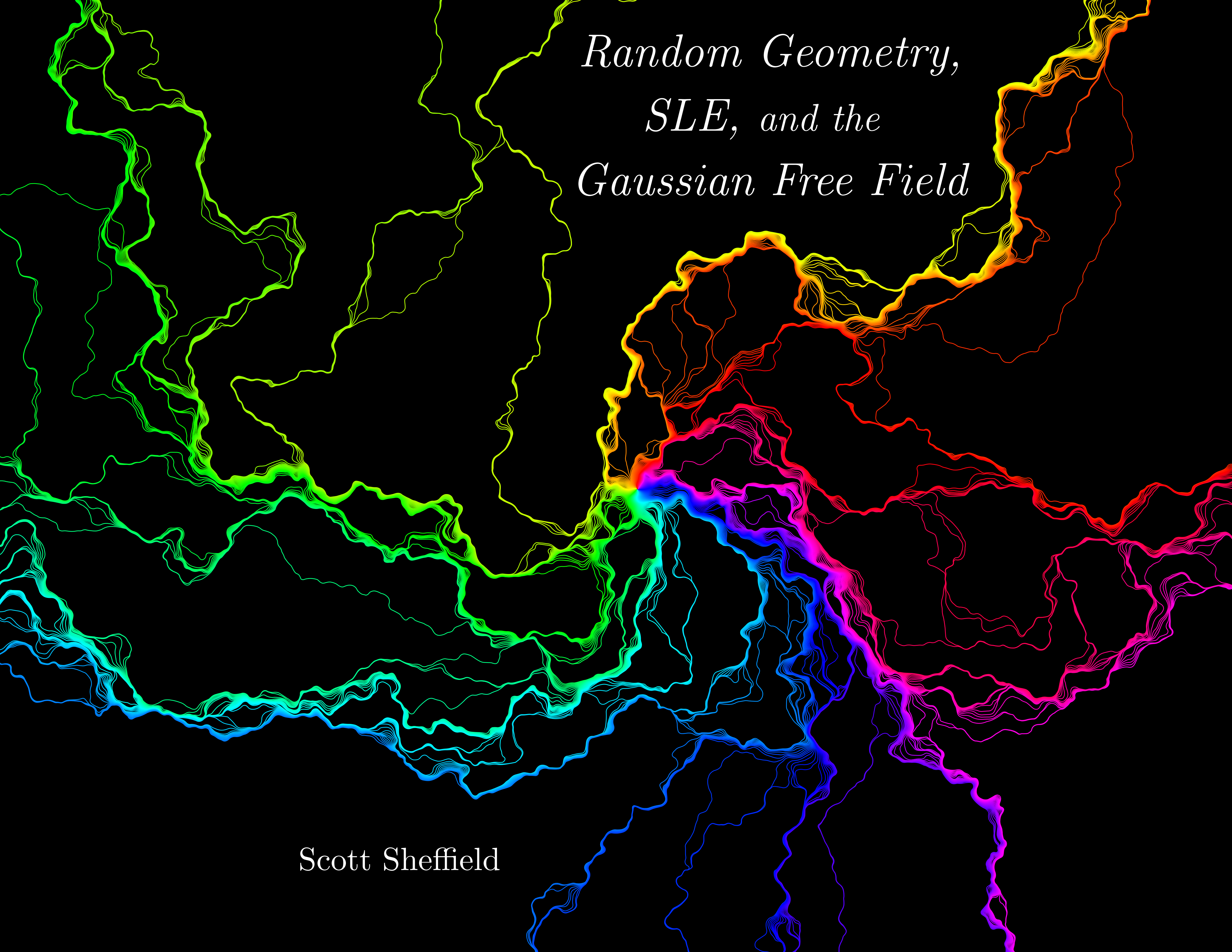


Random Geometry, SLE, and the Gaussian Free Field

Scott Sheffield

References (see arXiv.org):

1. *Gaussian Free Fields for Mathematicians*, S.
2. *Harmonic Explorer and its convergence to SLE(4)*, S. and Schramm
3. *Contour lines of the two-dimensional discrete Gaussian free field*, S. and Schramm
4. *Exploration trees and conformal loop ensembles*, S.
5. *Conformal radii for conformal loop ensembles*, S., Schramm, and Wilson
6. More figures and talks: math.nyu.edu/faculty/sheff

A few SLE motivated goals

1. Construct a "random geometry" in which the SLE curves and SLE loops (the **conformal loop ensembles**) appear naturally as geometric objects.

2. Understand **Duplantier duality** — the relationship between SLE(κ) and SLE($16/\kappa$) — by realizing both objects within the same geometry.

3. Understand what is special about the self-dual value $\kappa = 4 = 16/\kappa$.

4. Understand the time reversal symmetry of SLE.

The *standard Gaussian* on n-dimensional Hilbert space

has density function $e^{-(v,v)/2}$ (times an appropriate constant). We can write a sample from this distribution as

$$\sum_{i=1}^{n} \alpha_i v_i$$

where the v_i are an orthonormal basis for $\mathbb{R}^n$ under the given inner product, and the α_i are mean zero, unit variance Gaussians.

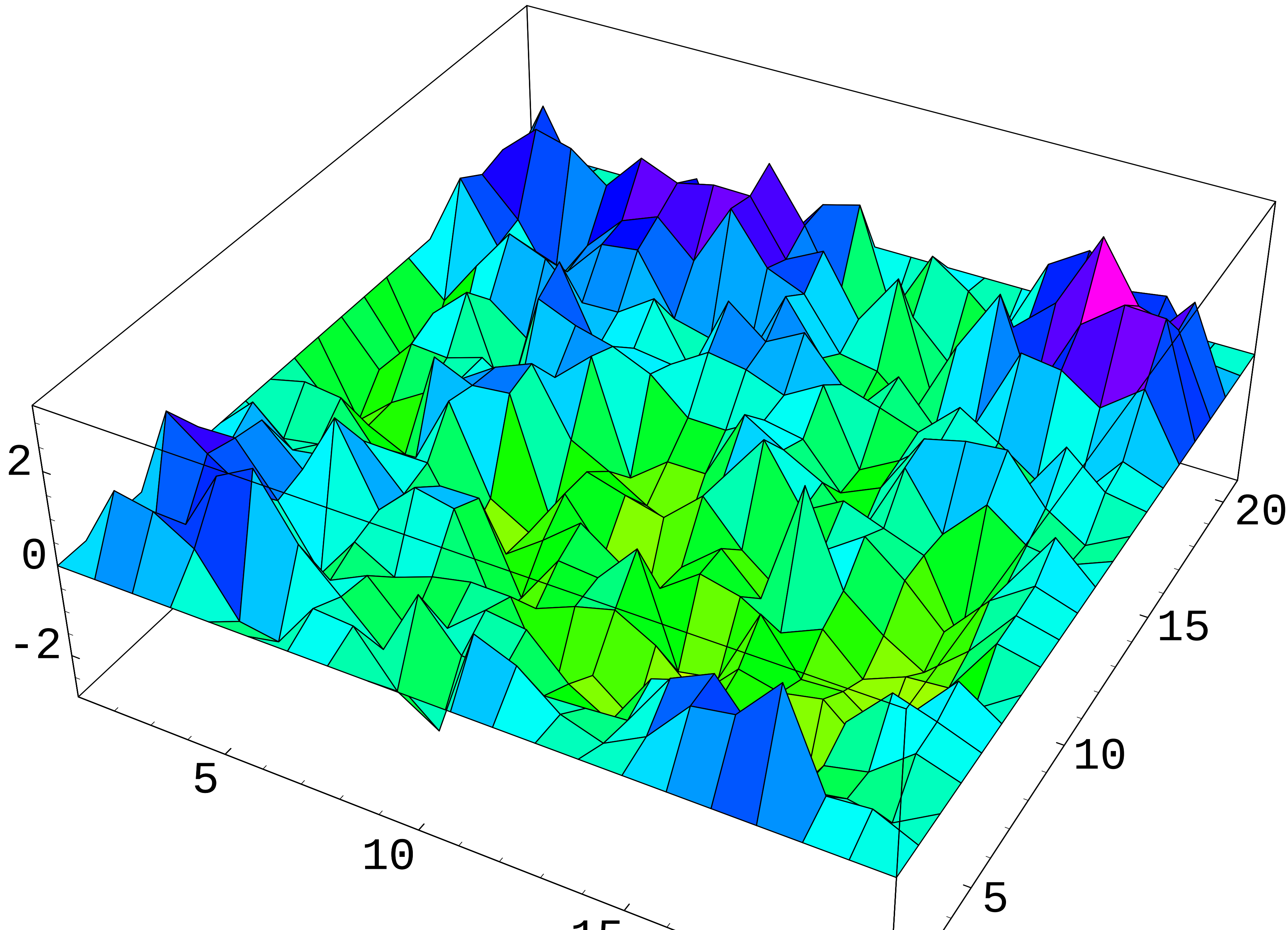
The discrete Gaussian free field

Let f and g be real functions defined on the vertices of a planar graph Λ. The **Dirichlet inner product** of f and g is given by

$$(f, g)_\nabla = \sum_{x \sim y} (f(x) - f(y))\,(g(x) - g(y))\,.$$

The value $H(f) = (f, f)_\nabla$ is called the **Dirichlet energy of f.** Fix a function f_0 on boundary vertices of Λ. The set of functions f that agree with f_0 is isomorphic to $\mathbb{R}^n$, where n is the number of interior vertices. The **discrete Gaussian free field** is a random element of this space with probability density proportional to $e^{-H(f)/2}$.

Discrete GFF on 20×20 grid, zero boundary



Some DGFF properties:

Zero boundary conditions: The Dirichlet form $(f, f)_\nabla$ is an inner product on the space of functions with zero boundary, and the DGFF is a standard Gaussian on this space.

Other boundary conditions: DGFF with boundary conditions f_0 is the same as DGFF with zero boundary conditions *plus* a deterministic function, which is the (discrete) harmonic interpolation of f_0 to Λ.

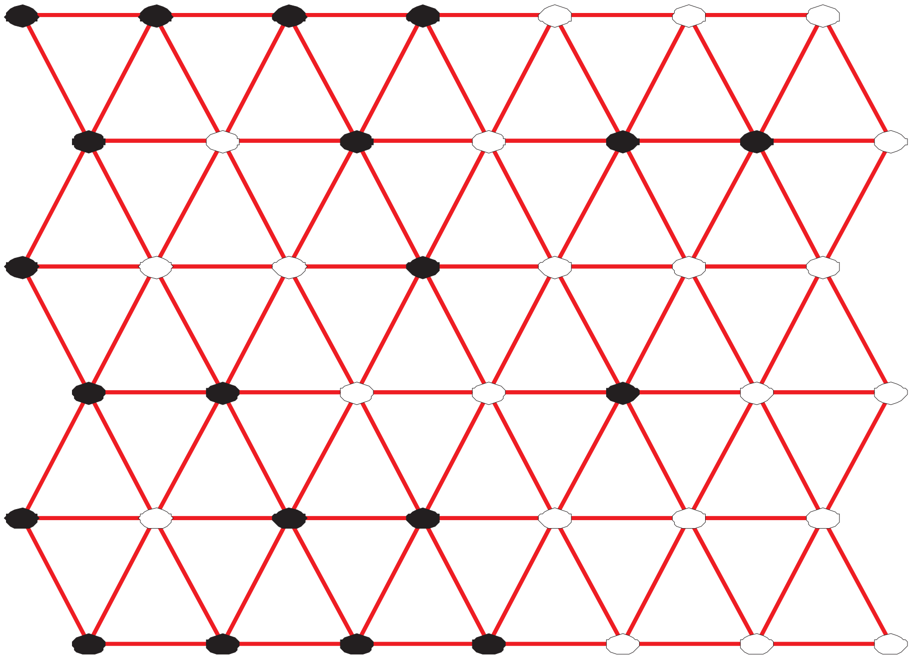
Markov property: **Given** the values of f on the boundary of a subgraph Λ' of Λ, the values of f on the remainder of Λ' have the law of a DGFF on Λ', with boundary condition given by the observed values of f on $\partial\Lambda'$.

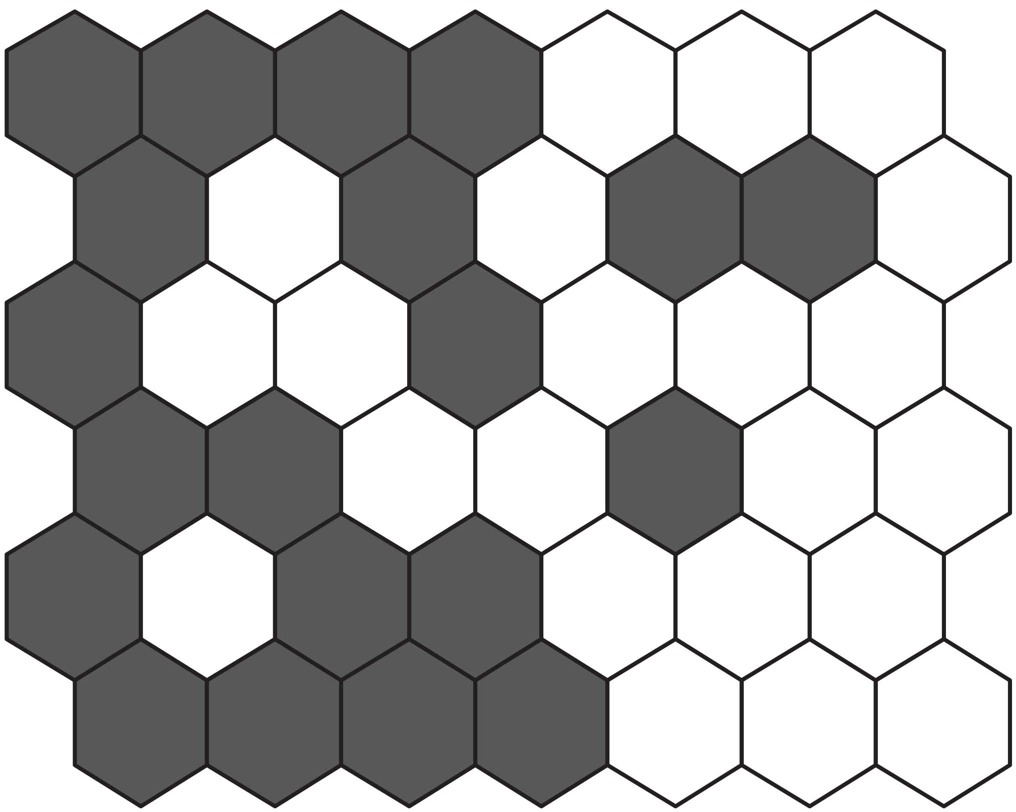
The continuum Gaussian free field

is a "standard Gaussian" on an *infinite* dimensional Hilbert space. Given a planar domain D, let $H(D)$ be the Hilbert space closure of the set of smooth, compactly supported functions on D under the conformally invariant *Dirichlet inner product*

$$(f_1, f_2)_\nabla = \int_D (\nabla f_1 \cdot \nabla f_2) dxdy.$$

The GFF is the formal sum $h = \sum \alpha_i f_i$, where the f_i are an orthonormal basis for H and the α_i are i.i.d. Gaussians. The sum does not converge point-wise, but h can be defined as a *random distribution*—inner products (h, ϕ) are well defined whenever ϕ is sufficiently smooth.



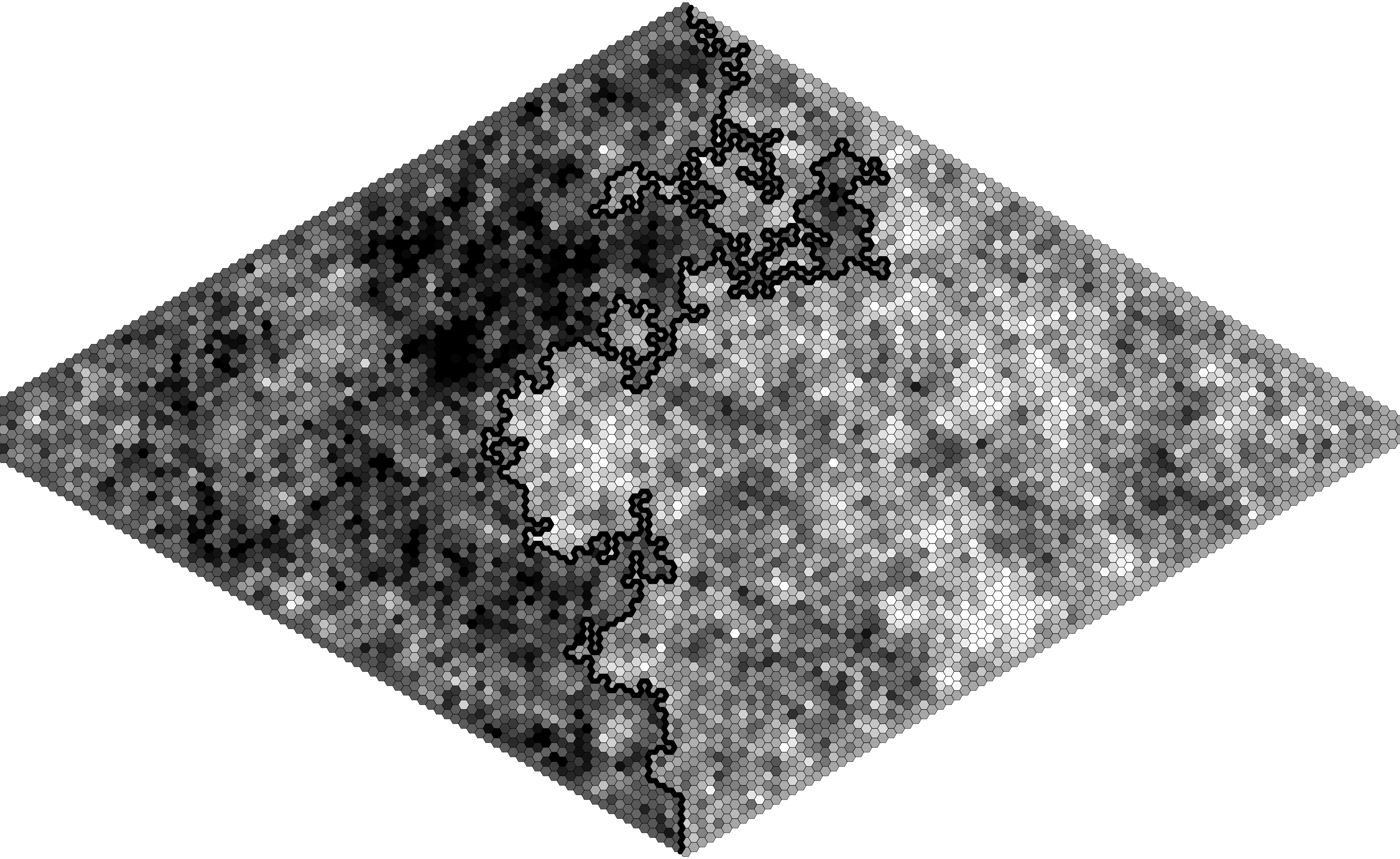


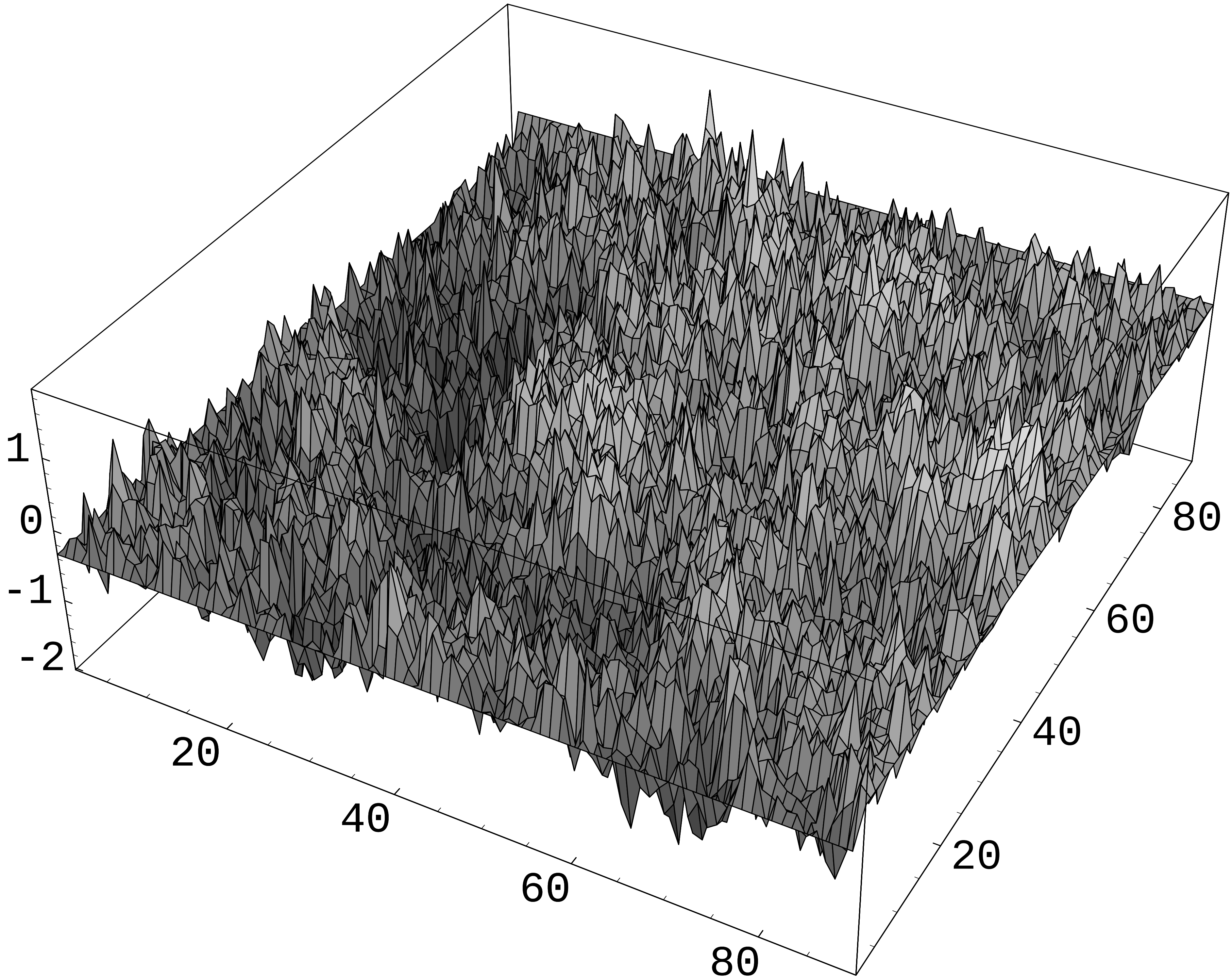
Scaling limit of zero-height contour line

Theorem (Schramm, S): If initial boundary heights are λ on one boundary arc and $-\lambda$ on the complementary arc, where λ is the constant $\sqrt{\frac{\pi}{8}}$, then the scaling limit of the zero-height interface (as the mesh size tends to zero) is SLE_4.

If the initial boundary heights are instead $-(1+a)\lambda$ and $(1+b)\lambda$, then as the mesh gets finer, the laws of the random paths described above converge to the law of $\mathrm{SLE}_{4,a,b}$.

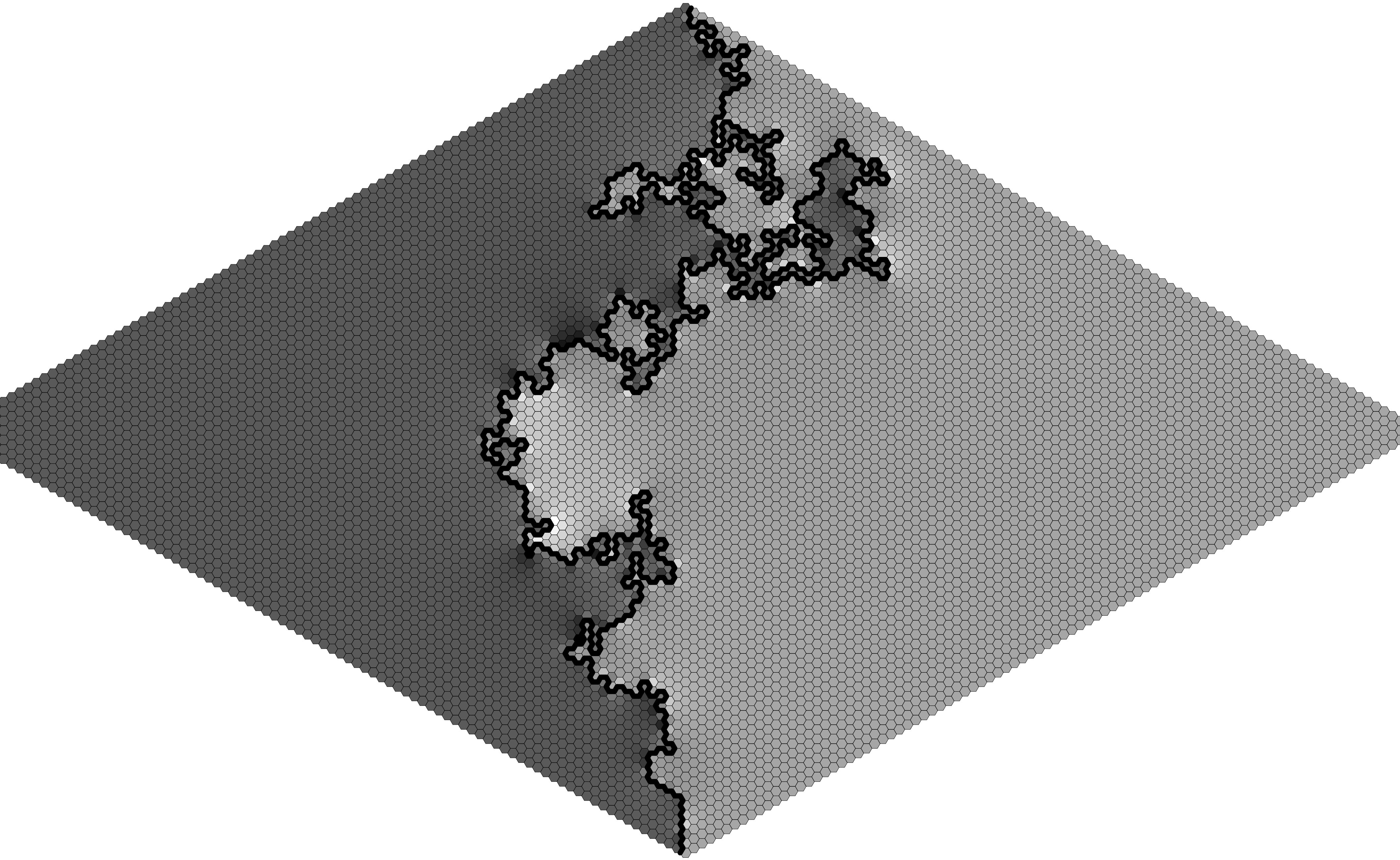
DGFF with $\pm\lambda$ boundary conditions

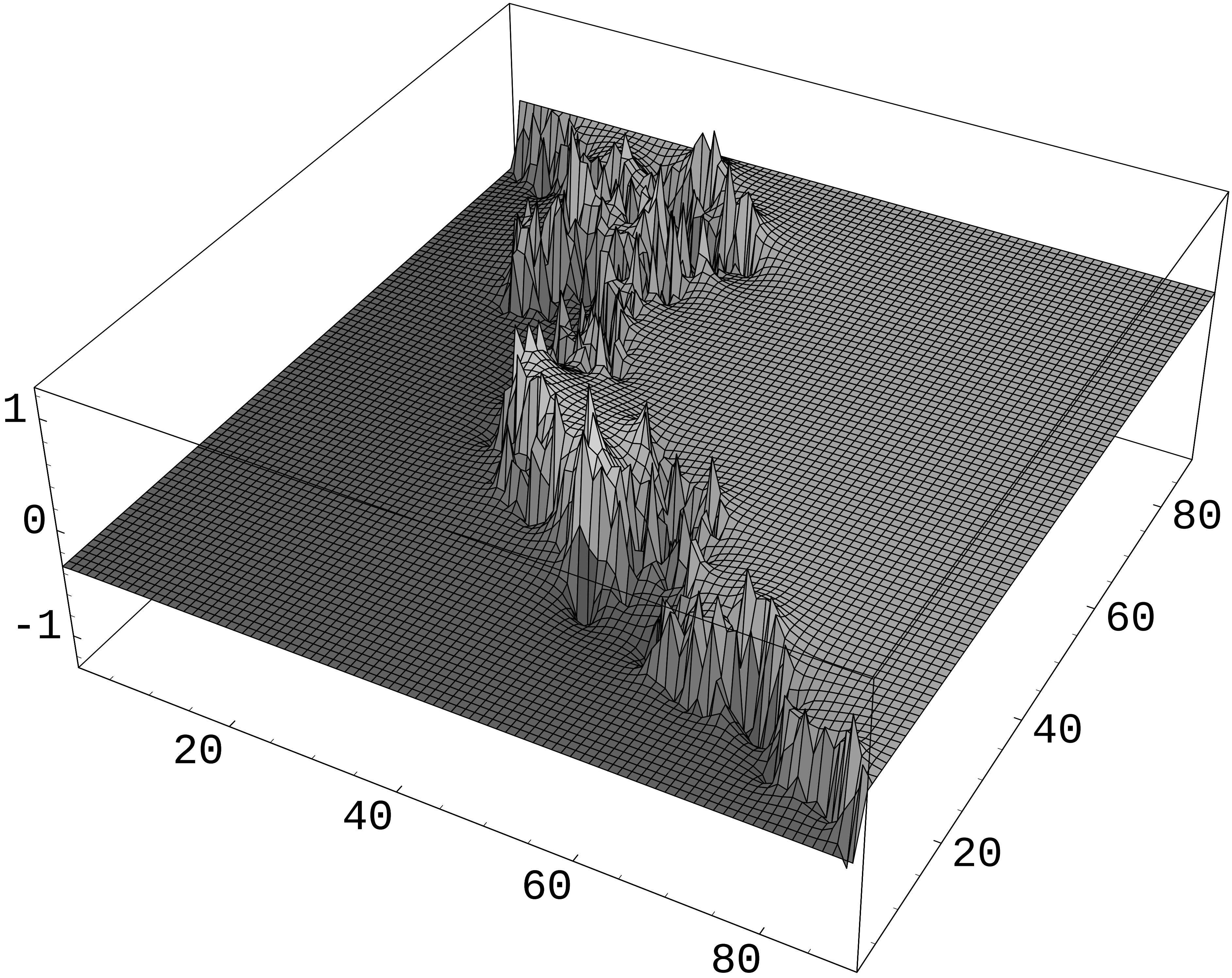




1
0
-1
-2
20
40
60
80
20
40
60
80

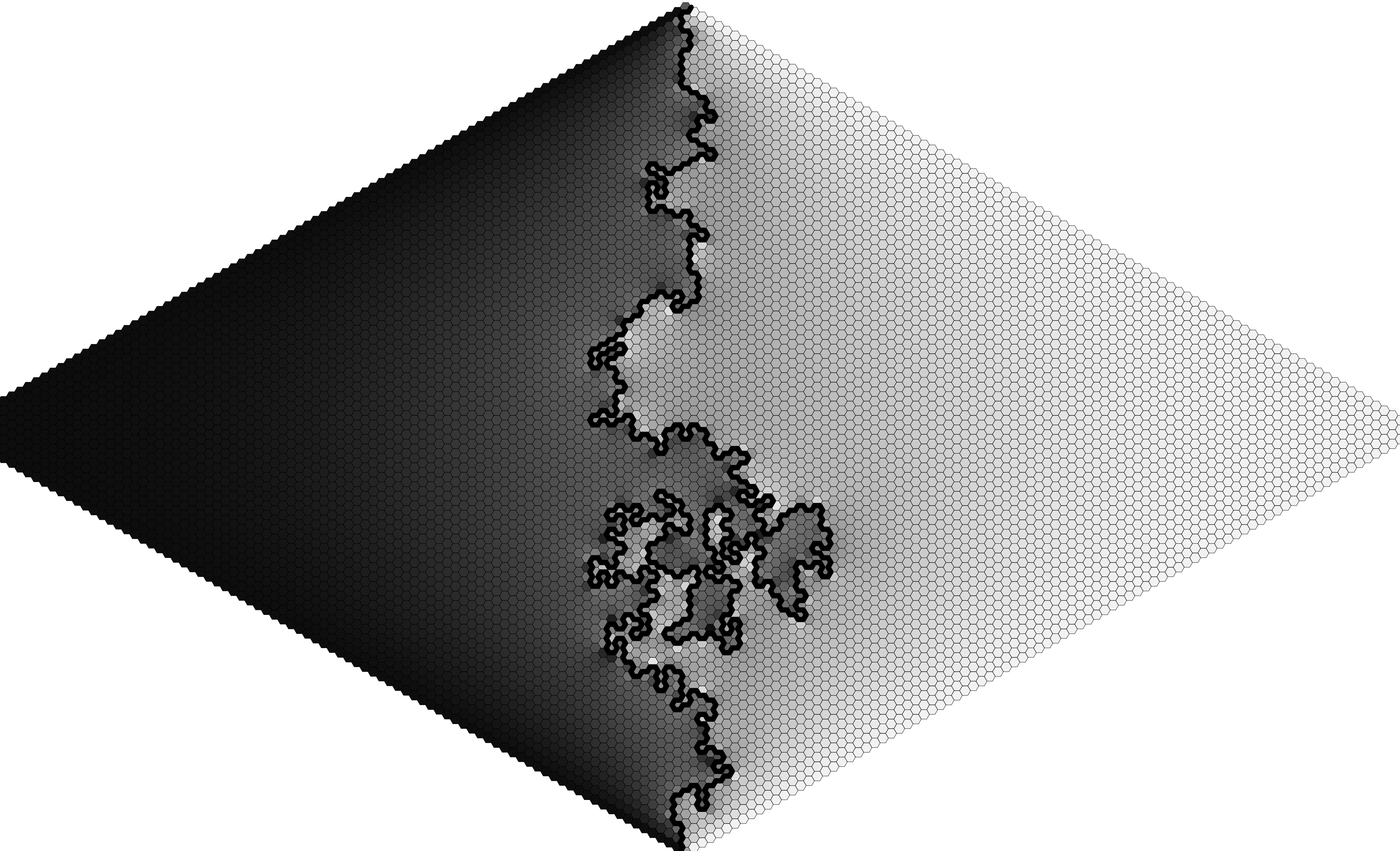
Expectations given values along interface

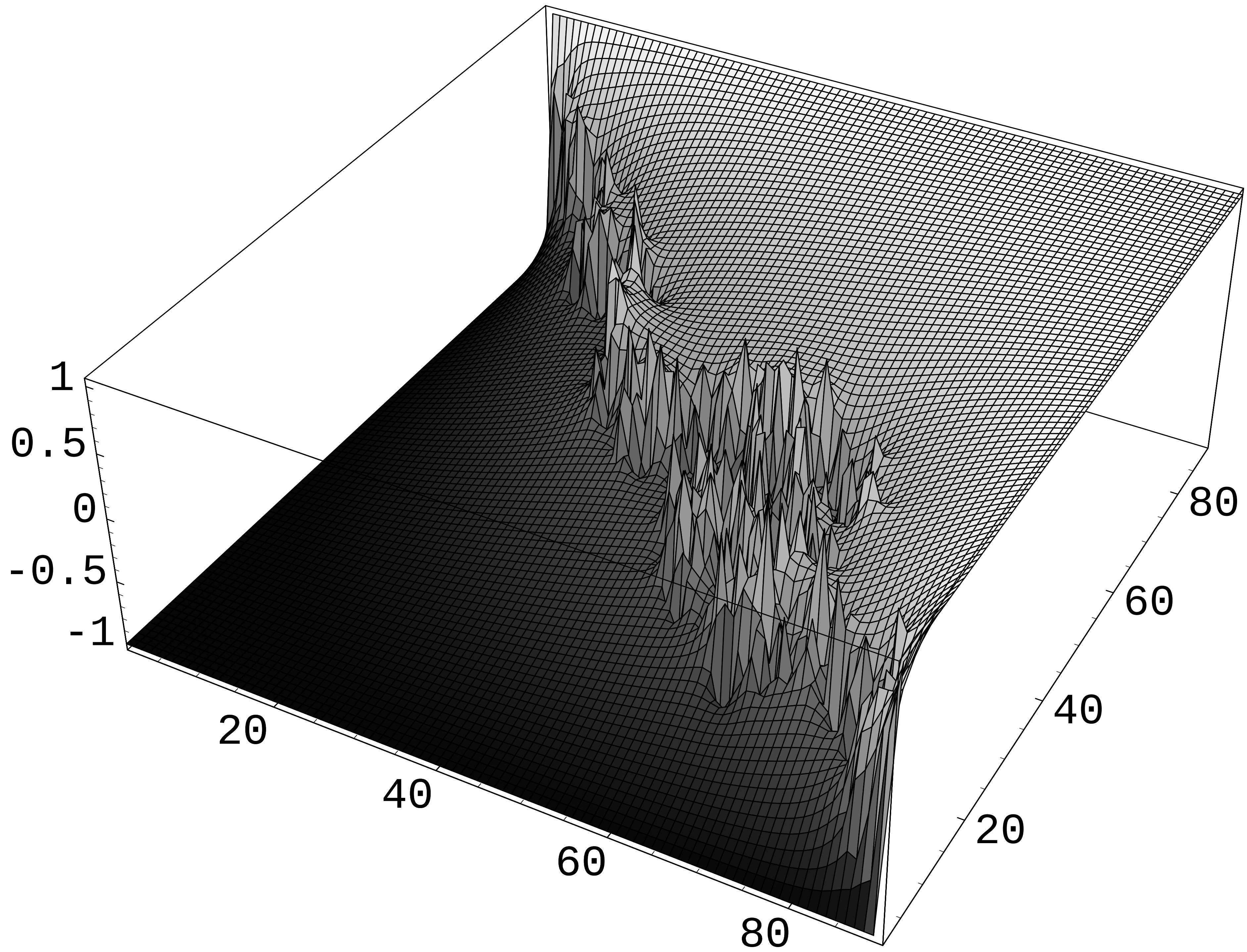




1
0
-1
20
40
60
80
20
40
60
80

Expectations given interface, $\pm 3\lambda$ boundary conditions

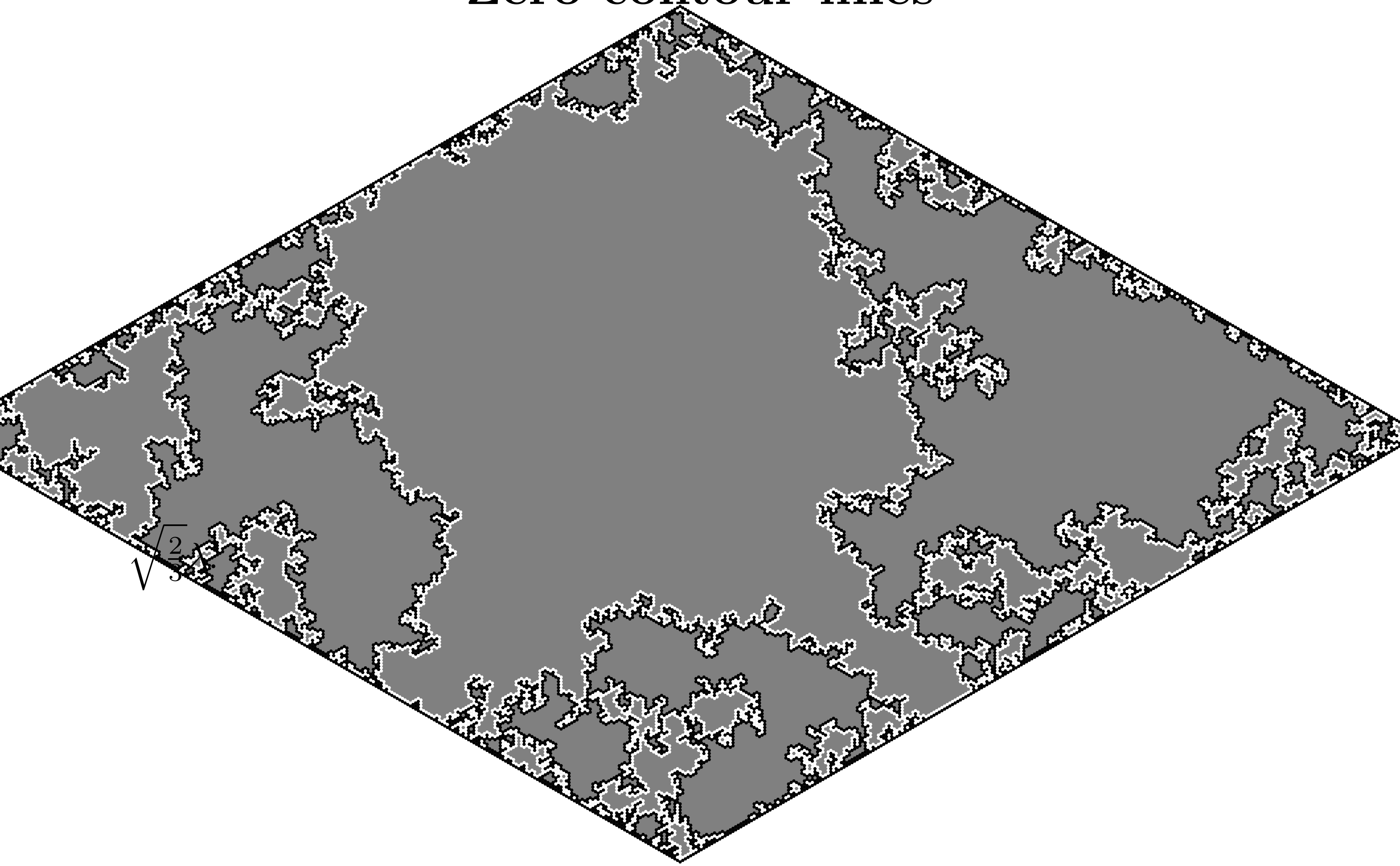



1
0.5
0
-0.5
-1
20
40
60
80
20
40
60
80

Sketch of proof

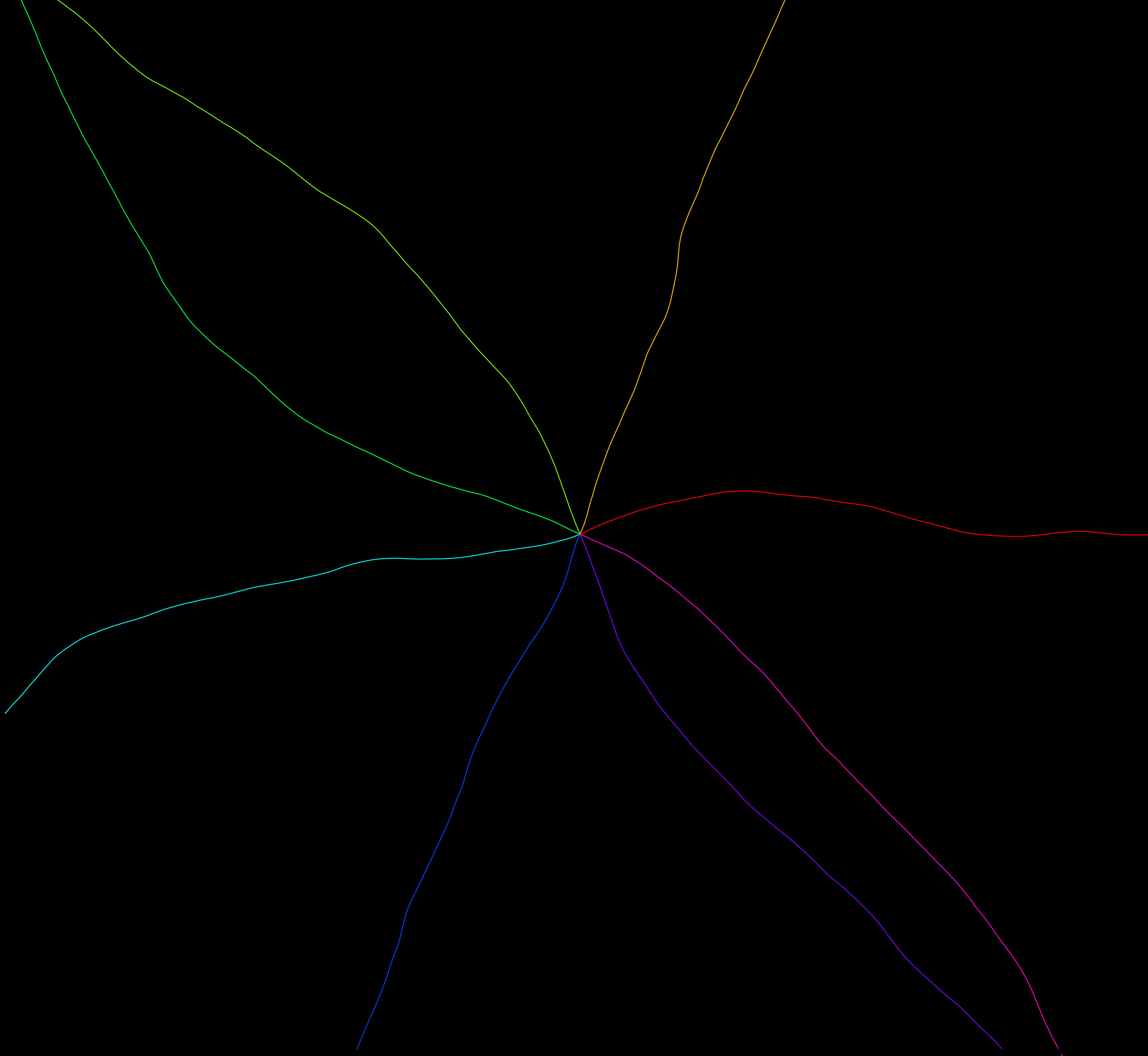
1. Observe that SLE_4 is the only random path γ with the following property: *Given* $\gamma([0,t])$, the probability that γ passes z on right equals the probability that Brownian motion started at z first hits $\mathbb{R} \cup \gamma[0,t]$ on the left side of $\gamma(t)$.
2. Show that any scaling limit of discrete paths has to have the same property.

Zero contour lines

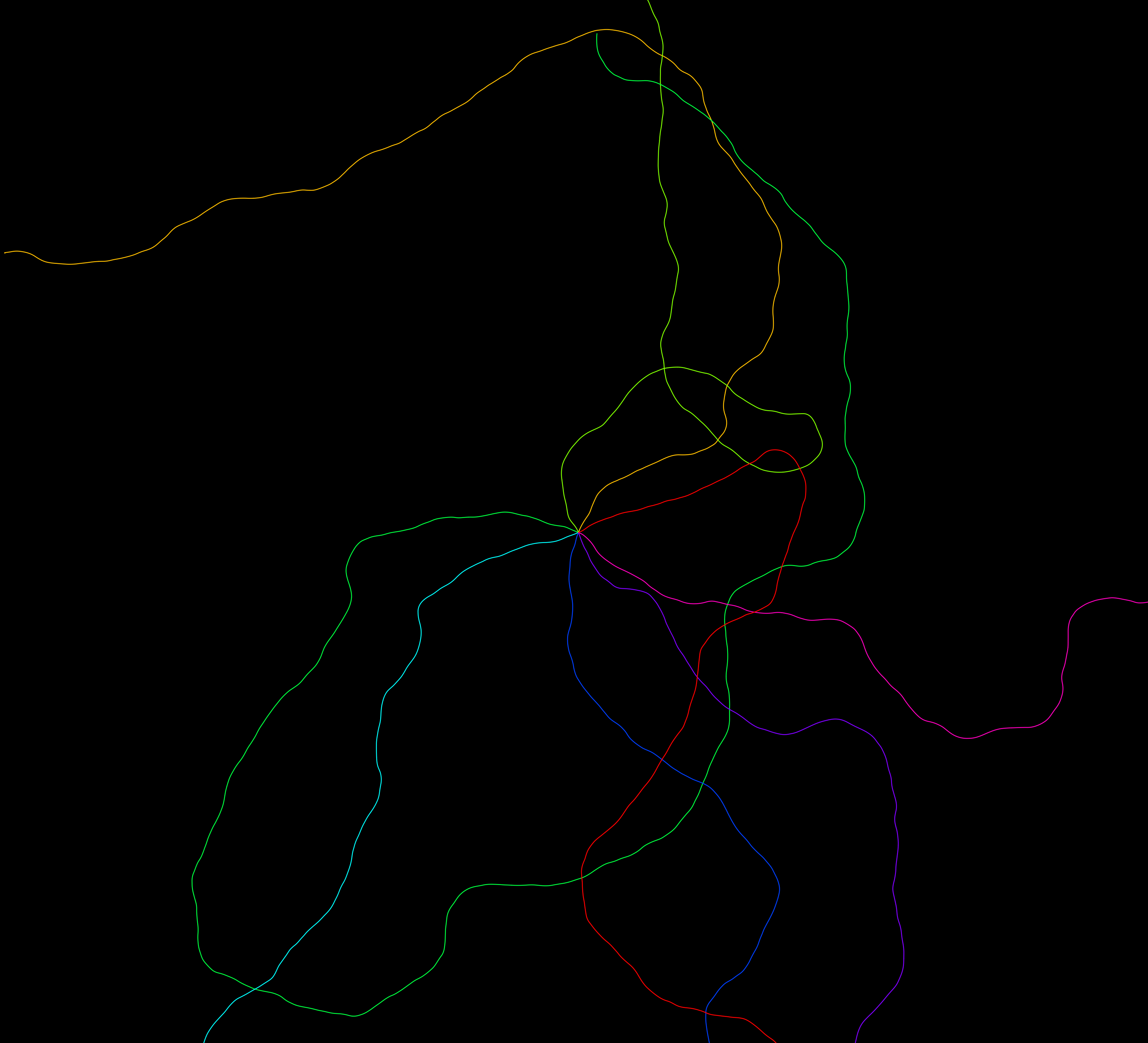


How do we get SLE$_\kappa$ from the GFF when $\kappa \neq 4$?

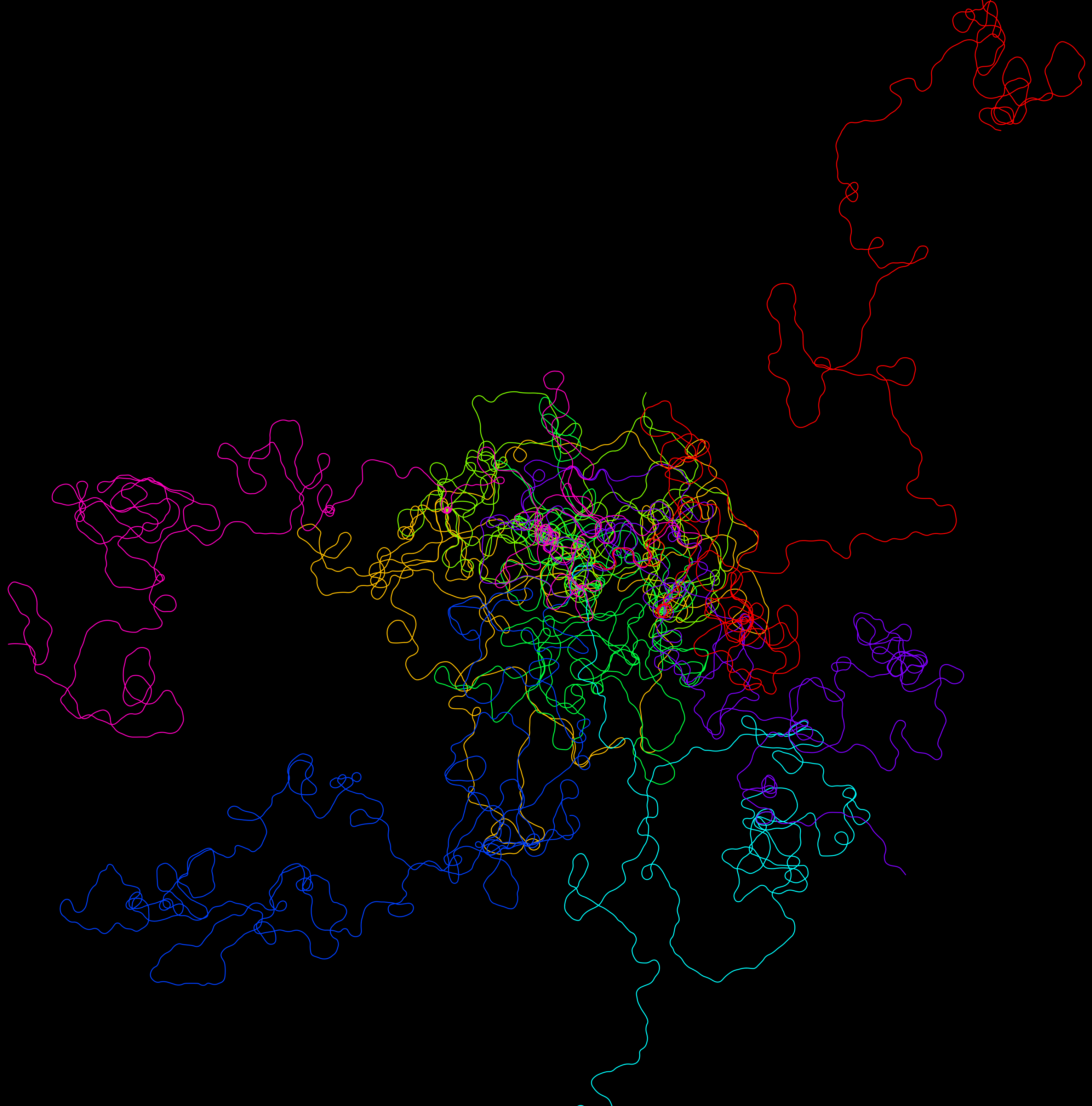
One obvious way to construct a random Riemannian geometry from a continuous approximation h of the GFF: endow the disc with the metric $e^h dL$, where L represents Euclidean length. This geometry is flat if and only if h is harmonic.



Geodesics flows of metric $e^h dL$ where h is .05 times the GFF.

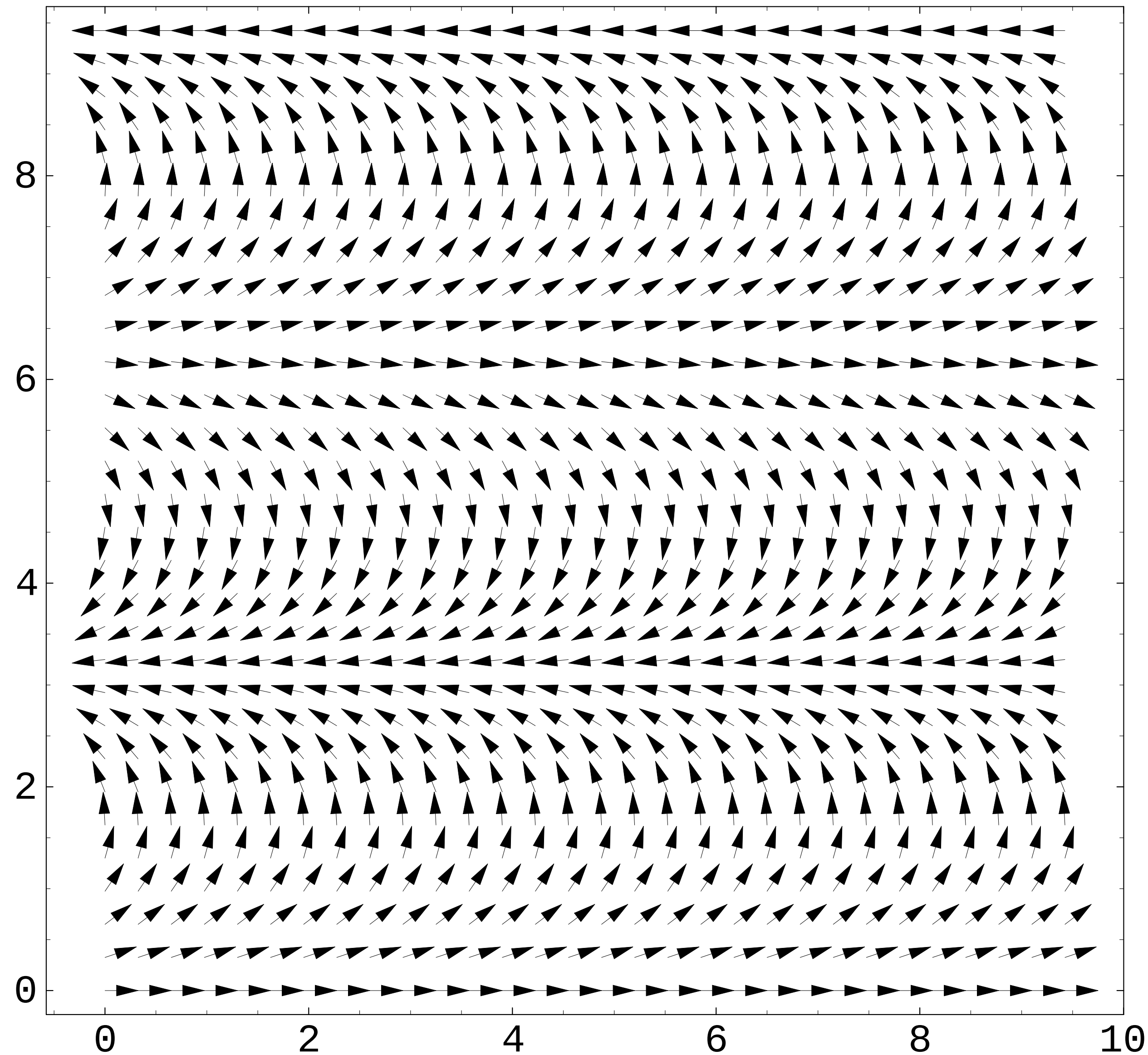


Geodesics flows of metric $e^h dL$ where h is .2 times the GFF.

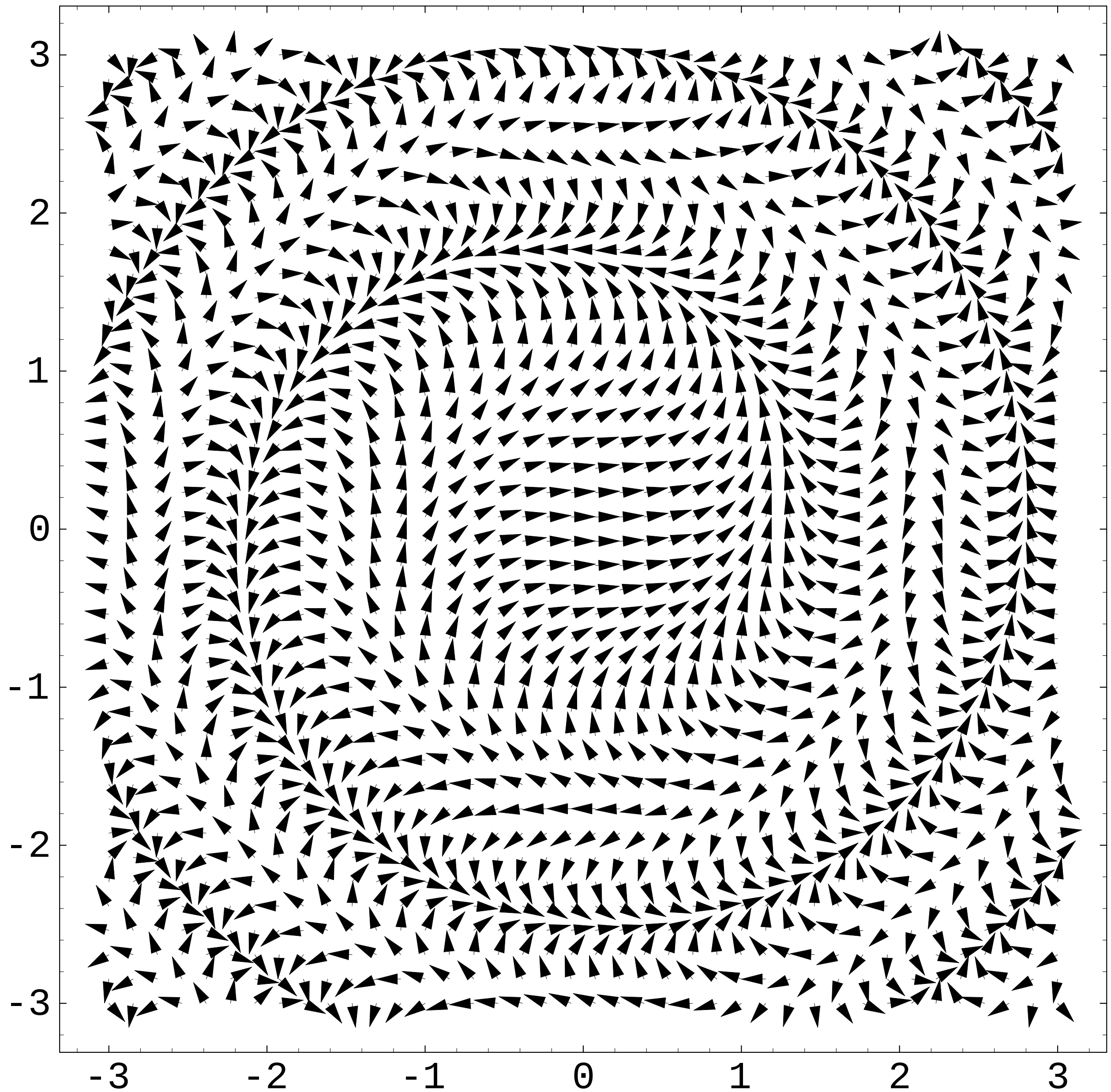


Geodesics flows of metric $e^h dL$ where h is 1 times the GFF.

Vector Field e^{ih}, with $h(x,y) = y$



Vector Field e^{ih} where $h(x,y) = x^2 + y^2$



Altimeter compass geometry

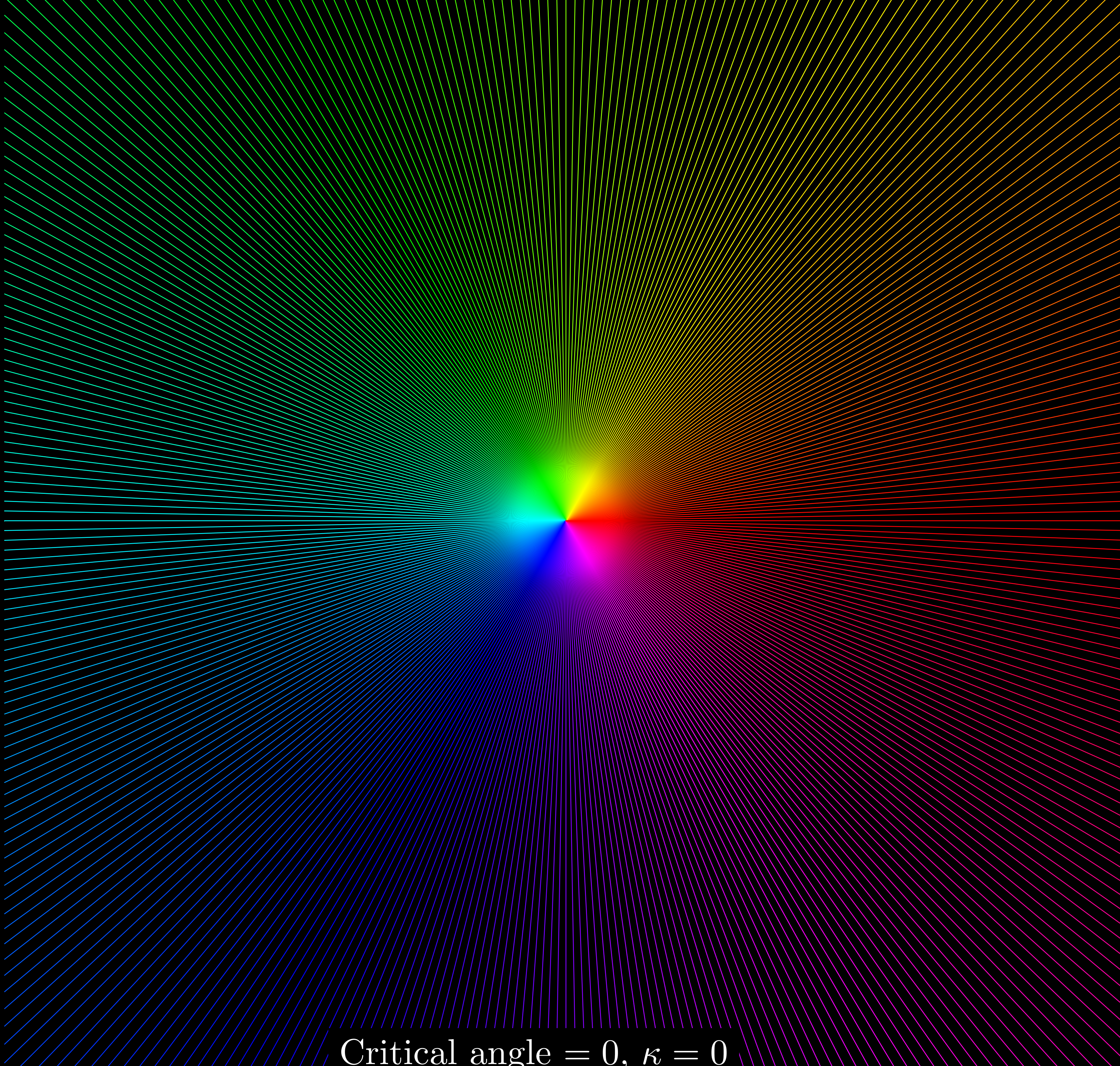
A ray in the altimeter compass geometry is a flow line of $e^{2\pi i(\alpha+h/\chi)}$ for some α.
Now let's modify our sense of direction. Call the direction $e^{2\pi i(\alpha+h/\chi)}$

1. East if $\alpha = 0$.
2. North if $\alpha = .25$.
3. West if $\alpha = .5$.
4. South if $\alpha = .75$.

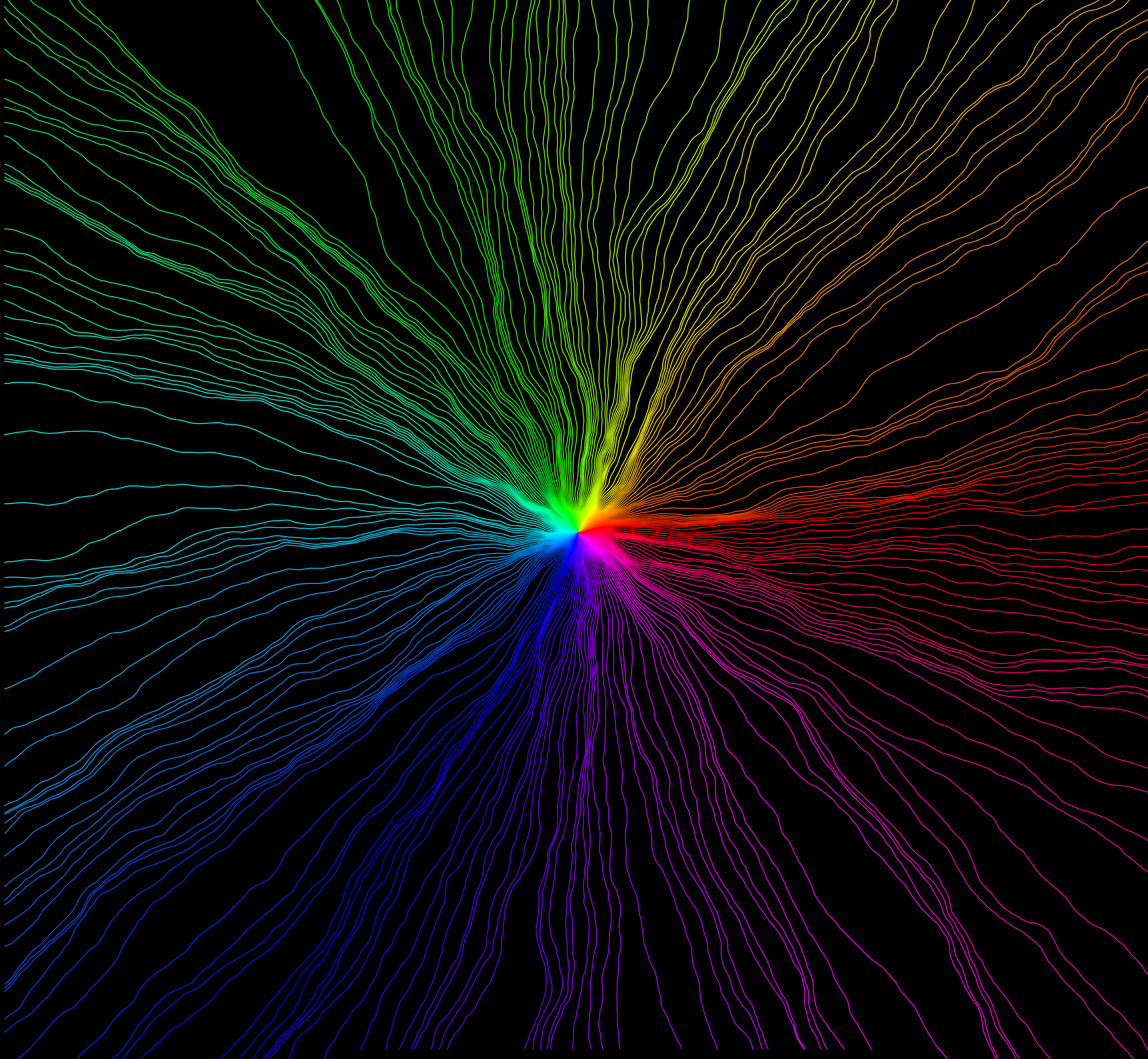
If $h = 0$, then the rays of the AC geometry are those of ordinary Euclidean geometry. More generally, if h is Lipschitz, then the flow line of $e^{2\pi i(\alpha+h/\chi)}$ starting at a given point exists and is uniquely defined.

AC geometry of the GFF

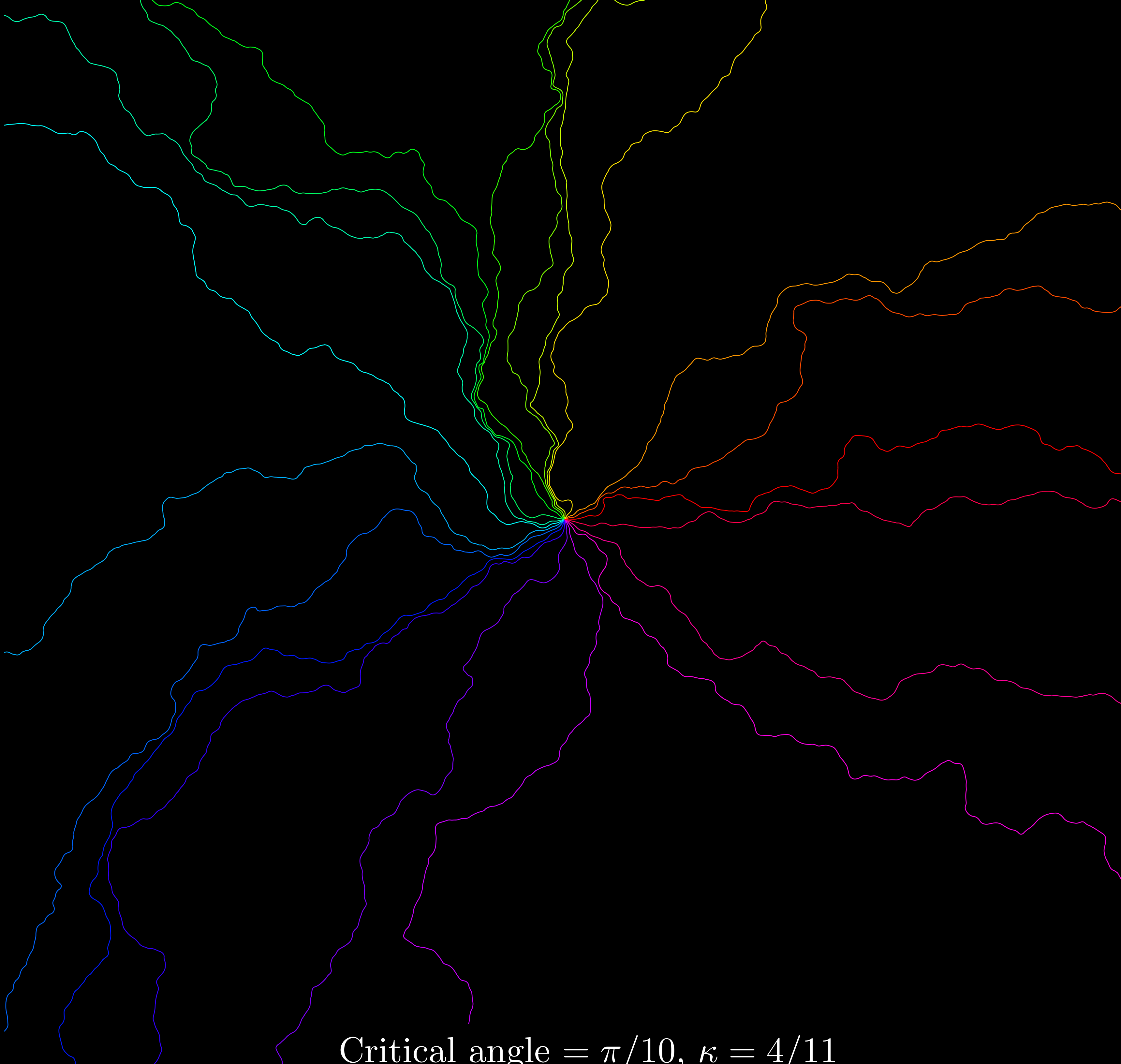
Question: Is there a natural way to define the set of "flow lines" of $e^{ih/\chi}$ when χ is a constant and h is the continuous Gaussian free field?
Answer: Yes. The flow lines are forms of SLE_κ where $0 < \kappa < 4$ and $\chi = \frac{(\kappa/4)^{-1/2} - (\kappa/4)^{1/2}}{2}$. As in the case of contour lines, there is a constant "height gap" between one side of the flow line and the other. We may view this gap as an "angle gap." In radians, the gap is $\frac{\kappa\pi}{4-\kappa}$, i.e., $\frac{\kappa}{2(4-\kappa)}$ revolutions. This gap is called the **critical angle**.



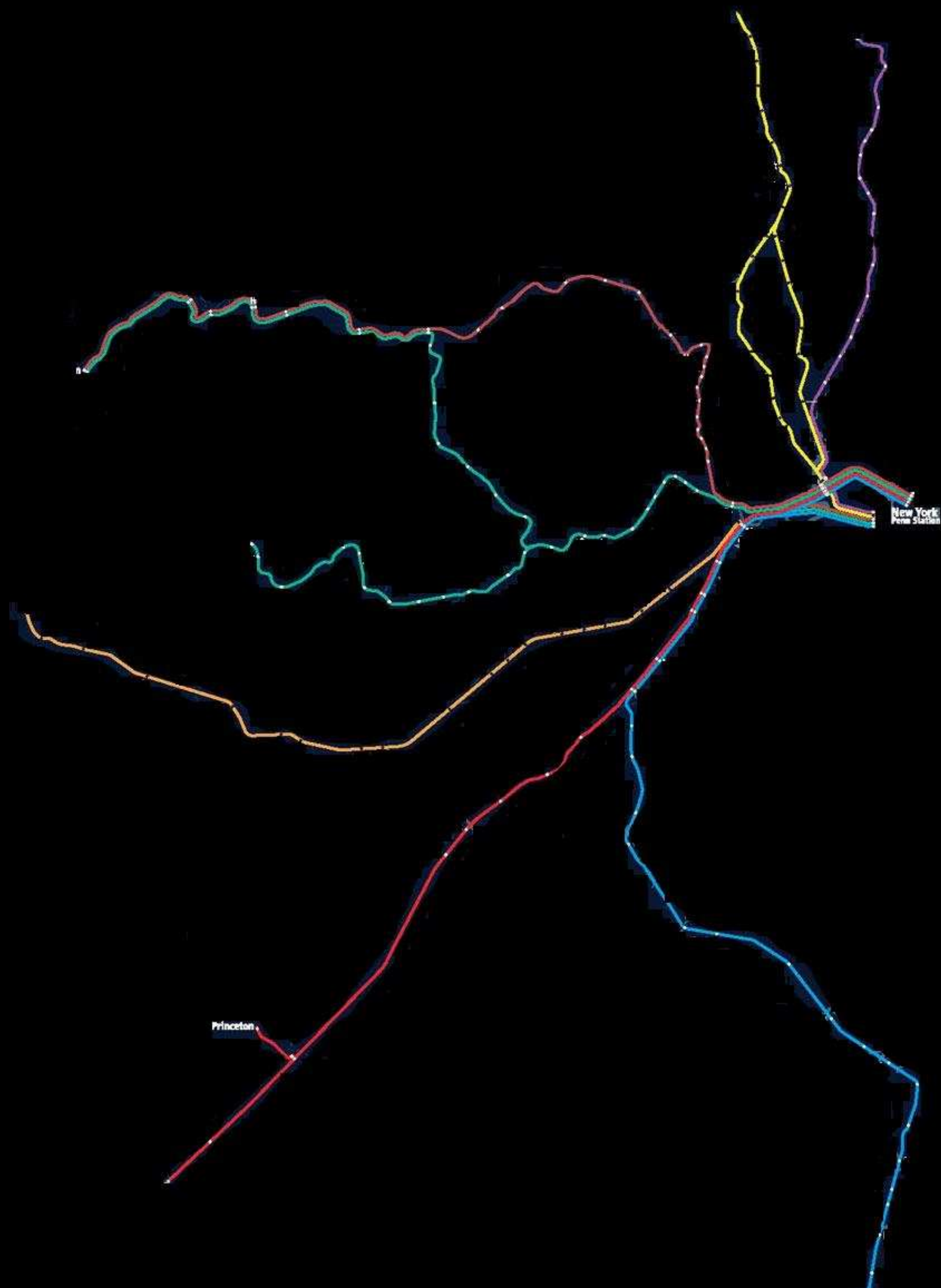
Critical angle $= 0$, $\kappa = 0$



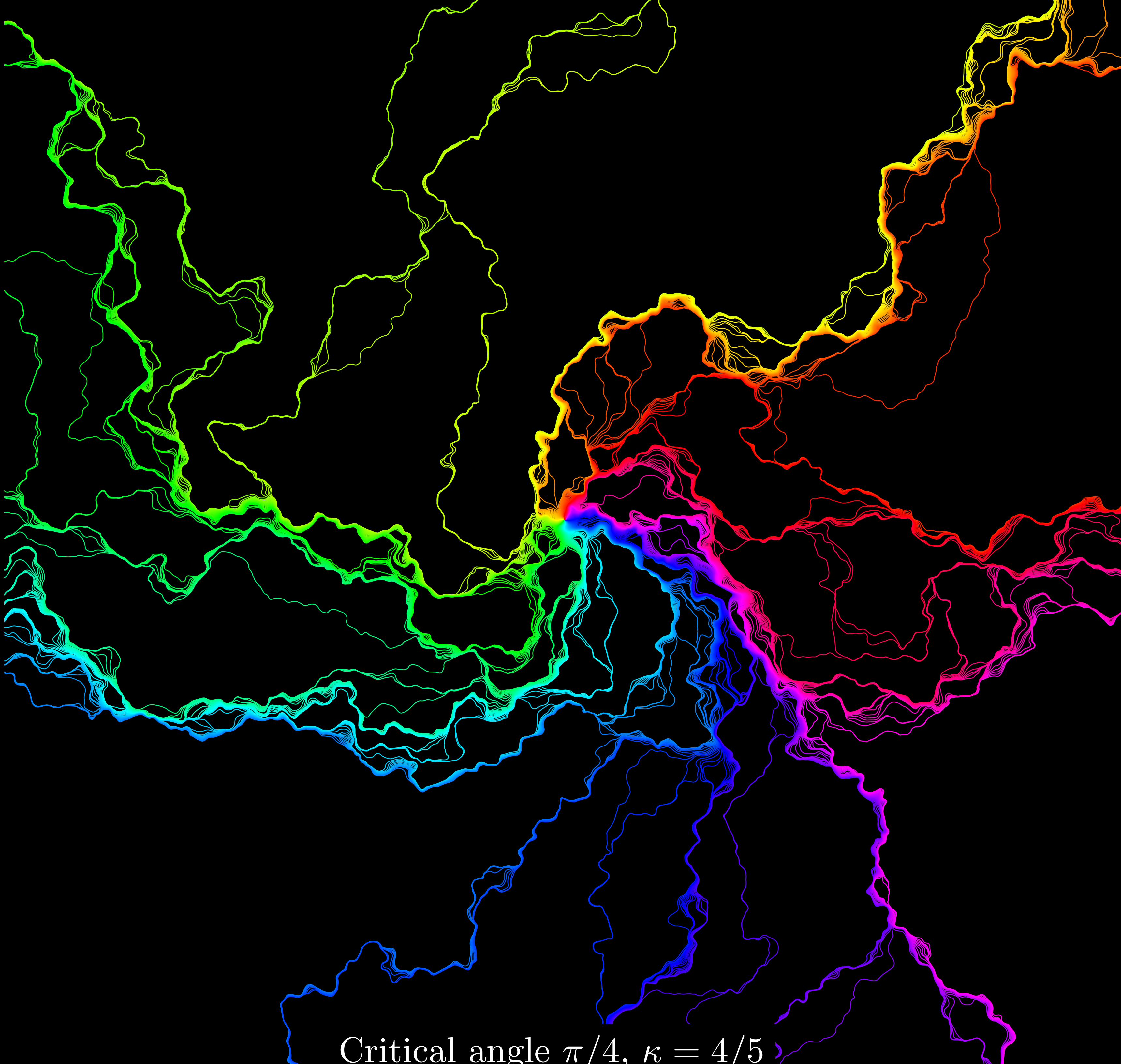
Critical angle = $\pi/100$, $\kappa = 4/101$



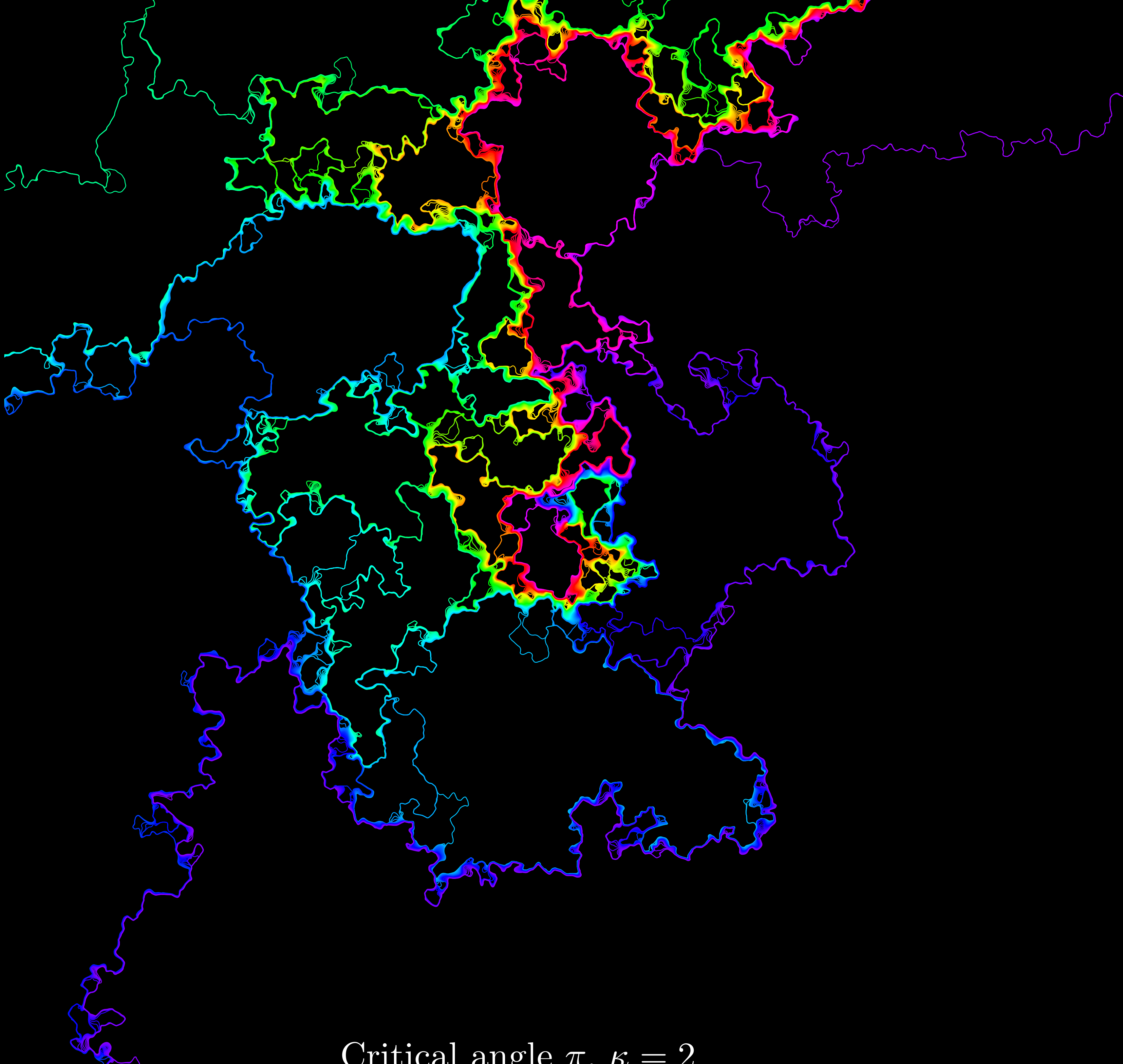
Critical angle $= \pi/10$, $\kappa = 4/11$



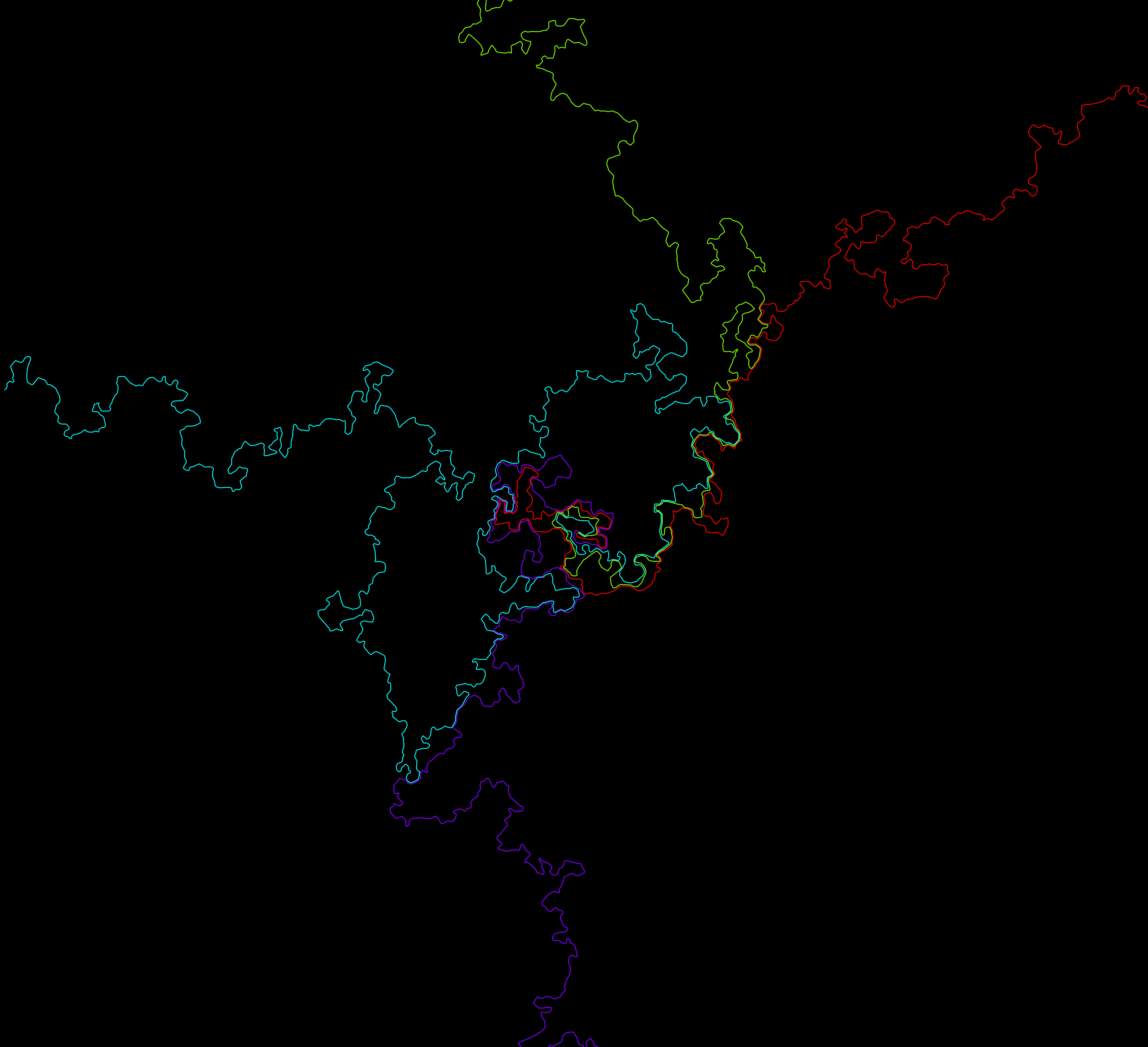
Critical angle $\pi/10$?



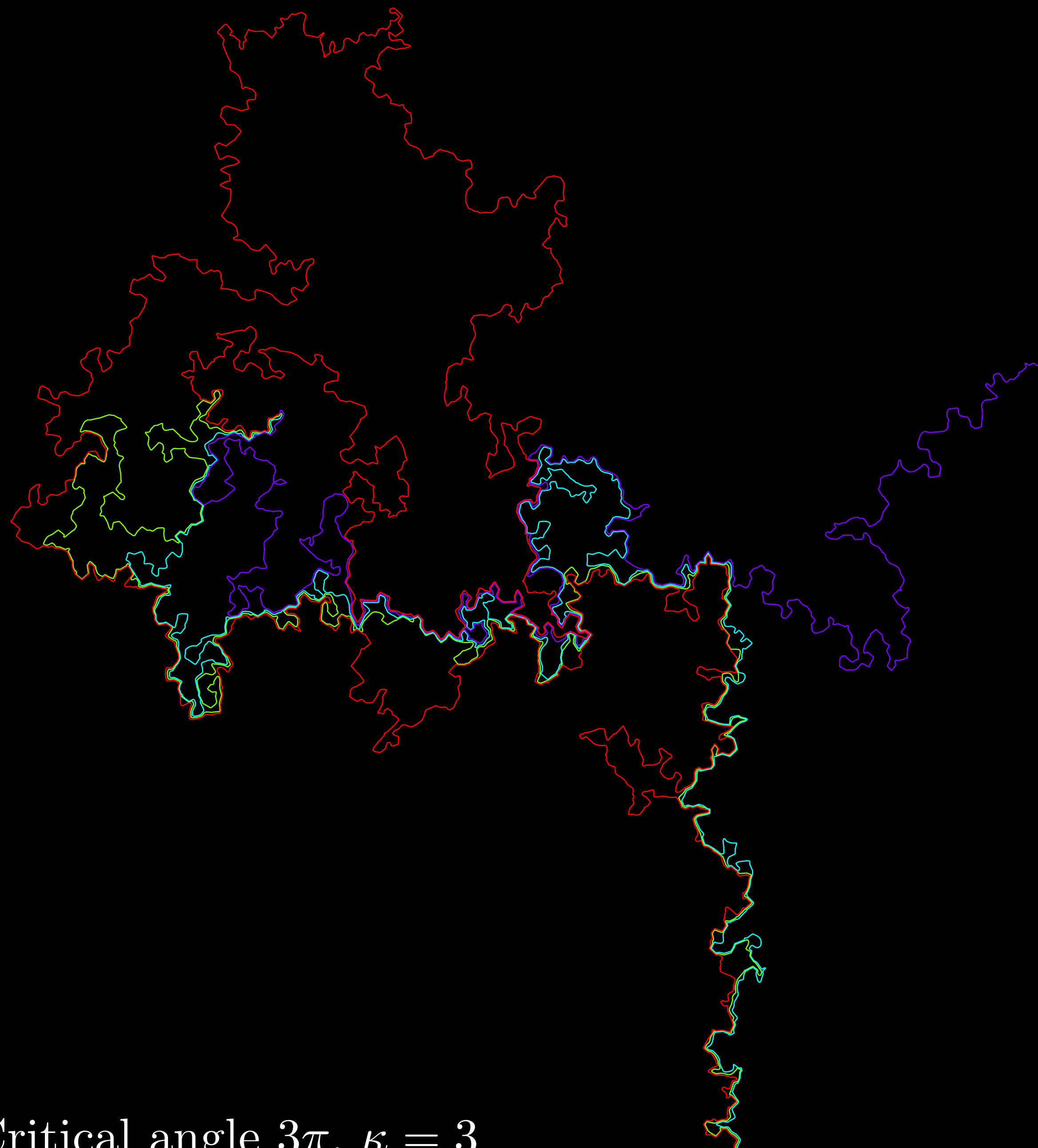
Critical angle $\pi/4$, $\kappa = 4/5$



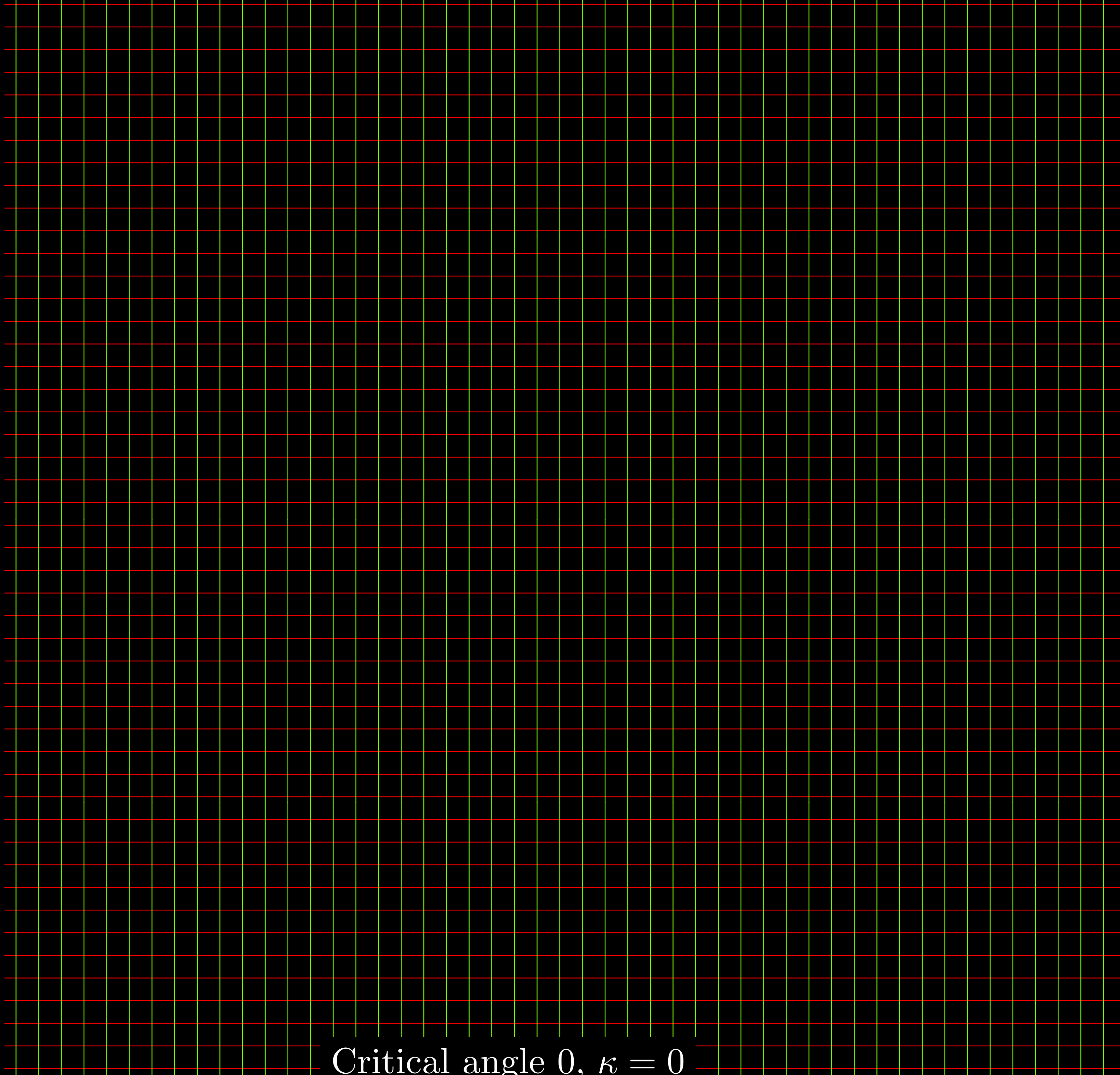
Critical angle π, $\kappa = 2$



Critical angle 2π, $\kappa = 8/3$

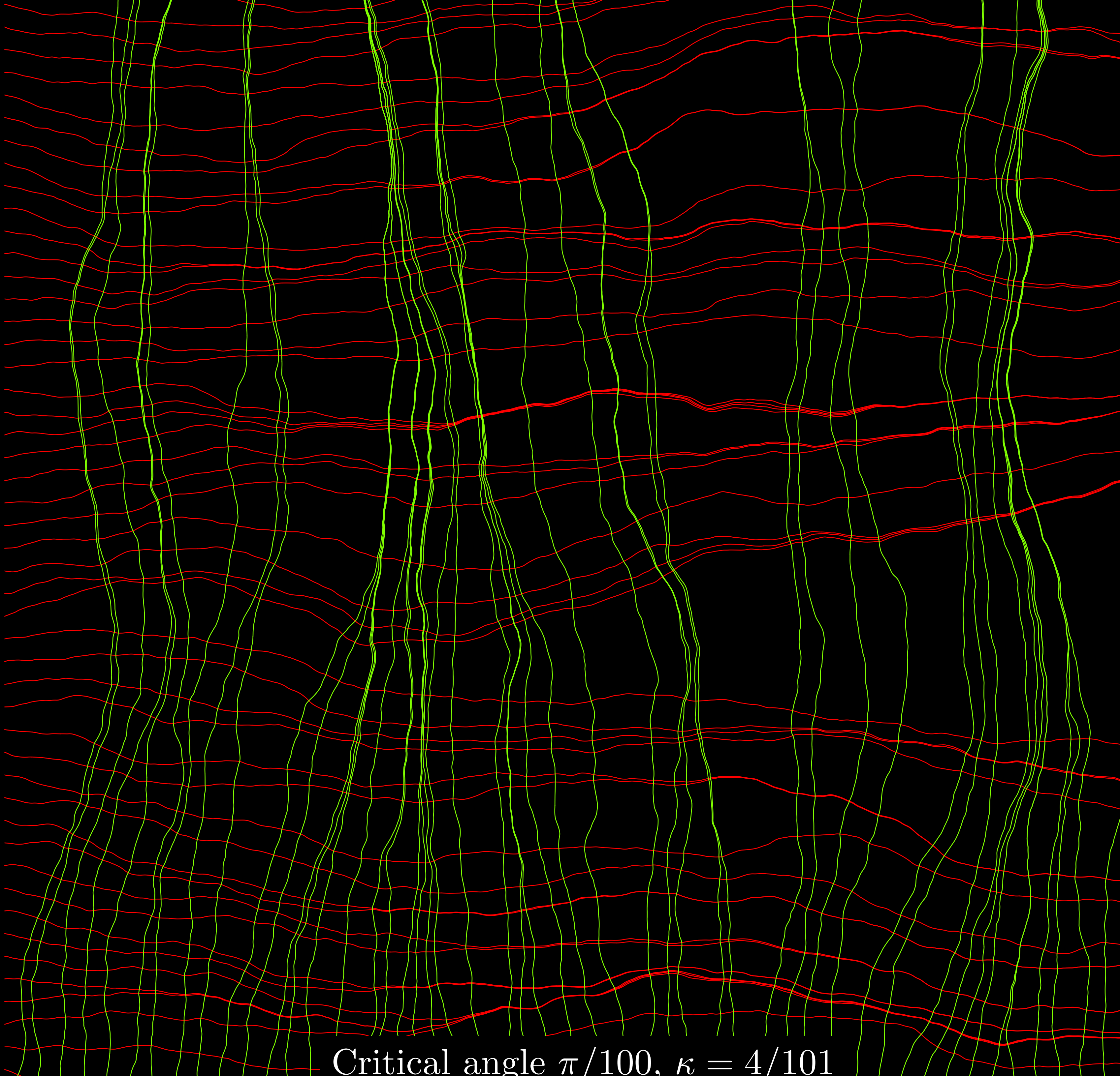


Critical angle 3π, $\kappa = 3$

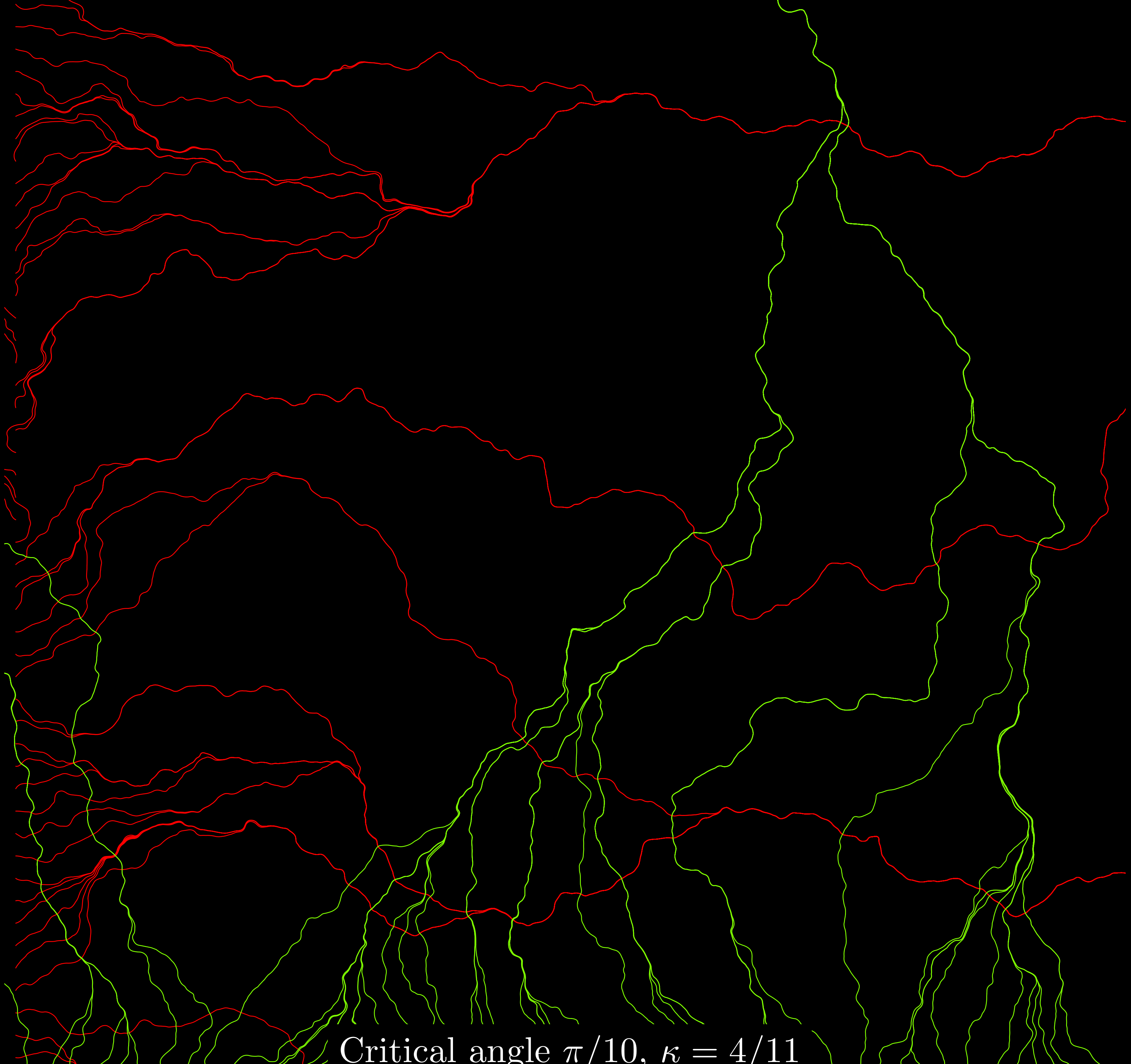


Critical angle 0, $\kappa = 0$

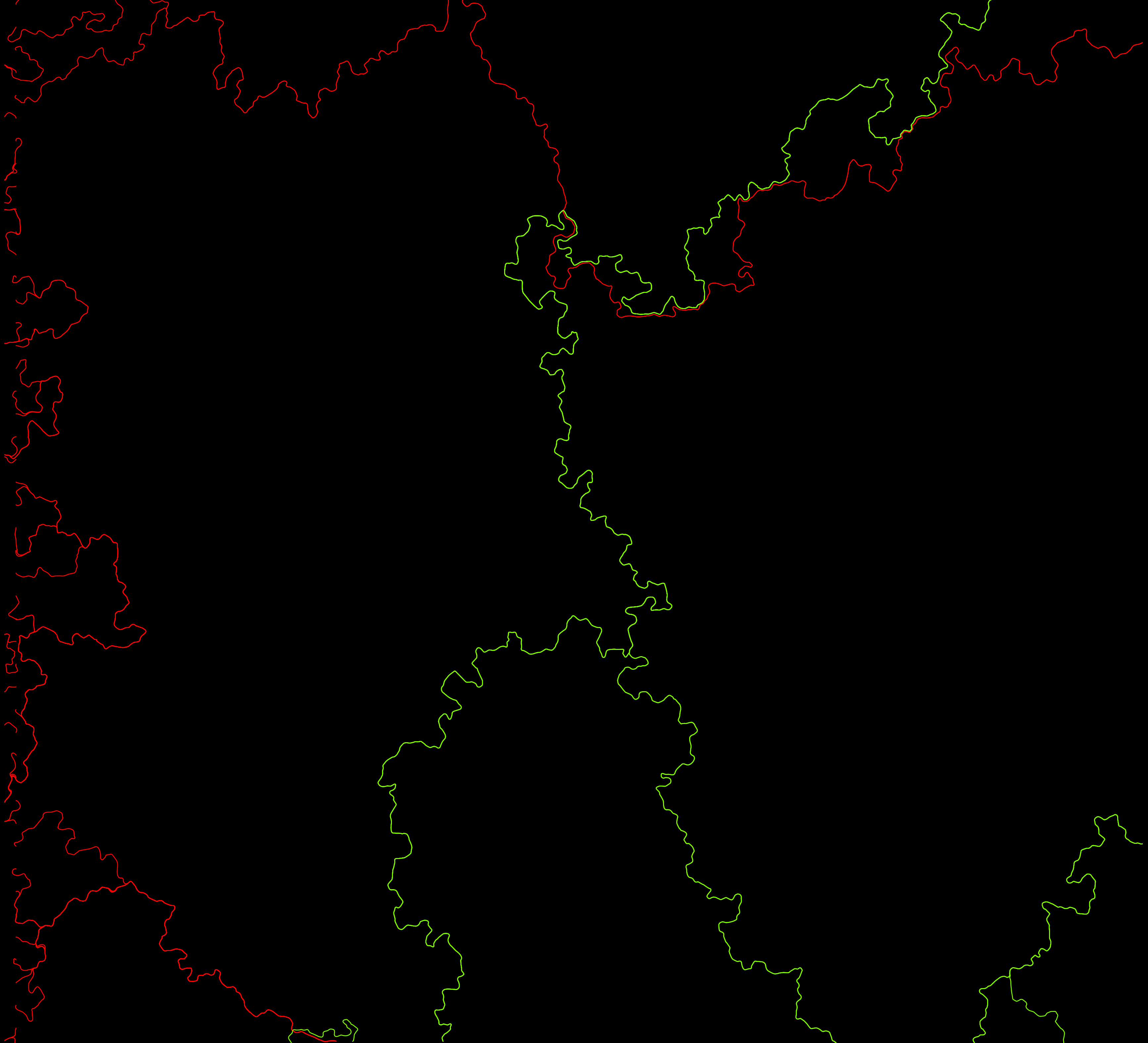
Critical angle $\pi/1000$, $\kappa = 4/1001$



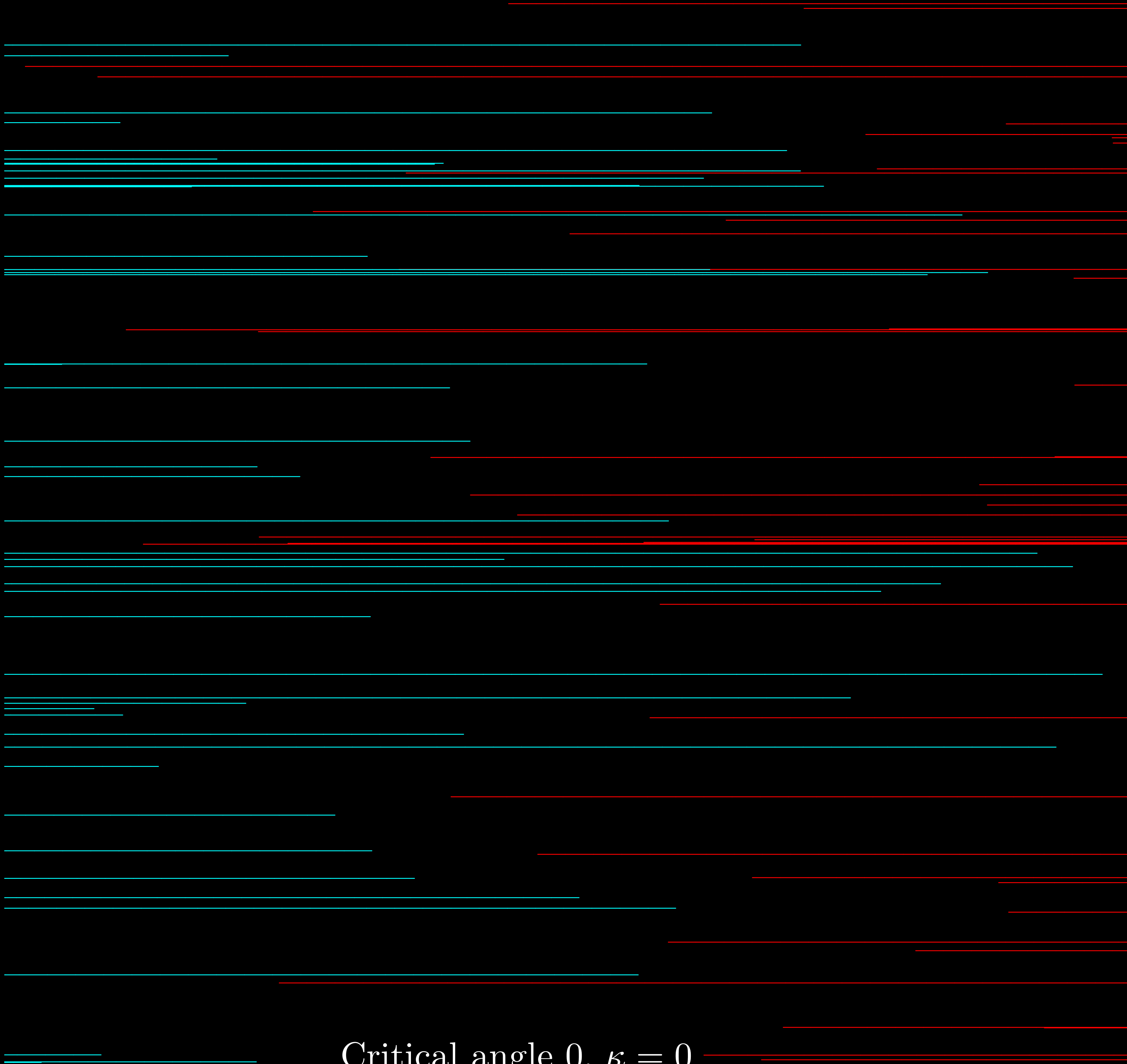
Critical angle $\pi/100$, $\kappa = 4/101$



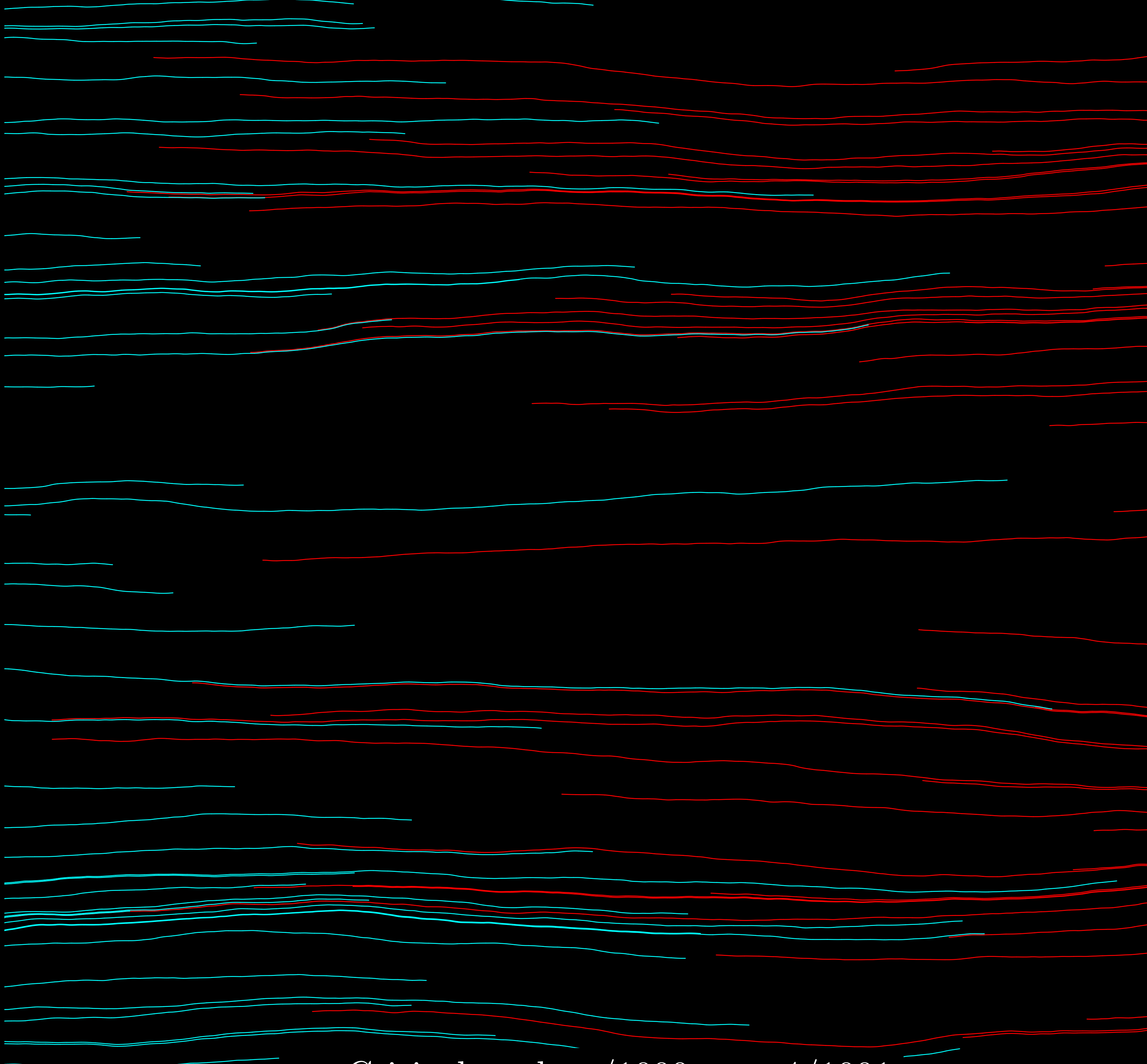
Critical angle $\pi/10$, $\kappa = 4/11$



Critical angle π, $\kappa = 2$

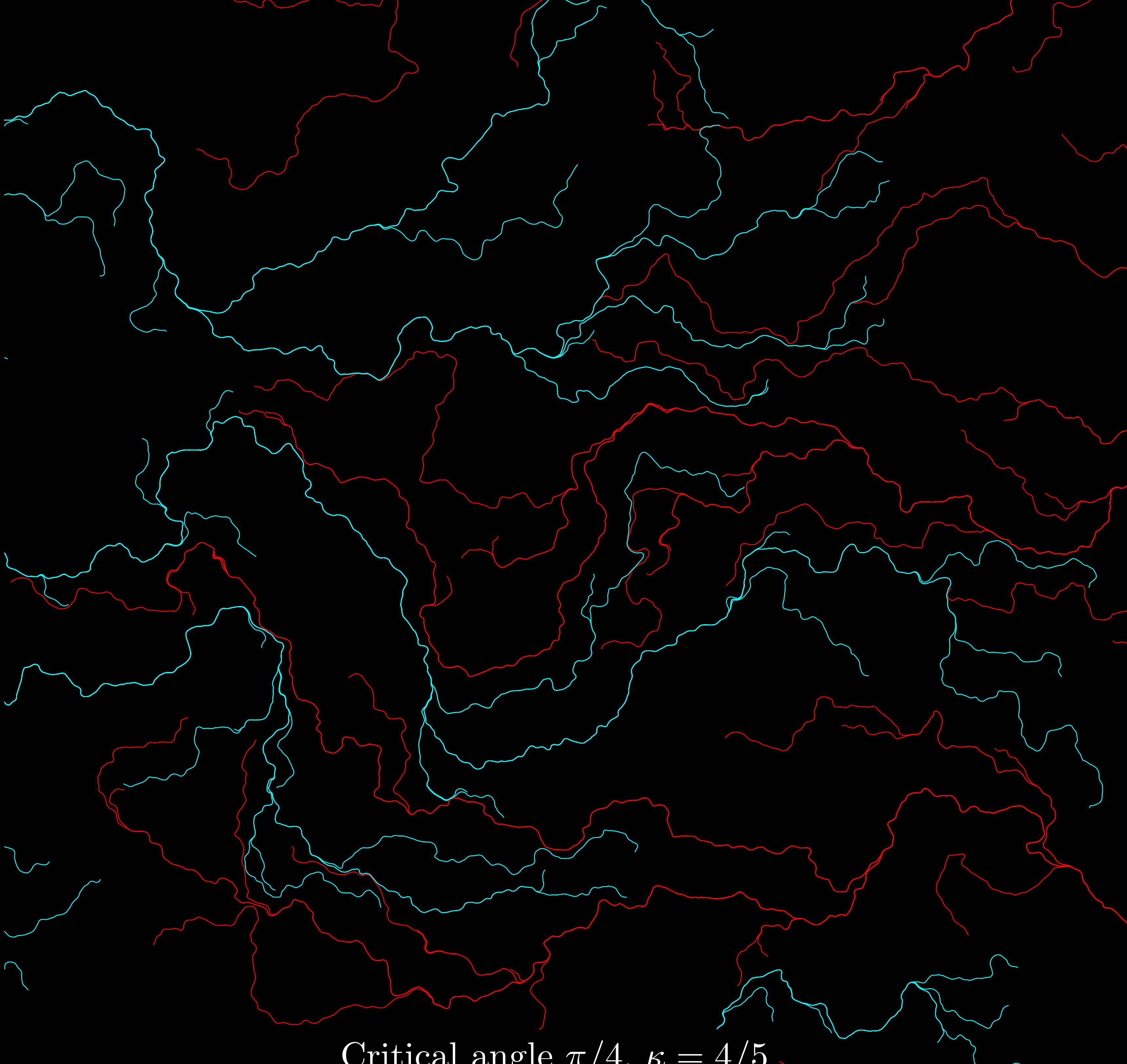


Critical angle 0, $\kappa = 0$

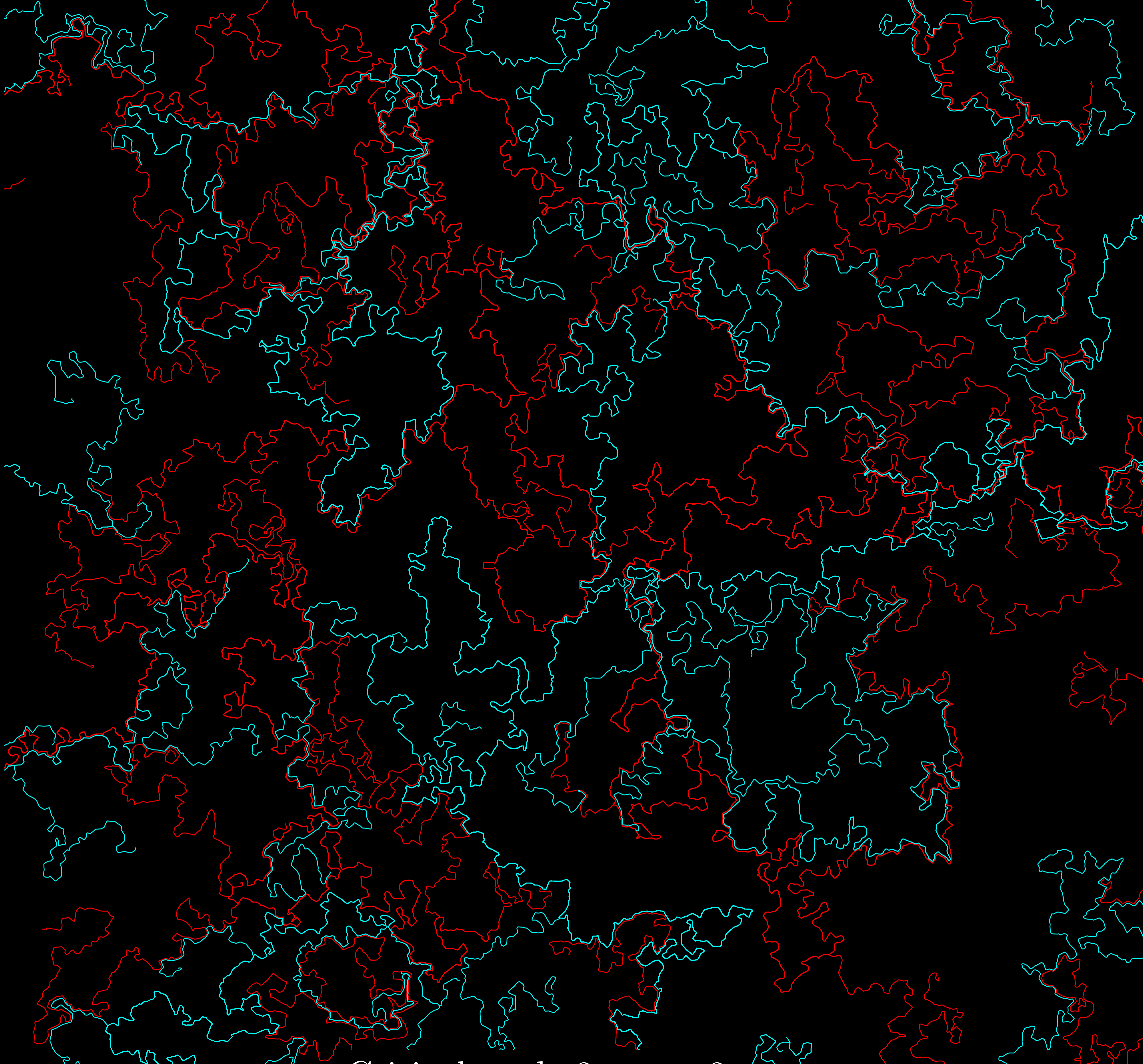


Critical angle $\pi/1000$, $\kappa = 4/1001$

Critical angle $\pi/10$, $\kappa = 4/11$



Critical angle $\pi/4$, $\kappa = 4/5$



Critical angle 3π, $\kappa = 3$

Critical angles in statisical physics (proved or conjectured)

- π: $\kappa \in \{2, 8\}$. *Loop erased random walk (2), uniform spanning tree boundary (8).*
- 2π: $\kappa \in \{8/3, 6\}$. *Brownian motion boundary (8/3), self-avoiding-walk (8/3), critical percolation (6).*
- 3π: $\kappa \in \{3, 16/3\}$. *Critical Ising cluster boundaries (3), critical Ising FK-cluster boundaries (16/3).*
- 5π: $\kappa \in \{10/3, 24/5\}$. *Supercritical 3-state Potts cluster boundaries (24/5).*
- 6π: $\kappa \in \{24/7, 14/3\}$. *Critical 3-state Potts cluster boundaries (24/7), critical 3-state Potts FK (14/3).*
- $\infty\pi$: $\kappa = 4$. *Harmonic explorer (4), GFF level lines (4), double dimer model (4), critical 4-state Potts cluster boundaries (4).*