A particle representation for the heat equation solution

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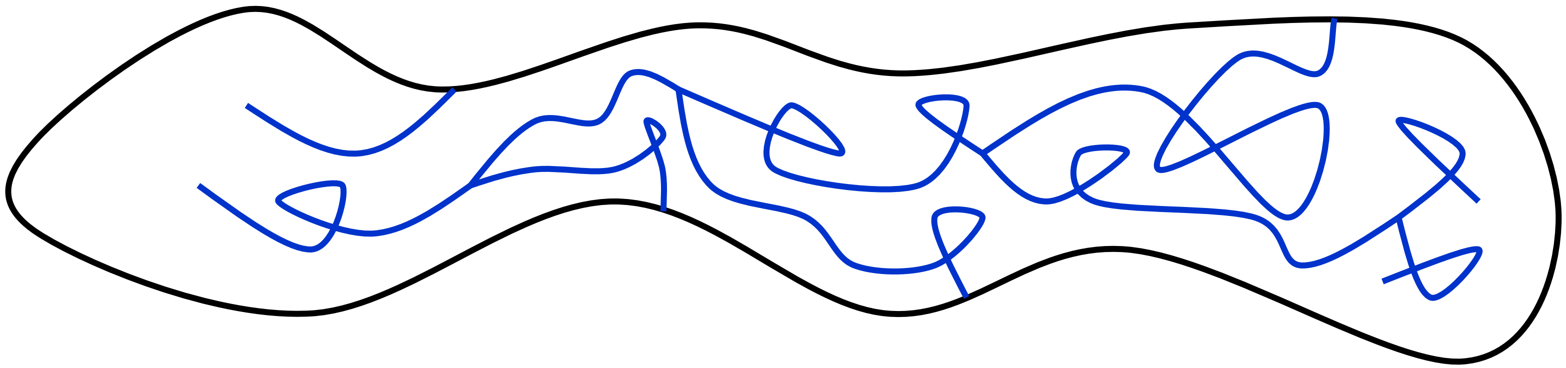
Fleming-Viot model



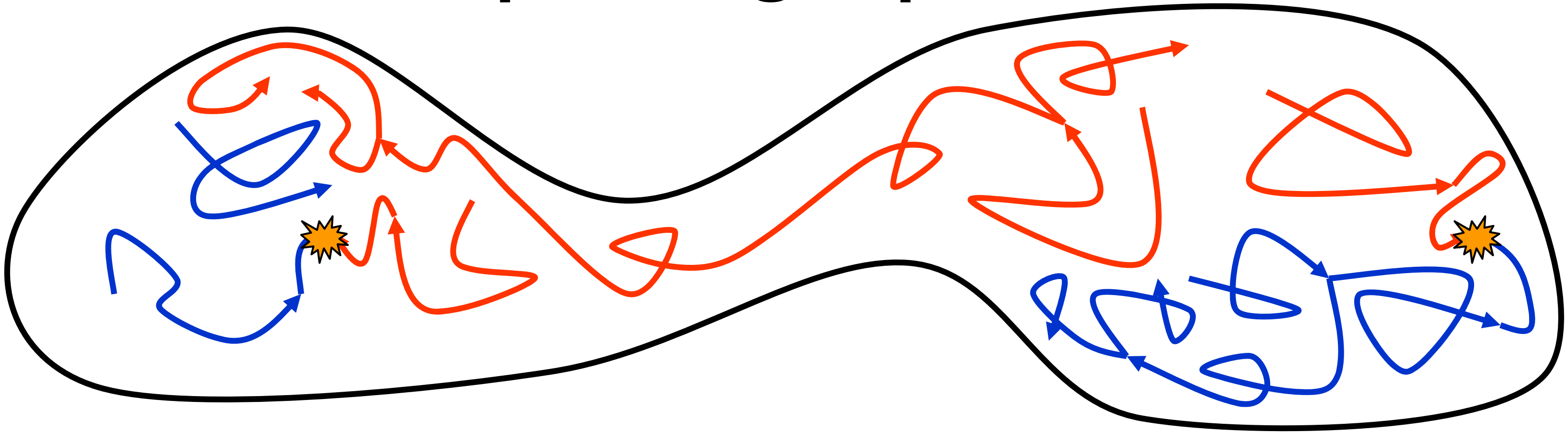
N = population size (constant in time)
= ecological niche carrying capacity

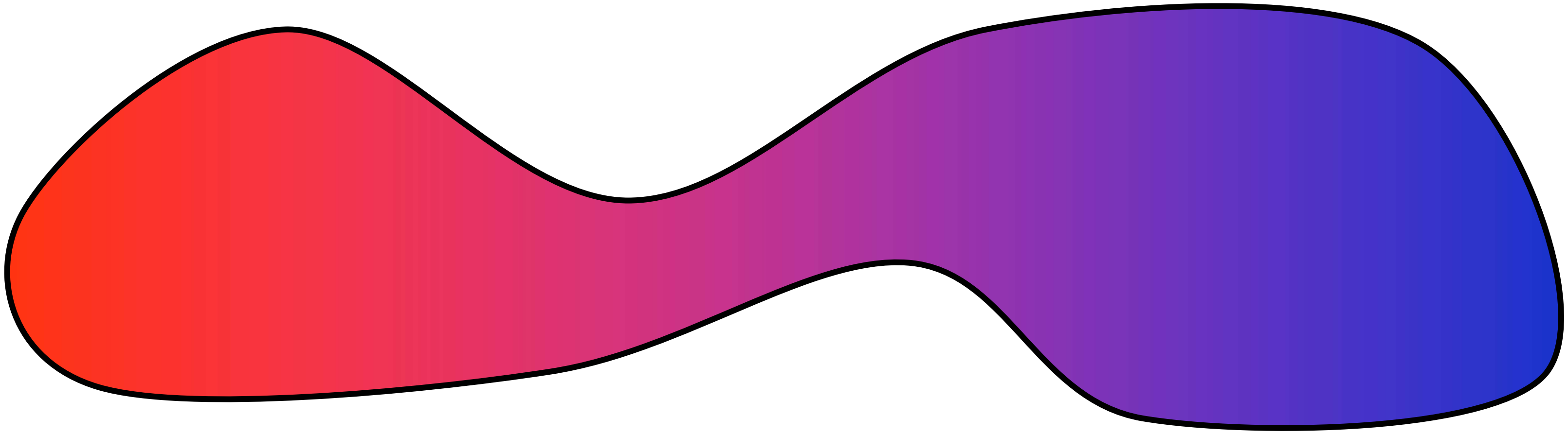
Let N go to infinity.

Free space
Population distribution
converges to a fractal structure

Bounded domain
Population distribution converges to a density
= heat equation solution

Competing species

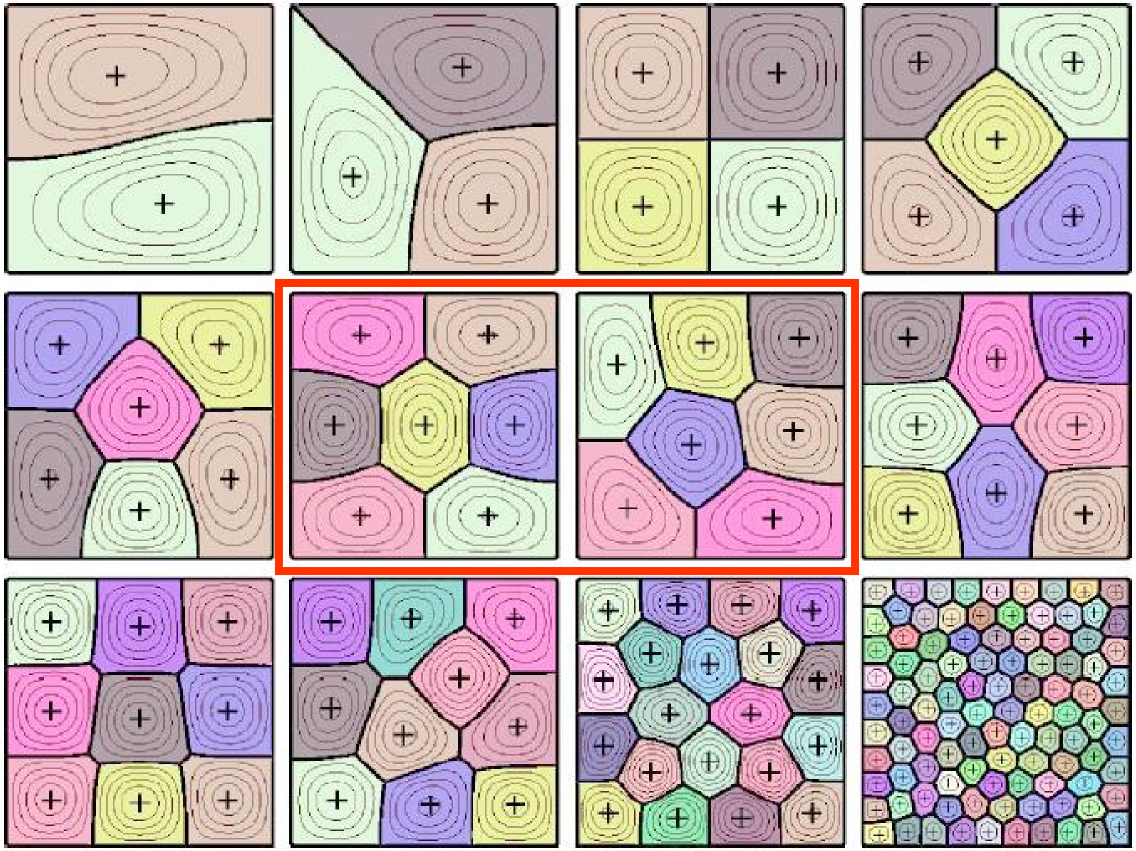




Selected references

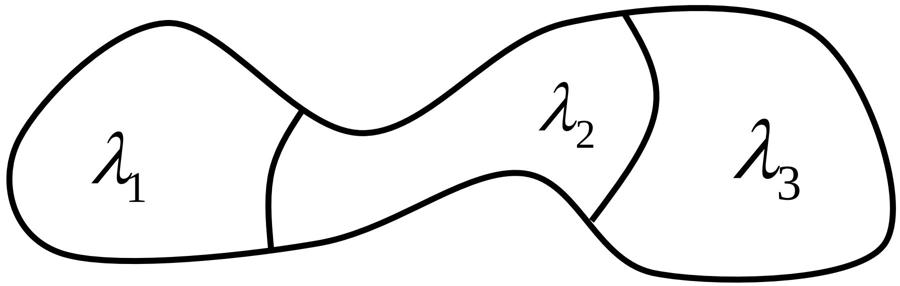
- B, Holyst, Ingerman, March (1996)
- B, Holyst, March (2000)
- B, Quastel (2007)
- Grigorescu, Kang (2004, 2006, 2006)

Multiple populations



"Minimization of the Renyi entropy production in the space-partitioning process"
Cybulski, Babin, and Holyst, Phys. Rev. E 71, 046130 (2005)

The stationary distribution



λ_k - first Dirichlet eigenvalue in k-th region

Conjecture 1: The stationary distribution minimizes

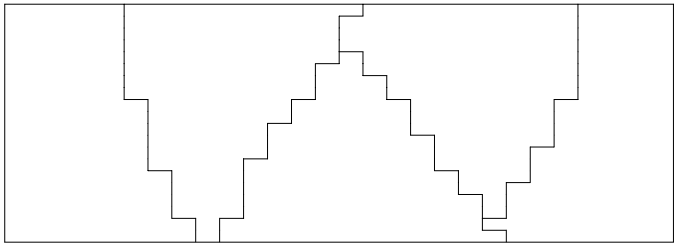
$$\lambda_1 + \lambda_2 + \ldots + \lambda_n \stackrel{df}{=} \lambda^*$$

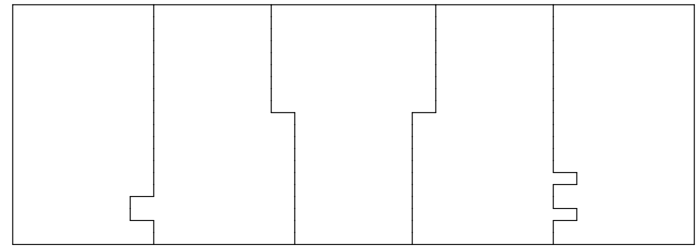
Bucur, Buttazzo and Henrot "Existence results for some optimal partition problems" Adv. Math. Sci. Appl. 8 (1998) 571—579
Conti, Terracini and Verzini "On a class of optimal partition problems related to the Fucık spectrum and to the monotonicity formulae" Calc. Var. 22, 45–72 (2005)

Conjecture 2: The honeycomb pattern minimizes λ^*

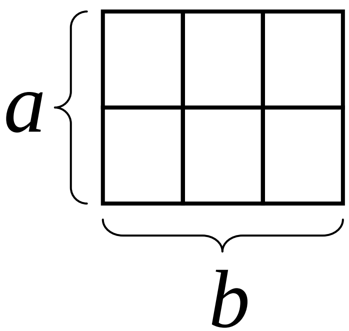
Special thanks to Luis Caffarelli!

The stationary distribution (2)

(*) 



B, Holyst, Ingerman and March “Configurational transition in a Fleming-Viot-type model and probabilistic interpretation of Laplacian eigenfunctions” J. Phys. A 29, 1996, 2633-2642



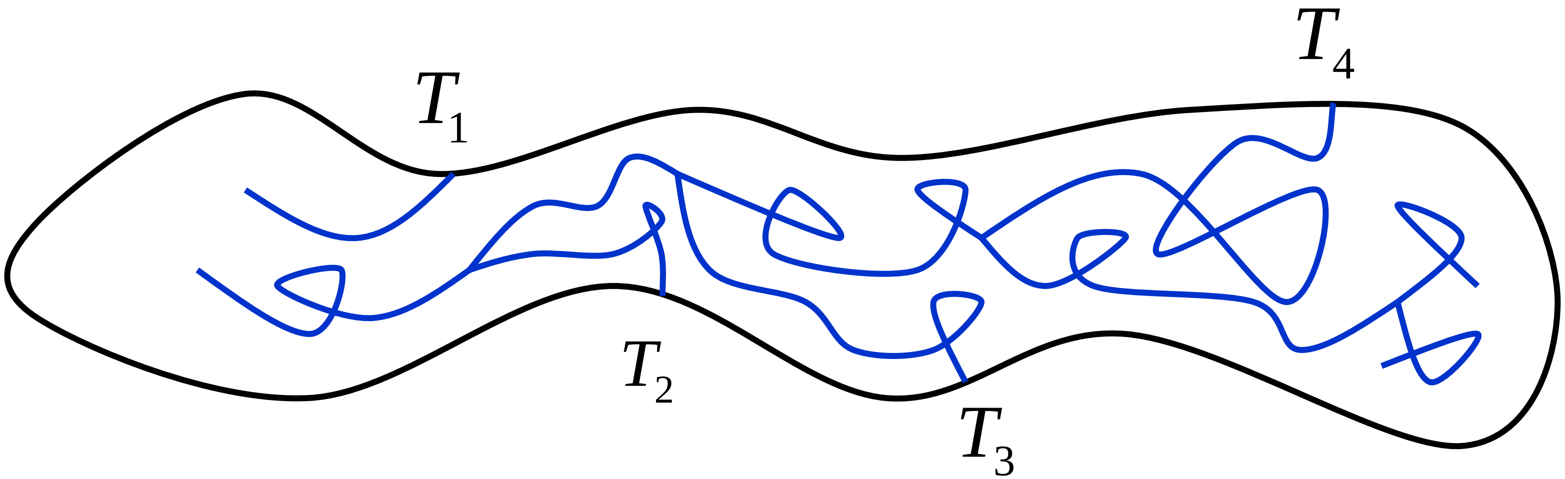
$$\lambda_{j,k} = (\pi / a)^2 (j^2 + (k/r)^2), \quad r = b/a$$

$$\lambda_{2,3} = (\pi / a)^2 (2^2 + (3/r)^2)$$

Conjecture 3: The critical ratio r for (*) and m populations satisfies

$$\lambda_{1,m} = \lambda_{2,1}$$

Rigorous results – one population



Theorem (B, Holyst, March, 2000) Suppose that the individual trajectories are independent Brownian motions. Then

$$T_n \xrightarrow{n\to\infty} \infty, \qquad a.s.$$

Idea of the proof

A parabolic function (harmonic in space-time):

$$h(t,x) = P(X_s \in D, s \in [t,1] \mid X_t = x)$$

A martingale plus a process with positive jumps:

$$M_t = \sum_{k=1}^{N} h(t, X_t^k)$$

One population – convergence to the heat equation solution

N - population size

$1/N$ - individual particle mass

$Q_N(t)$ - empirical density at time t

- individual trajectories follow Brownian motions

Theorem (B, Holyst, March, 2000) If

$$Q_N(0)(dx) \xrightarrow{N\to\infty} u_0(x)dx$$

then

$$Q_N(t)(dx) \xrightarrow{N\to\infty} u(t,x)dx, \quad t>0$$

where $u(t,x)$ is the normalized heat equation solution with

$$u(0,x) = u_0(x)$$

One population – convergence of stationary distributions

N - population size

$1/N$ - individual particle mass

$Q_N(t)$ - empirical density at time t

- individual trajectories follow Brownian motions

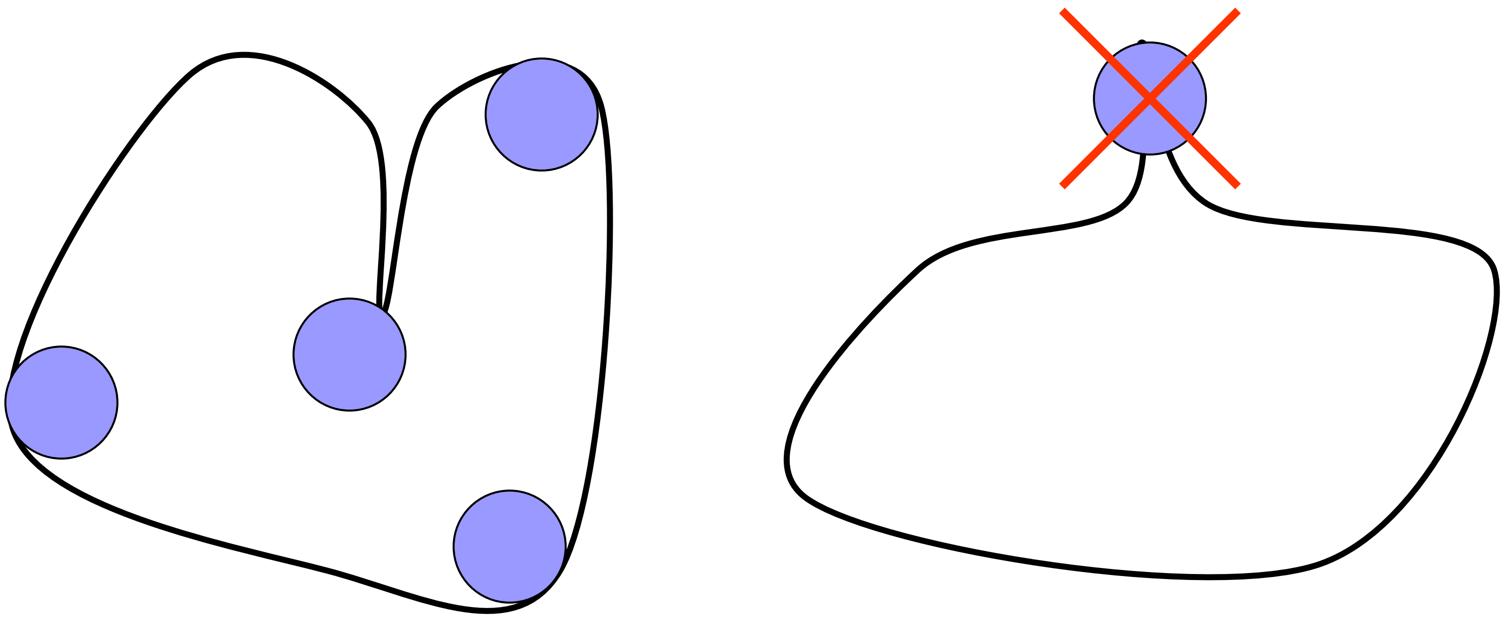
Theorem (B, Holyst, March, 2000) The process $Q_N(t)$ has a stationary distribution Λ_N. Moreover,

$$\Lambda_N(dx) \xrightarrow{N\to\infty} \varphi(x)\,dx$$

where $\varphi(x)$ is the first Dirichlet eigenfunction.

One population – convergence of stationary distributions – assumptions

Assumption: The uniform internal ball condition



Two populations – convergence to the heat equation solution

N - population size (same for population I and II)
$1/N$ - individual particle mass (population I)
$-1/N$ - individual particle mass (population II)
$Q_N(t)$ - empirical density at time t
- individual trajectories follow random walks

Theorem (B, Quastel, 2007) If

$$Q_N(0)(dx) \xrightarrow{N\to\infty} u_0(x)dx$$

then

$$Q_N(t)(dx) \xrightarrow{N\to\infty} u(t,x)dx, \quad t>0$$

where $u(t,x)$ is the normalized heat equation solution with

$$u(0,x) = u_0(x)$$

Two populations – convergence to the heat equation solution – assumptions

(i) Trajectories – simple random walks
(ii) Trajectories reflect at the domain boundary
(iii) The two populations have equal sizes
(iv) The domain has an analytic boundary

Idea of the proof

φ_n - n-th Neumann eigenfunction

$$\hat{u}_n(t) = \pm \frac{1}{N} \sum_{k=1}^{N} \varphi_n(X_t^k)$$

$$d\hat{u}_n(t) = \Delta \hat{u}_n(t)dt + M_t$$

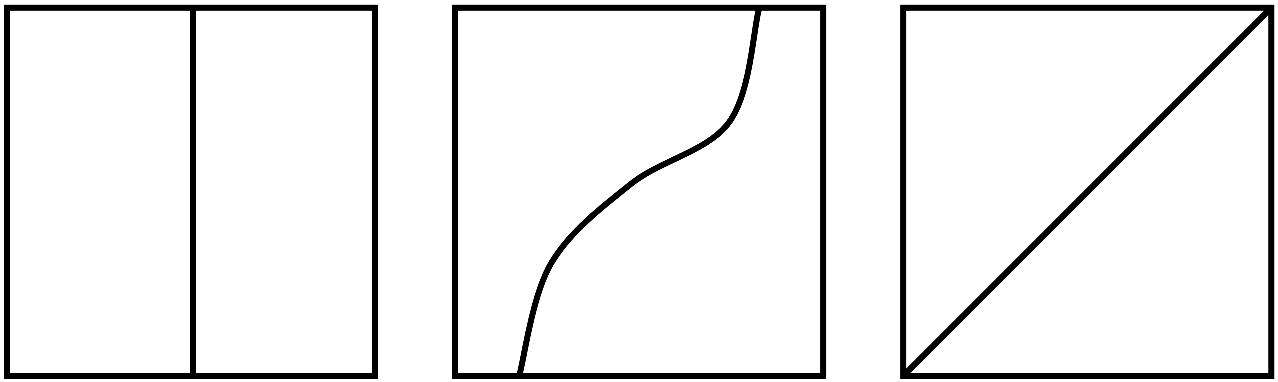
Main technical challenge: bound the clock rate

Spectral representation and L^1

Lemma. Suppose that D is a domain with C^2 smooth boundary, φ_n is the n-th eigenfunction for the Laplacian with Neumann boundary conditions and μ is a signed measure with a finite total variation.

$$\forall n \quad \int \varphi_n(x)\,\mu(dx) = 0 \quad \Rightarrow \quad \mu \equiv 0$$

Diffusion in eigenfunction space



Open problem: What is the speed of diffusion?

Invariance principle for reflected random walks

Theorem. Reflected random walk converges to reflected Brownian motion.

(i) C^2-domains, Stroock and Varadhan (1971)

(ii) Uniform domains, B and Chen (2007)

Example: Von Koch snowflake is a uniform domain.

Counterexample (B and Chen, 2007): Reflected random walk does not converge to reflected Brownian motion in a planar fractal domain.

Myopic conditioning

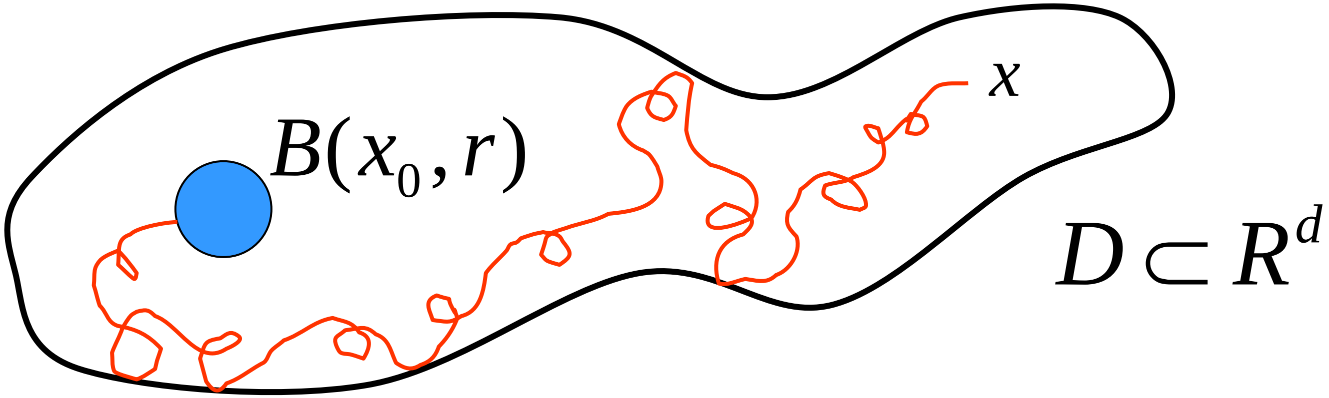
$D \subset R^d$ - open, connected, bounded set

$\{X_t^n, t \geq 0\}$ - Markov process

$\{X_t^n, t \in [k/n, (k+1)/n]\}$ - Brownian motion conditioned by

$$\{X_t^n \in D, t \in [k/n, (k+1)/n]\}$$

Theorem (B and Chen, 2007). When $n \to \infty$,
$\{X_t^n, t \geq 0\}$ converge to reflected Brownian motion in D.



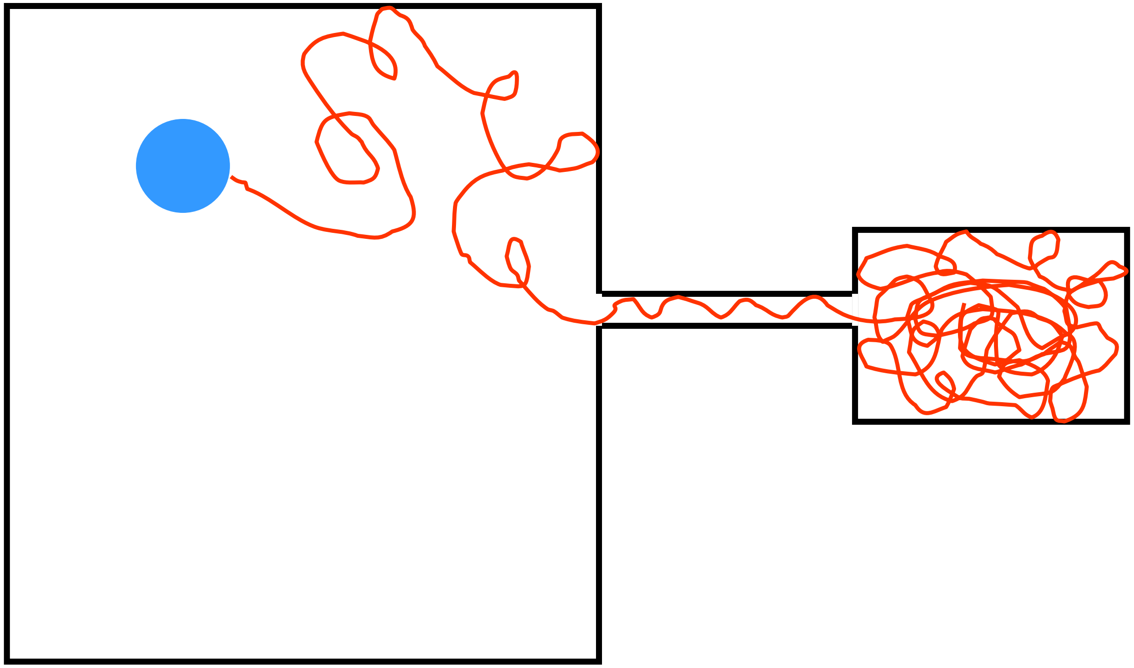
X_t - reflected Brownian motion in D

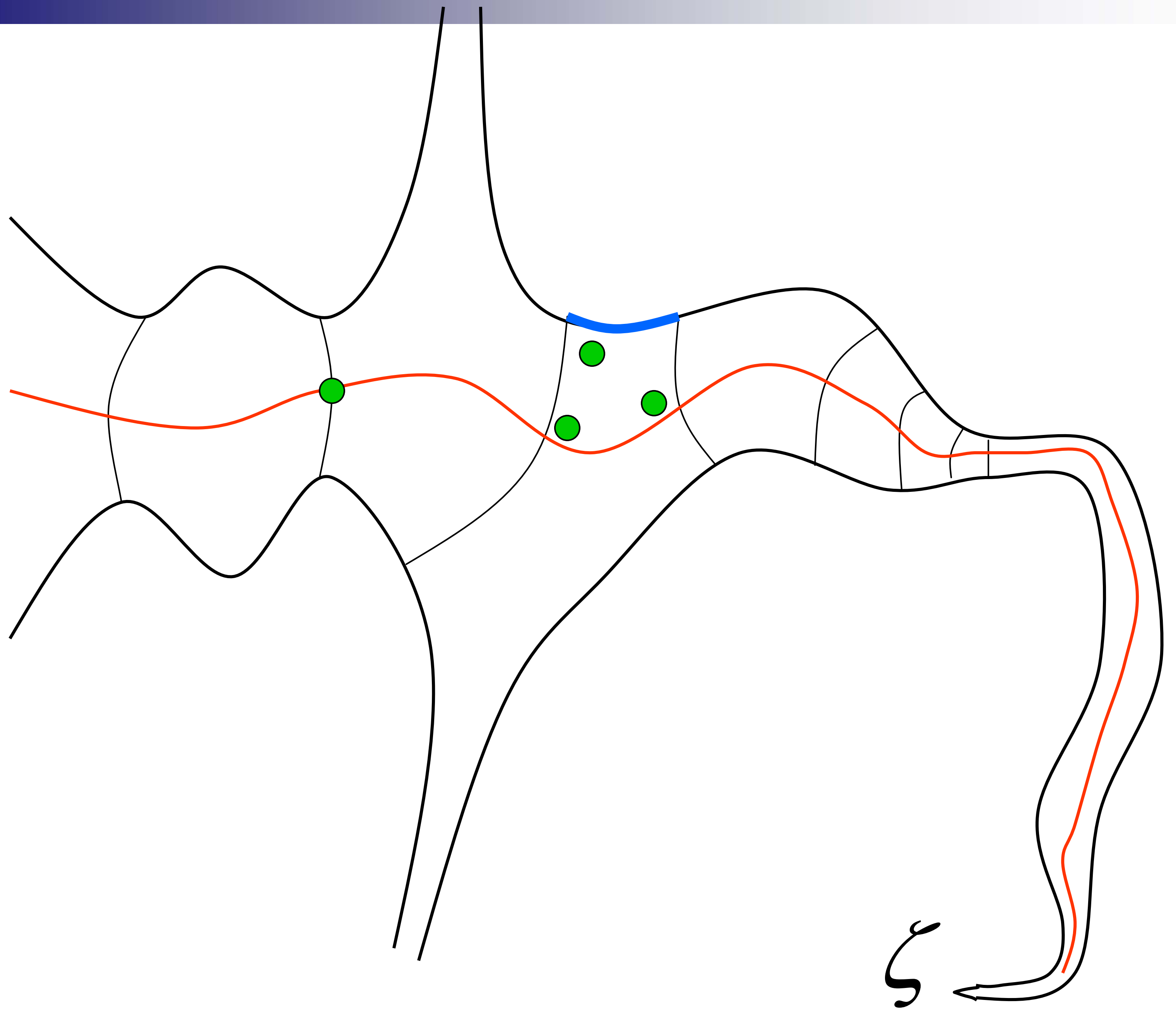
T_B - hitting time of $B(x_0, r)$

Problem: $\sup_{x \in D} E^x T_B < \infty$?

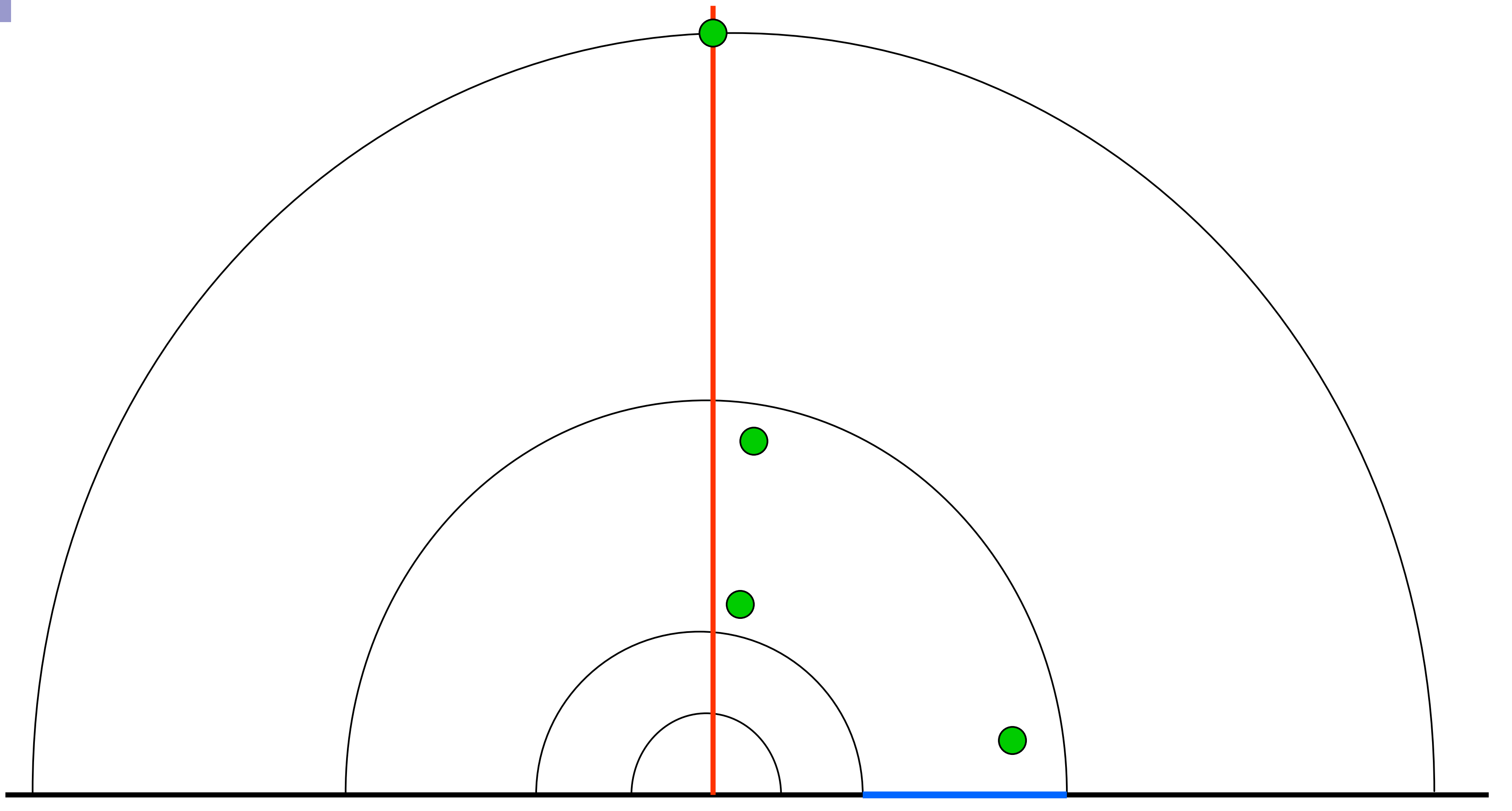
Definition: We will call a bounded set D a <u>trap</u> domain if

$$\sup_{x \in D} E^x T_B = \infty.$$



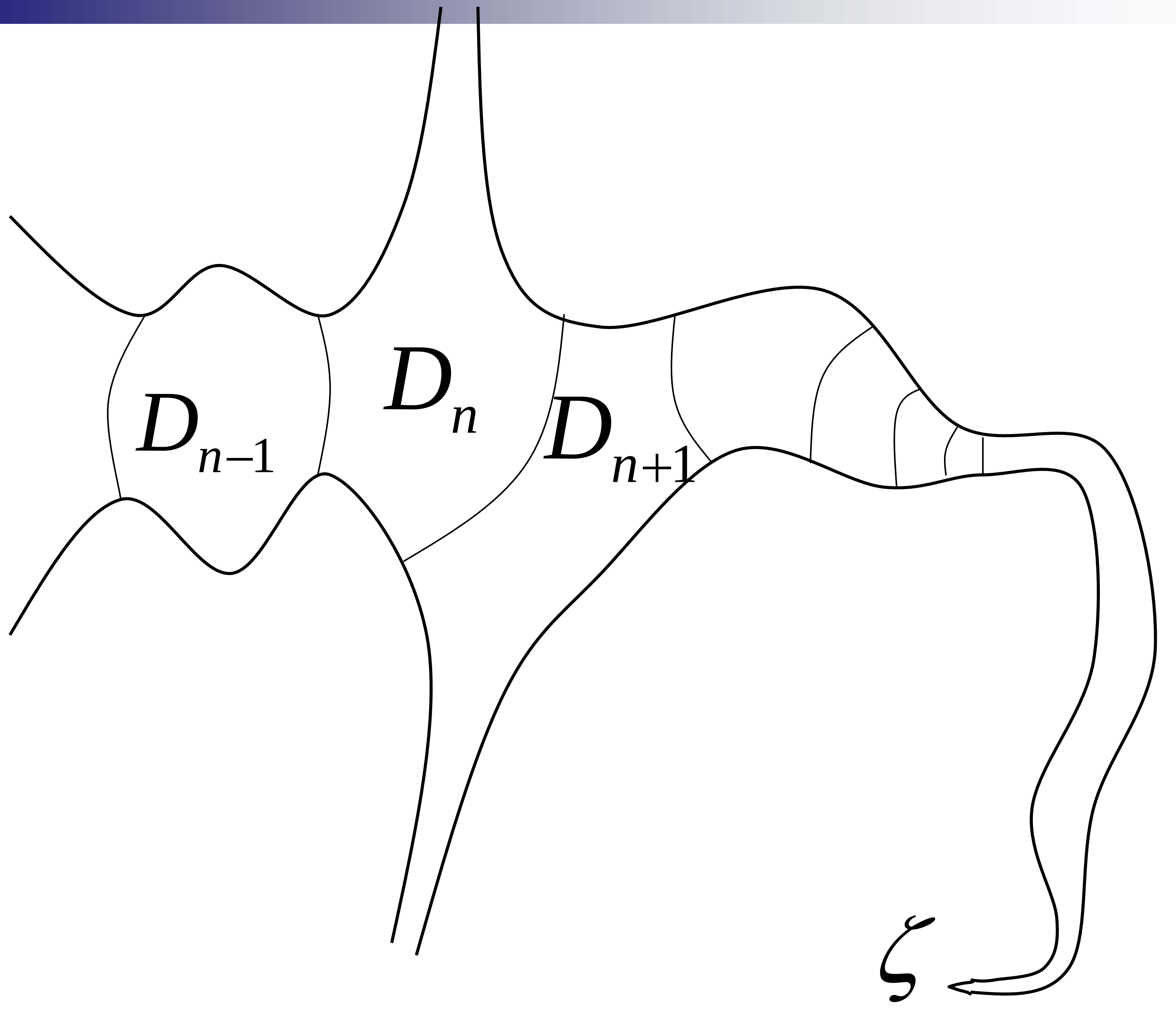


ζ



$$\varphi(\zeta)$$

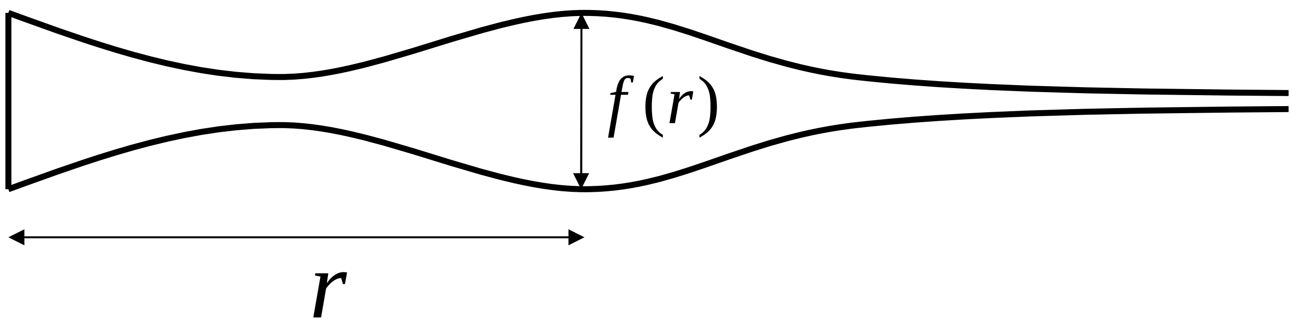
Hyperbolic blocks


D_{n-1}
D_n
D_{n+1}
ζ

Theorem (B, Chen and Marshall, 2006): A simply connected planar domain D is a trap domain if and only if

$$\sup_{\zeta} \sum_{n \geq 1} n \cdot Area(D_n) = \infty.$$

Horn domain:



Corollary: D is a trap domain iff

$$\int_1^{\infty}\left(\int_1^{x}\frac{1}{f(z)}\,dz\right) f(x)\,dx = \infty.$$

Example: $f(r) = \exp(-r^{\alpha})$

Trap domain $\Leftrightarrow$ $\alpha \le 2$