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RANDOM QUASICONFORMAL MAPPINGS

joint with

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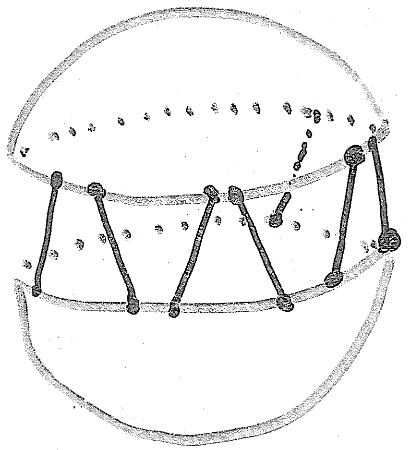
QUASICONFORMAL MAPPINGS:

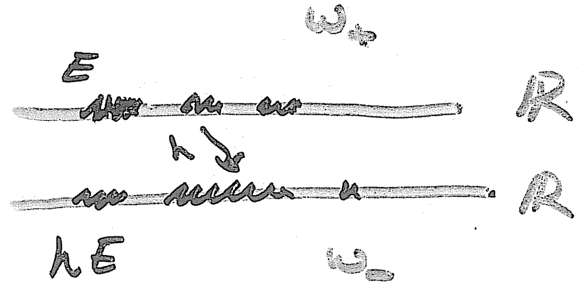
- (The) method to produce complex structures, by

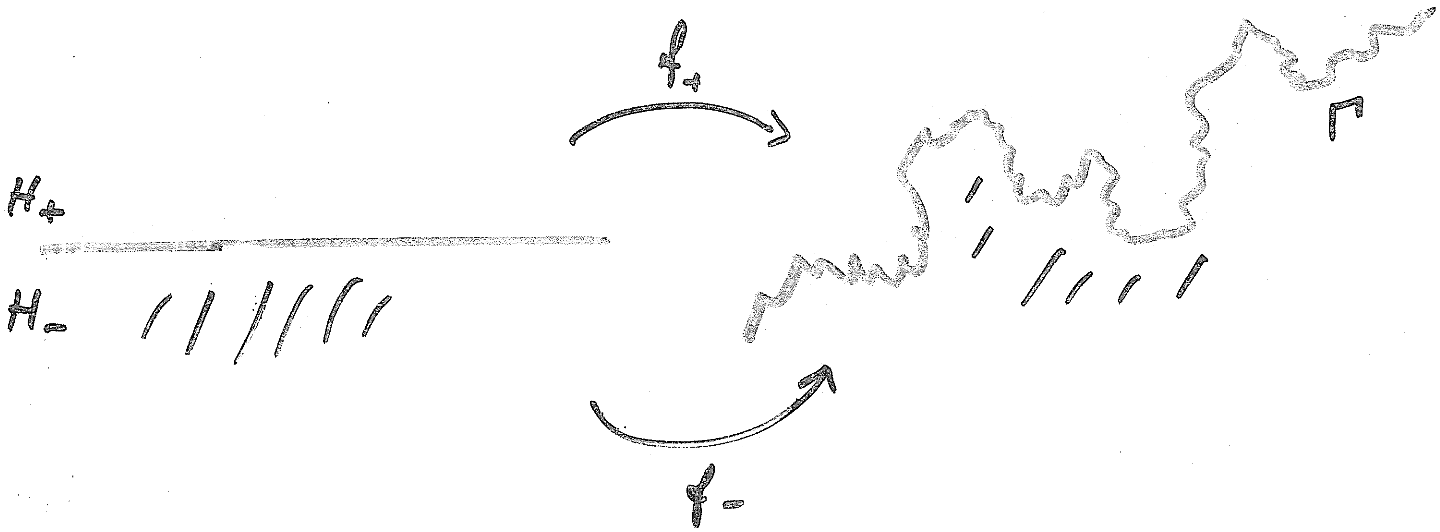
- Existence and uniqueness of solutions to (certain, elliptic) PDE's

Example: Conformal welding

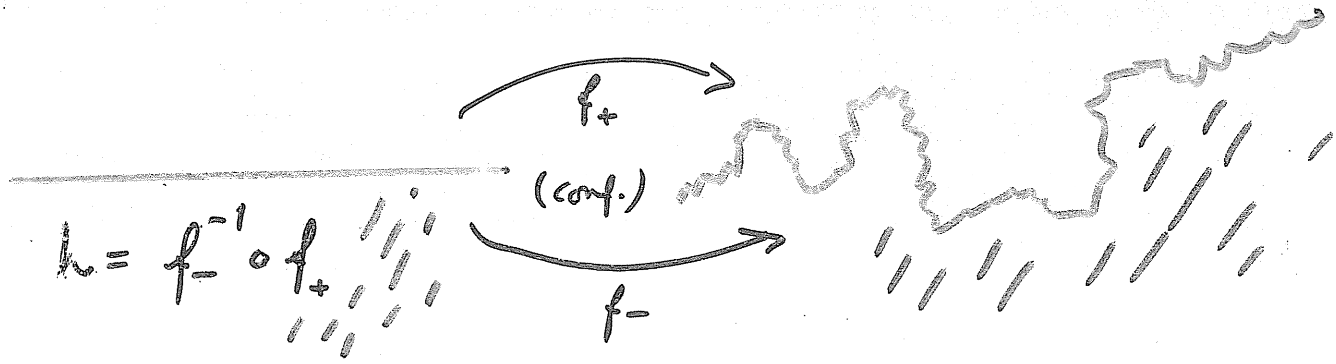
$$h : \mathbb{R} \to \mathbb{R} \text{ homeo}$$







$$h = f_-^{-1} \circ f_+ \big|_{\mathbb{R}} \;, \quad f_\pm \text{ conf. in } H_\pm \quad ?$$



$$h = f_-^{-1} \circ f_+$$

- $h : \mathbb{R} \to \mathbb{R}$; extend to $H : \mathbb{C} \to \mathbb{C}$ w. $\bar{\partial} H = \mu \, \partial H$

$$\mu_+ := \chi_{\{\operatorname{Im} z < 0\}} \, \mu$$

- $\bar{\partial} f_+ = \mu_+ \, \partial f_+$, $\quad f : \mathbb{C} \to \mathbb{C}$ homeo

Uniqueness of sol's to Beltrami equation $\Rightarrow$

$$f_+ = f_- \circ H, \quad f_-\big|_{\{\operatorname{Im} z < 0\}} \text{ conformal}$$

$$\boxed{h = H\big|_{\mathbb{R}} = f_-^{-1} \circ f_+}$$

Homeomorphic solutions to

$$\bar\partial f = \mu\, \partial f \qquad (B)$$

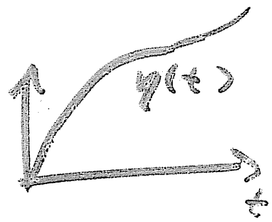
with μ random ?

Note: $\|\mu\|_\infty \leq k < 1 \iff$ (B) uniformly elliptic

$\iff f$ quasiconformal

$\iff$

$$\frac{|f(z) - f(w)|}{|f(z) - f(\xi)|} \leq \eta\left(\frac{|z - w|}{|z - \xi|}\right)$$



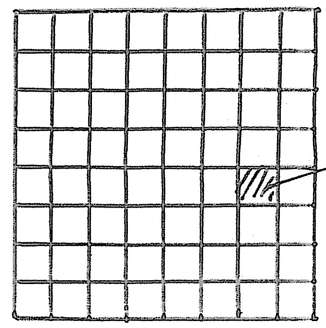
Note:

$|\mu(z)| < 1 \rightsquigarrow$ (degenerate) elliptic PDE's

Random Coefficients $\qquad \bar\partial f = \mu\, \partial f$

Partition $Q_0 = [0,1]^2 \subset \mathbb{C}$ into 2^{2k} squares

Q_0



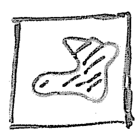
$\ell(Q_{j,k}) = 2^{-k}$

$j = 1, \ldots, 2^{2k}$

Let

$$\mu = \mu_k = \sum_{j=1}^{2^{2k}} \varepsilon_{j,k}\, \chi_{j,k}\, \phi\big(2^k(x - x_j)\big)$$

- $\chi_{j,k} \equiv \chi_{Q_{j,k}}$
- $\|\phi\|_\infty \le 1$, $\operatorname{supp}(\phi) \subset Q_0$



- $\varepsilon_{j,k}$ IID, $\mathbb{E}(\varepsilon_{j,k}) = 0$, $|\varepsilon_{j,k}| < 1$ a.s.

  (Independent Identically Distributed)

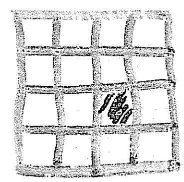
Basic cases:

I $\quad \mathbb{P}(\varepsilon_{j,k} = \pm \tfrac{1}{2}) = 1/2$

II $\quad \dfrac{\mu}{1-|\mu|} \sim N(0,1)$

Theorem

Let $$\mu_k = \sum_{j=1}^{2^{2k}} \varepsilon_{j,k}\, \chi_{Q_{j,k}}\, \phi(2^k(\cdot - x_j))$$



$\}\, 2^{-k}$

Suppose

$$\mathbb{P}(1 - |\varepsilon_{j,k}| < t) \leq c_0\, e^{-B/t}, \quad B \geq B_0 > 1$$

Then

(1a) almost surely, for every k

$$\bar\partial f = \mu_k\, \partial f$$

has a homeomorphic solution f_k

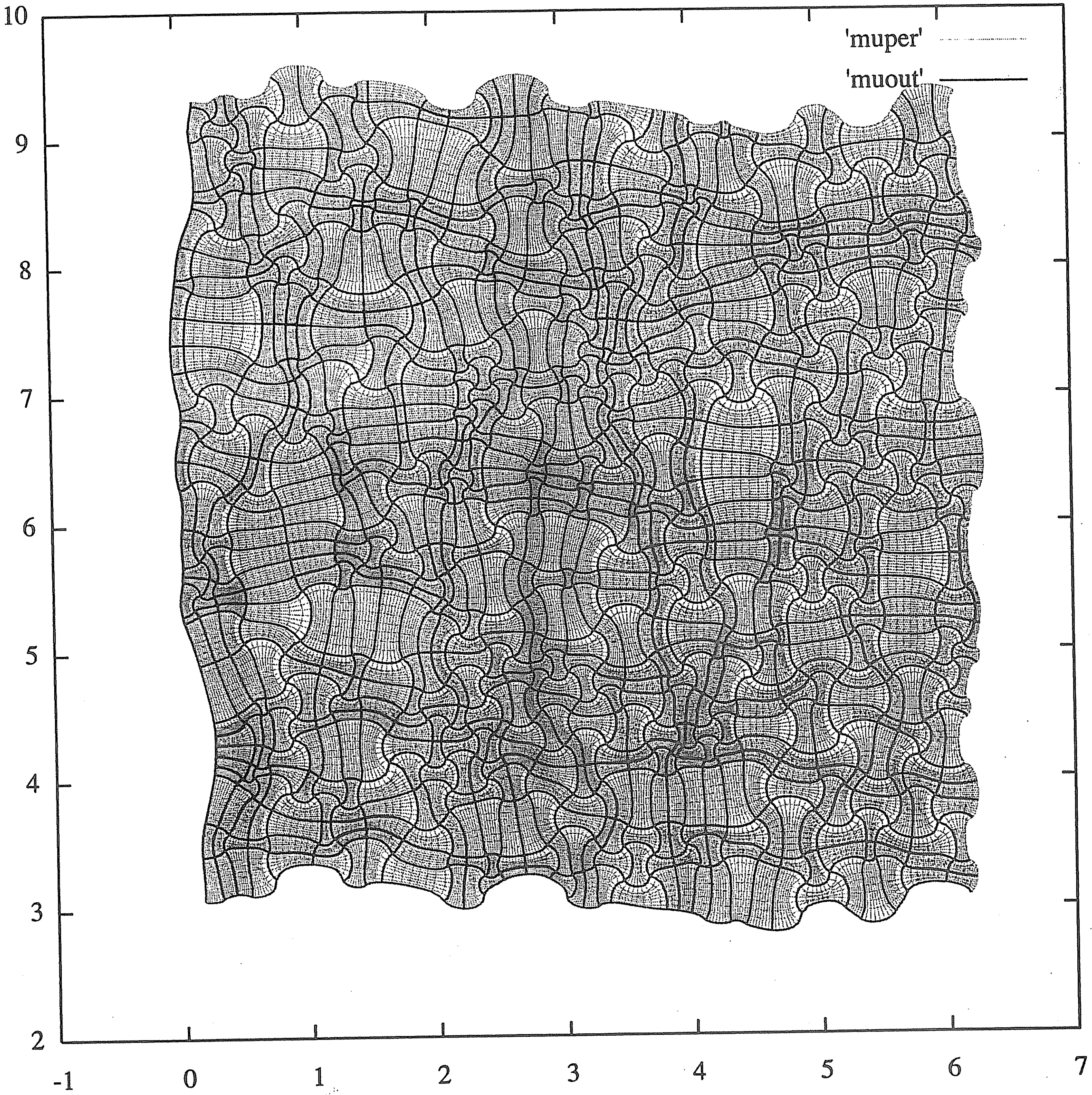
(1b) $\nabla f_k \in L^2_{loc}$, $\quad f(z) = z + \mathcal{O}(1/z)$

(2) In the scaling limit $k \to \infty$,

$$f_k(z) \longrightarrow F(z) = F_\phi(z) \quad \underline{\text{almost surely}}$$

$\uparrow$ deterministic

(3) $\phi \equiv 1 \rightsquigarrow F(z) \equiv z$; for ~~some~~ ϕ, $F_\phi(z) \neq z$

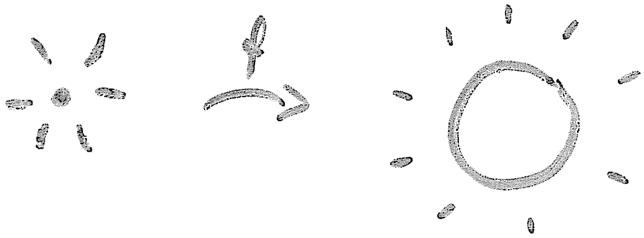


Picture by David White; program by D. Gaydashe

Regularity necessary

Example (of a bad behavior)

- $f(z) = z + \frac{z}{|z|}$



- $\nabla f \sim 1 + \frac{1}{|z|} \in$ weak-L^2

- $\mu_f(z) = \mu_0(z) \equiv -\frac{z}{\bar{z}} \frac{1}{1+2|z|}$

- every solution to $\bar{\partial} g = \mu_0 \partial g$ is non-continuous at $z = 0$.

Constructing f_k :

$$\bar\partial f = \mu\, \partial f \,, \qquad f(z) = z + \mathcal{O}(1/z)$$

- Beurling transform (Calderon - Zygmund theory)

$$T g(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(w)}{(w-z)^2}$$

- $\partial f_k - 1 = T(\bar\partial f_k) \implies \bar\partial f_k = \mu_k + \mu_k T(\bar\partial f_k)$

- $\bar\partial f_k = \mu_k + \mu_k T \mu_k + \cdots + (\mu_k T)^n \mu_k + \cdots$

- $f(z) = z - \frac{1}{\pi} \int_{\mathbb{C}} \frac{\bar\partial f(w)}{(w-z)}\, dm(w)$

Cauchy transform

$$\mu_k = \sum_{j=1}^{2^{2k}} \varepsilon_{j,k} \, \chi_{Q_{j,k}}$$

(Here assume $p \equiv 1$)

<u>1. Step</u> $\forall \tilde{Q} \subset Q_0$ dyadic subsquare, $\forall n$

$$\int_{\tilde{Q}} (\mu_k T)^n \mu_k \xrightarrow[k \to \infty]{P} 0$$

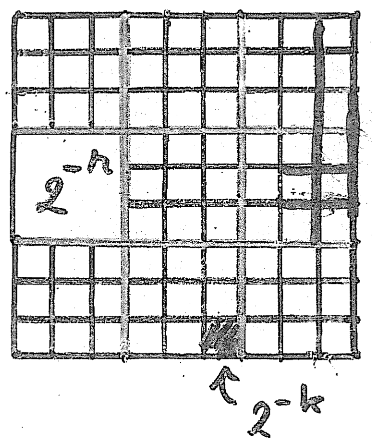
Proof by induction on n:

$n = 0$, e.g. by Law of Large Numbers (or Chebychev)

For induction step:

$T : L^{p_1}(Q_1) \to L^{p_2}(Q_2)$ <u>compact</u>

for $p_1 > p_2$, $Q_1 \cap Q_2$



$$\int_{\tilde{Q}} (\mu_k T)^n \mu_k = a_{m,n,k} + \sum_{\substack{s:\\ Q_{s,n} \subset \tilde{Q}}} \left(\mu_k \chi_{Q_{s,n}} T\right)^n \mu_k \chi_{Q_{s,n}}$$

$$\mu_k = \sum_j \varepsilon_{j,k} \chi_{Q_{j,k}}$$

$$\int_{\tilde{Q}} (\mu_k T)^m \mu_k =$$

$$a_{m,n,k} + \underbrace{\sum_{\substack{s:\\ Q_{s,n} \subset \tilde{Q}}} (\mu_k \chi_{Q_{s,n}} T)^m \mu_k \chi_{Q_{s,n}}}_{\sum_{p=1}^{2^{2(n-\tilde{k})}} Y_p}$$

$$a_{m,n,k} \xrightarrow[k \to \infty]{P} 0$$

$$Y_p \sim 2^{-2n} \int_{Q_0} (\mu_{\tilde{k}-n} T)^m \mu_{\tilde{k}-n} \quad ; \mathbb{E} Y_p = 0$$

<u>2. Step</u> $$\mathbb{E}\left(\sum_{k=1}^{\infty} \| (\mu_k T)^m \mu_k \|_{W^{-s,2}} \right) < \infty$$

$$\therefore \quad \| (\mu_k T)^m \mu_k \|_{W^{-s,2}} \xrightarrow[k \to \infty]{} 0 \quad \text{almost surely}$$

$$\therefore \quad \int_{\tilde{Q}} (\mu_k T)^m \mu_k \xrightarrow[k \to \infty]{} 0 \quad \text{almost surely}$$

Estimating the tail:

$$\sum_{n=0}^{\infty} \|(\mu_k T)^n \mu_k\|_{L^2} < \infty \quad \text{a.s.} \quad ?$$

Let

$$B_{n,k} = \{z \in Q_0 : |\mu_k| > 1 - \tfrac{2}{n}\} \,, \quad G_{n,k} = Q_0 \setminus B_{n,k}$$

- $$g_{n,k} := \chi_{G_{n,k}} \mu_k T g_{n-1,k}$$

$$\|g_{n,k}\|^2 = \int_{G_{n,k}} |\mu_k T g_{n-1,k}|^2 \le (1 - \tfrac{2}{n})^2 \|g_{n-1,k}\|^2 \le \cdots \le \frac{c}{n^2}$$

- $$\|g_{n,k} - (\mu_k T)^n \mu_k\|_2 \le \frac{c}{n^2} \sum_{j=1}^{n} j^2 \delta_{j,k} \qquad \delta_{j,k} = \|\chi_{B_{j,k}} (\mu_k T)^j \mu_k\|_2$$

Borel-Cantelli:

$$\sum_{k,n=1}^{\infty} \mathbb{P}(|B_{n,k}| > e^{-c_0 n}) < \infty$$