# Unversality and scaling in static granular matter

RSWS1

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# Granular Matter

## Contrast to atomic, molecular, or soft matter.

- Particles with a lot of internal structure
- Thermal energy insufficient for motion
  or structural change

## Consequences:

- Dissipation: loss of energy to finer scales
- No thermodynamic equilibrium
- Kinetics only under external drive
  or on a velocity scale that fades with time

# Examples of generic granular phenomena

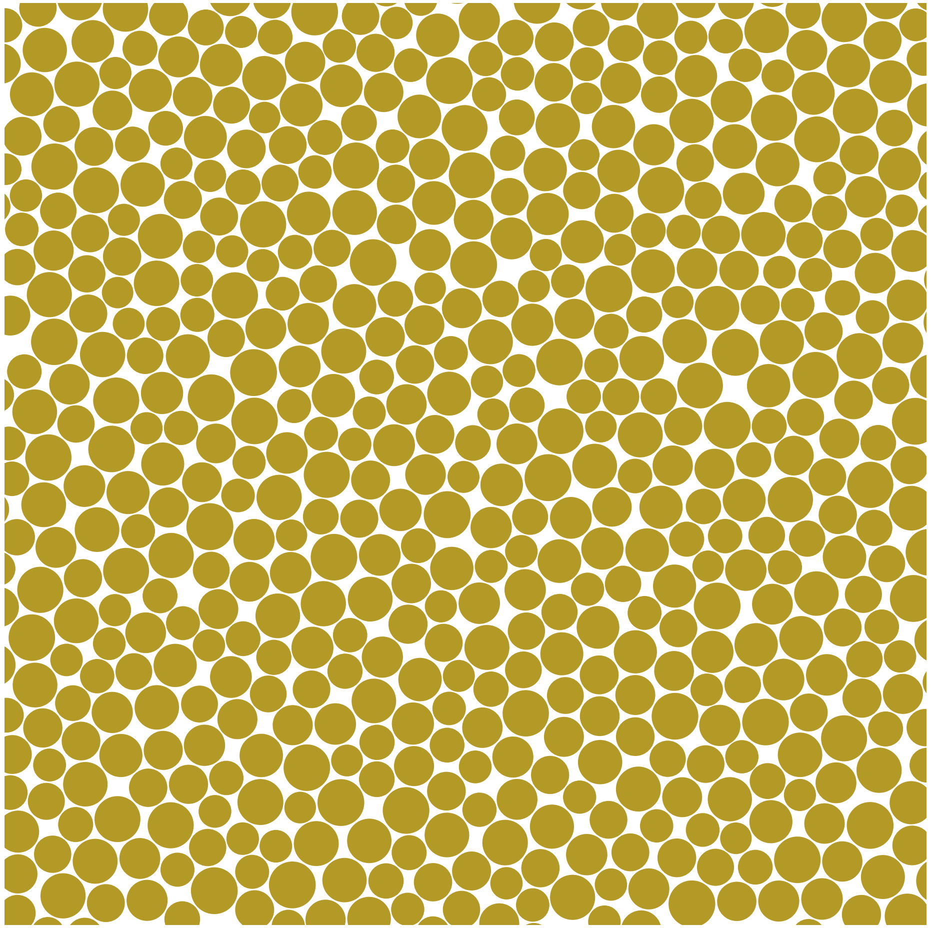
- gas: sand storms
- liquid: flowing sand in an hour glass
- solid: sandcastles, wet beach
- but also exotic phenomena:
  - Brasil nut effect
  - banding
  - oscillons, ...

# Static sand piles

- density fluctuations
- topology of the contact network
- probability distribution of local force
- spatial distribution of mean force
- probability measure of spatial force configuration

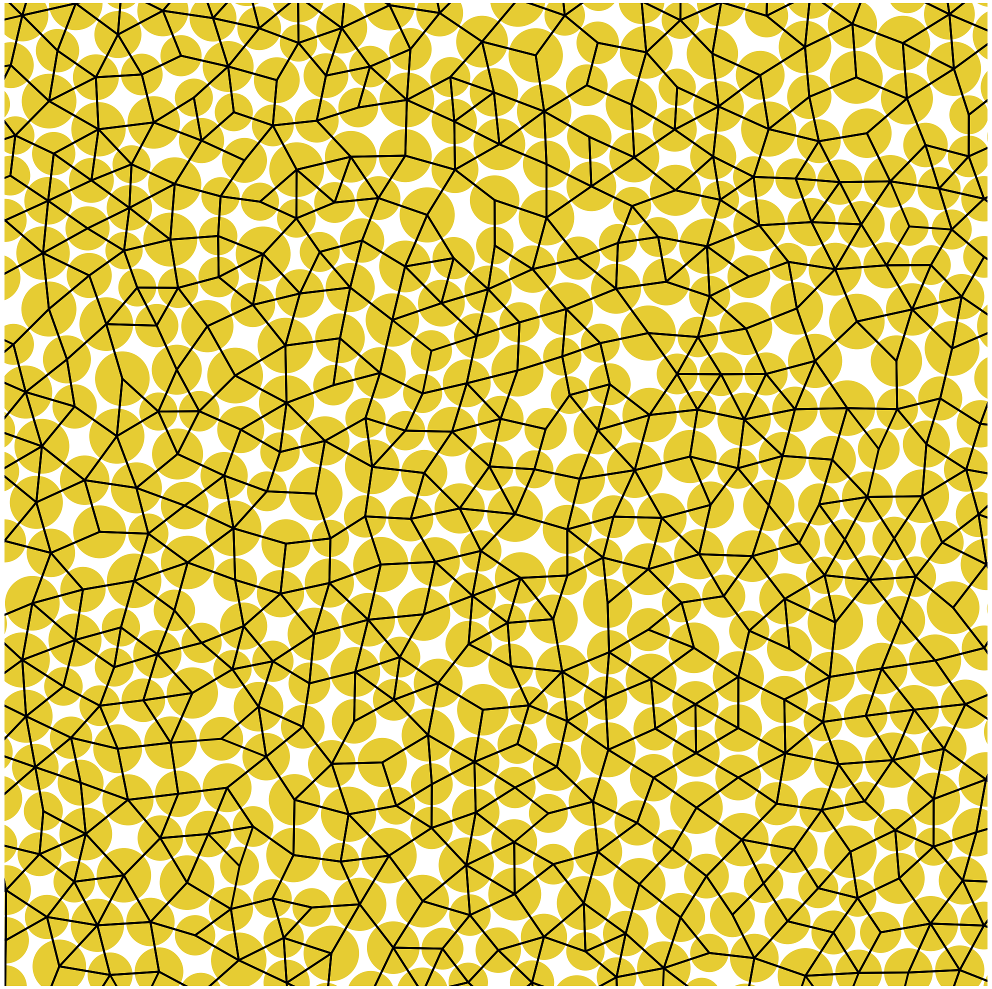
An example of a stack of granular particles.

How should such piles be characterized?



We see more structure when the contact network is shown.

One can study exceptional faces and vertices of the graph and their effective interactions.



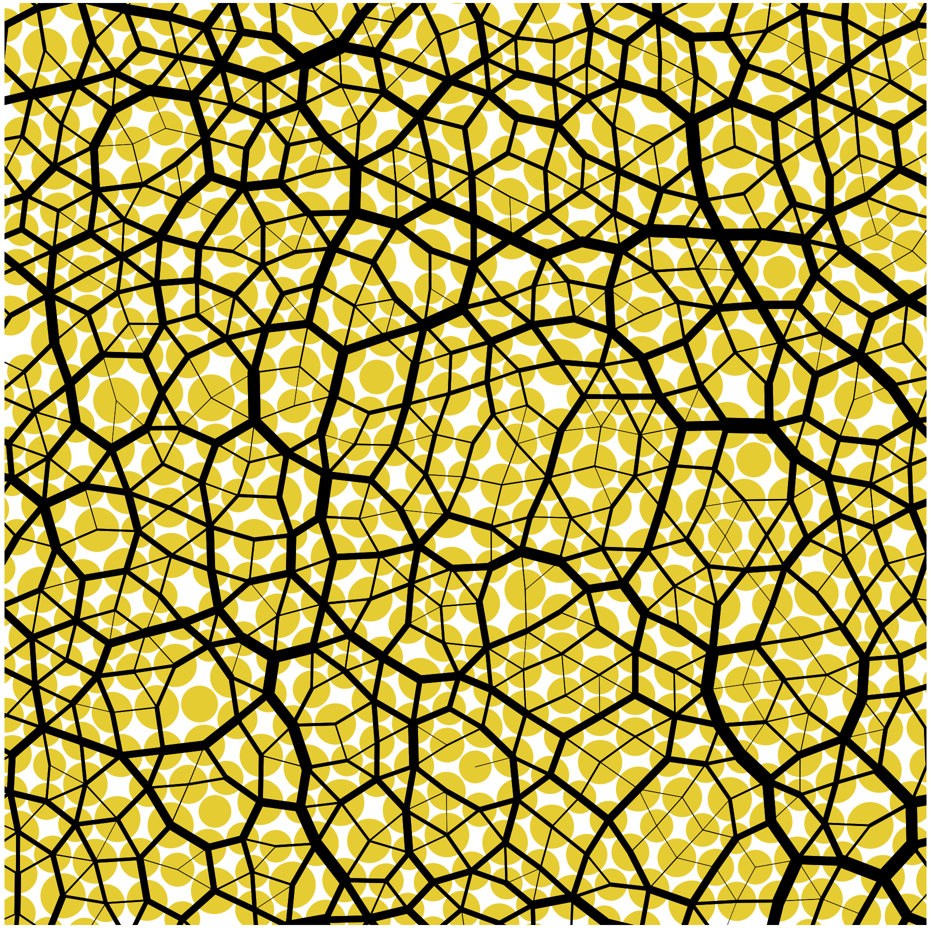
More interesting if forces are shown.

Force balance leads to force chains.

Do they have a typical scale?

If so, what is this scale?

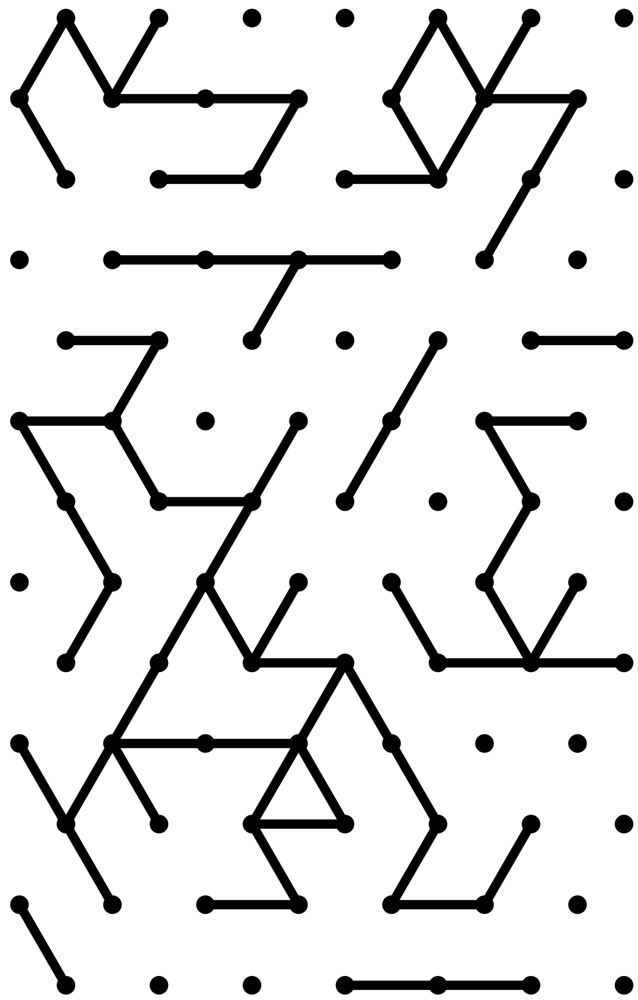
If not, could the force network be self-similar on different scales?



## Analogy:
## size of percolation clusters

In bond percolation all nearest neigbors are connected with probability $p$, and one studies the appearance of an infinite cluster as $p$ increases.
One may just as well associate a random number $r_j$ with each edge $j$, and connect the sites if $r_j < p$.
Now $p$ can be varied for fixed $\{r\}$.

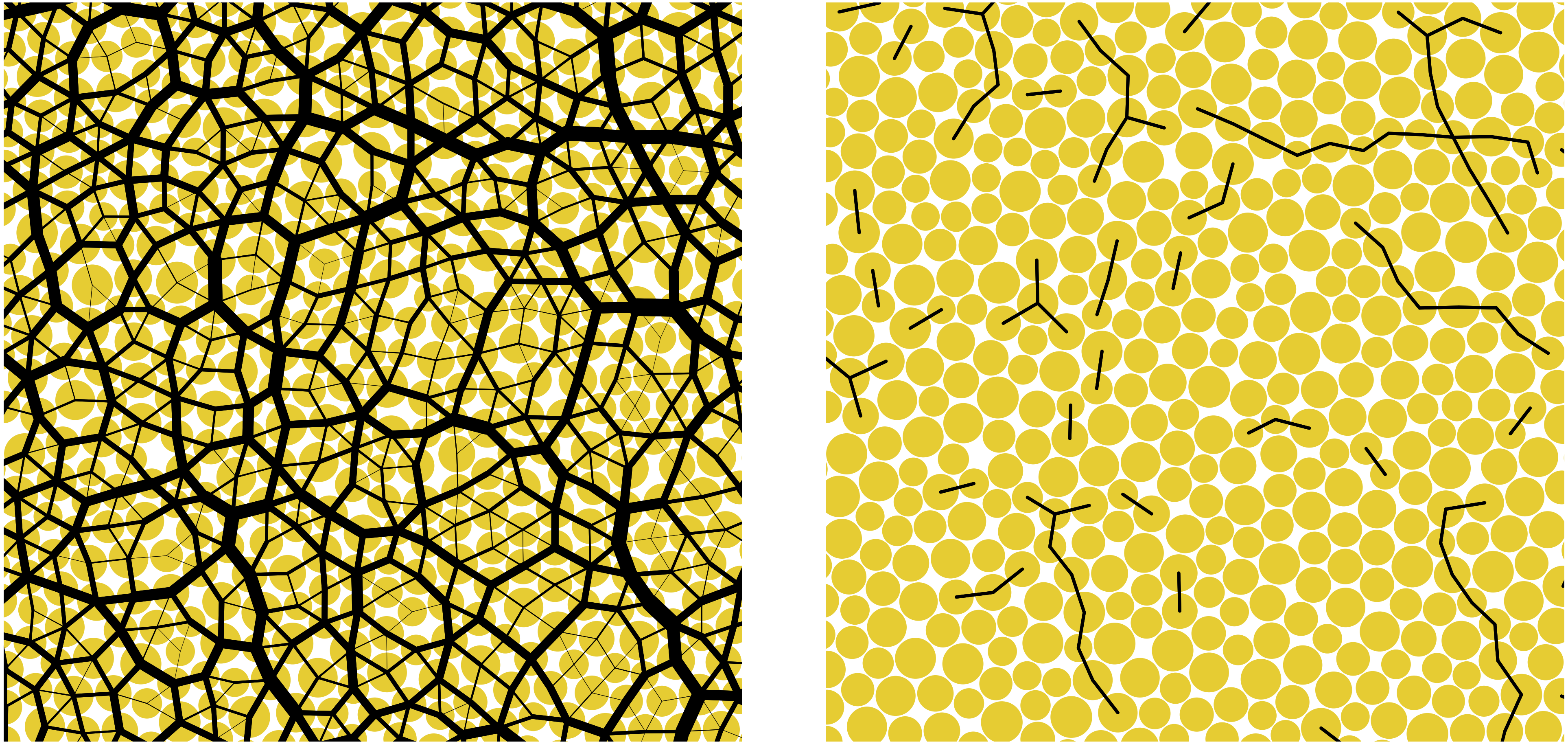
I propose to replace the $r_j$ (which are independent) by the force $f_j$ between two grains.
A threshold $f$ replaces $p$.



# Molecular dynamics to semi-hard spheres

- Polydisperse (20% in radius) to prevent crystallization.
- Elastic (Hertzian) forces: $F_{ij} = (R_i + R_j - r_{ij})^{3/2}\,\Theta(R_i + R_j - r_{ij})$
- Additional dashpot force during contact
- Dynamics under constant pressure
- Volume updates to keep up pressure
- Terminate simulation when all velocities under given threshold

The force clusters formed by all forces above a threshold $f$.



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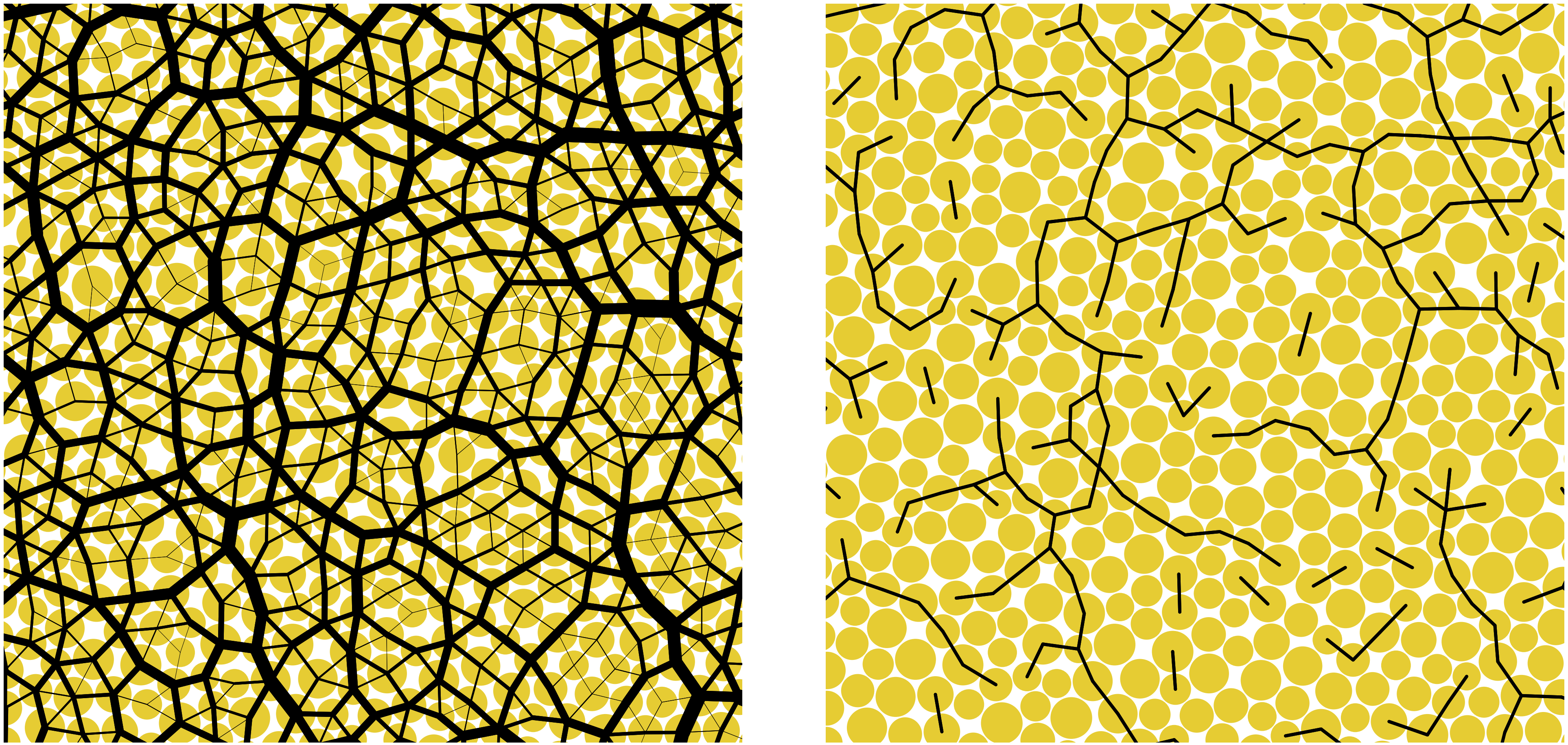
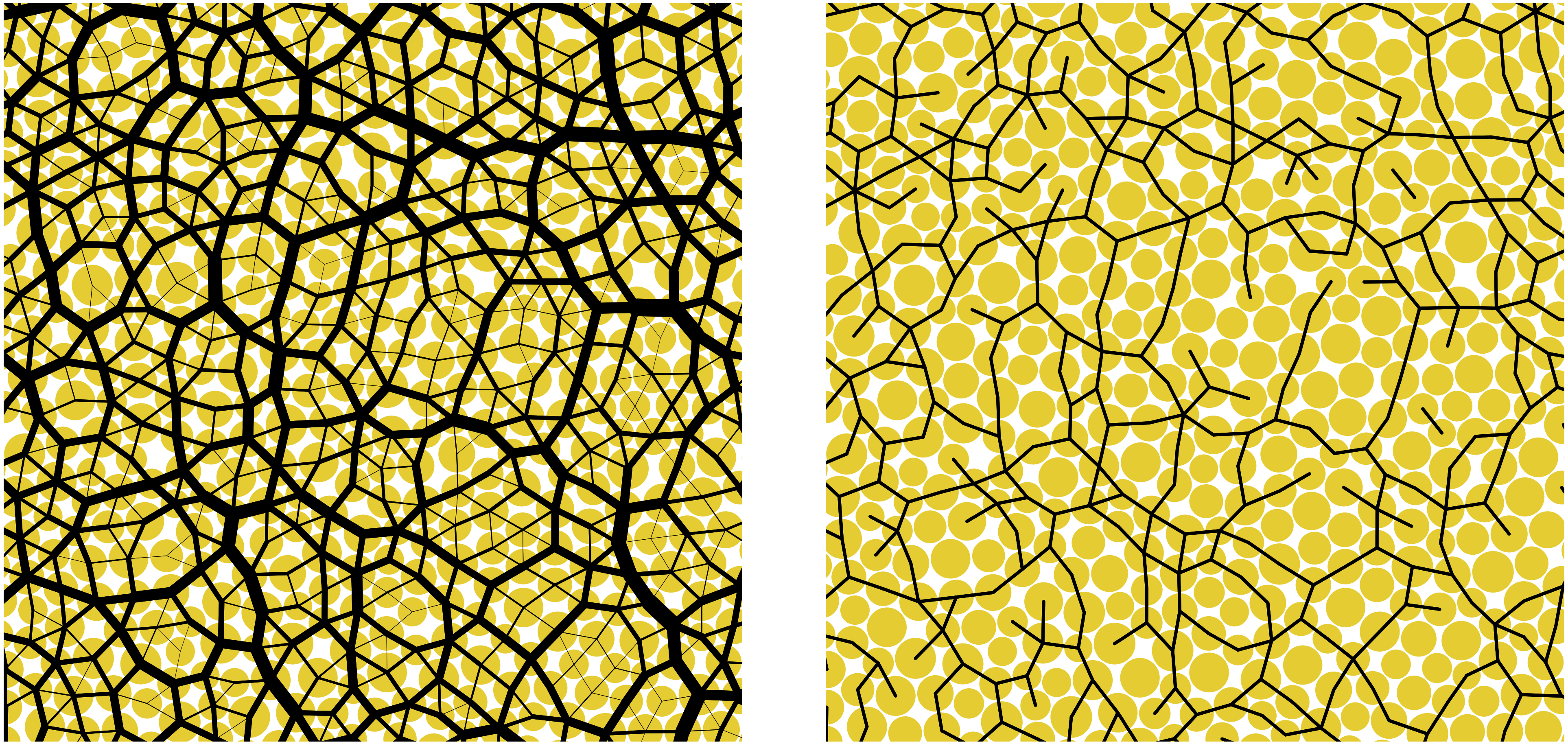
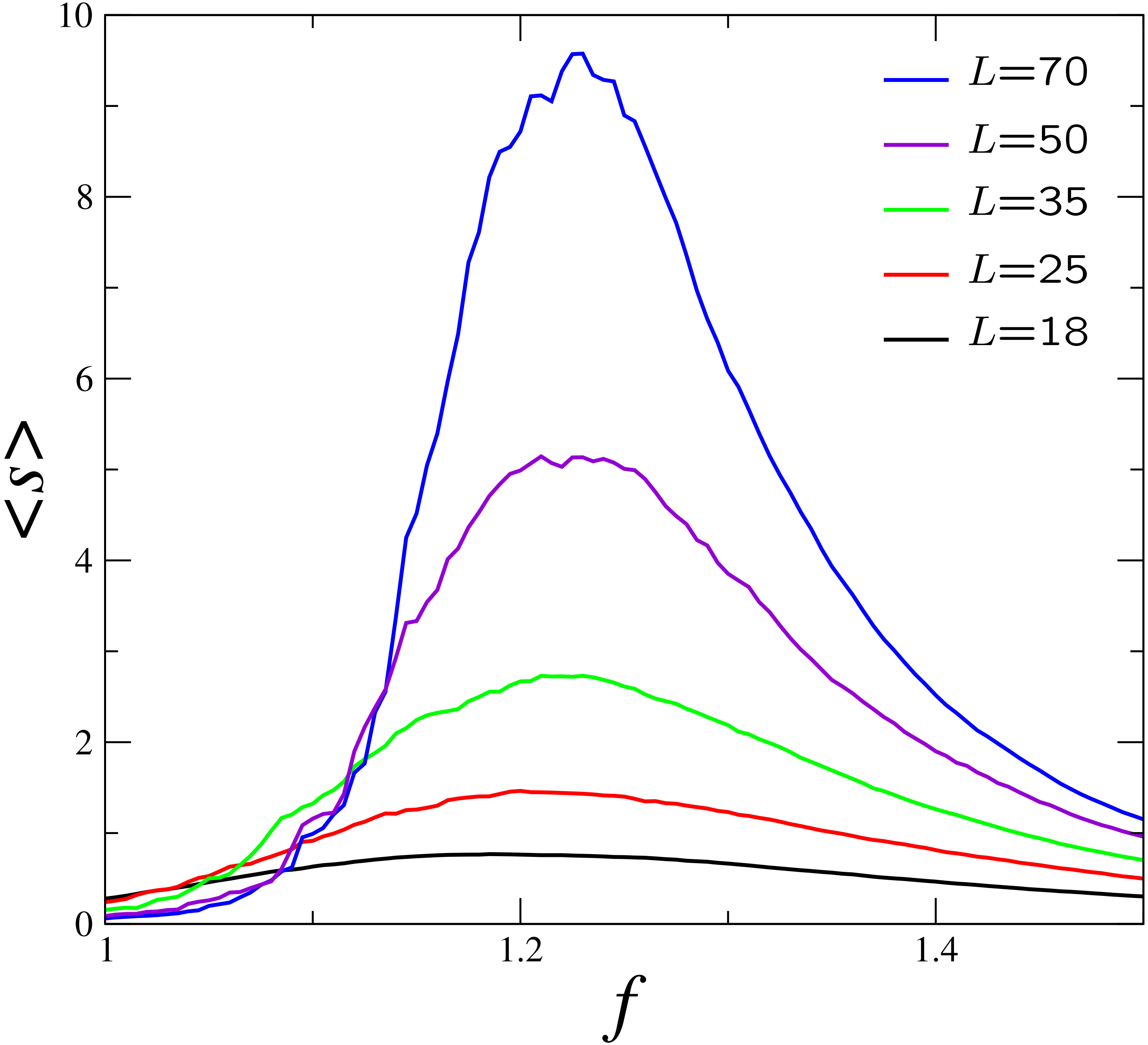


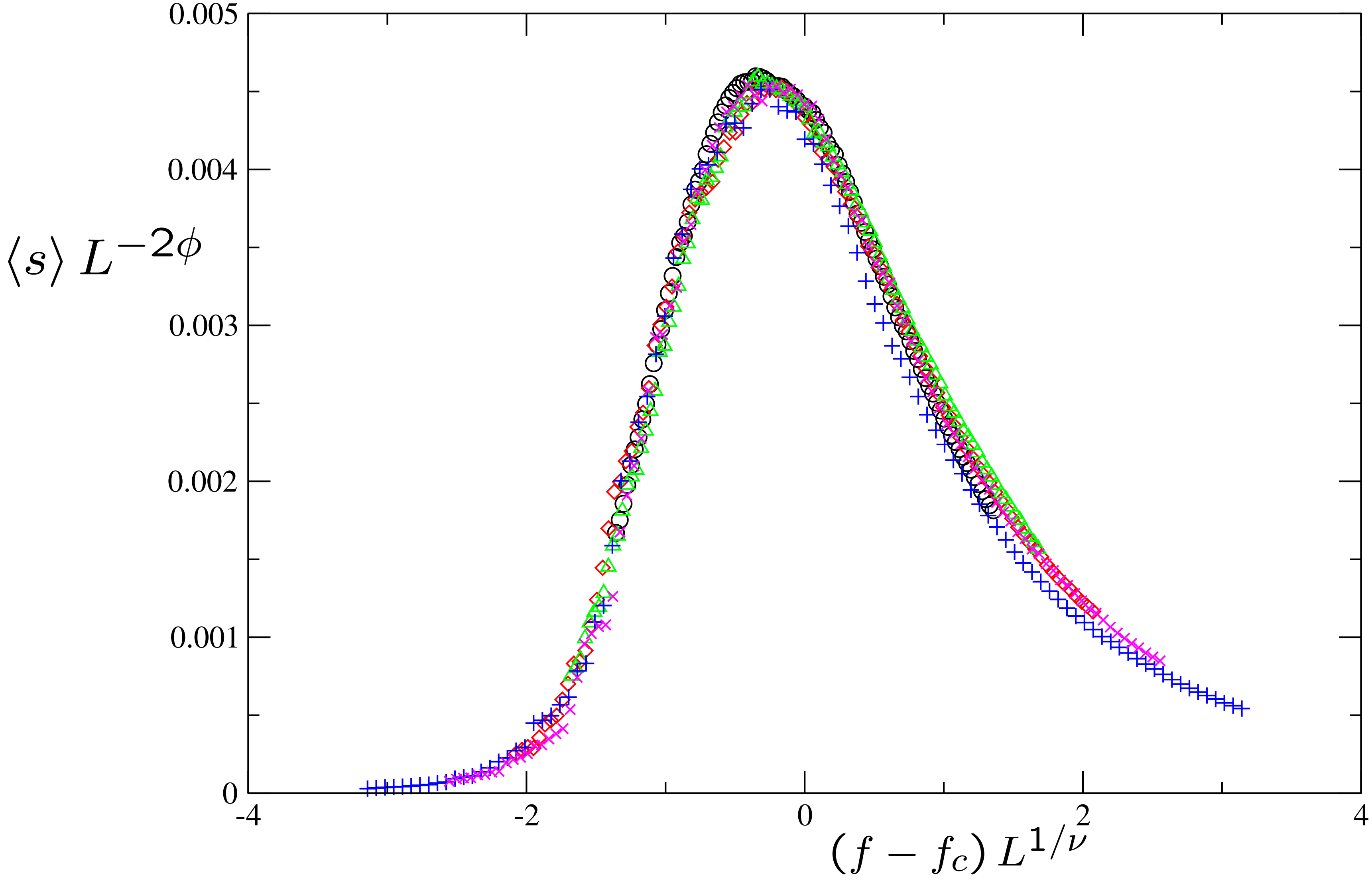
The force clusters formed by all forces above a threshold $f$.



Mean size of finite clusters vs threshold $f$ for different linear system sizes $L$.



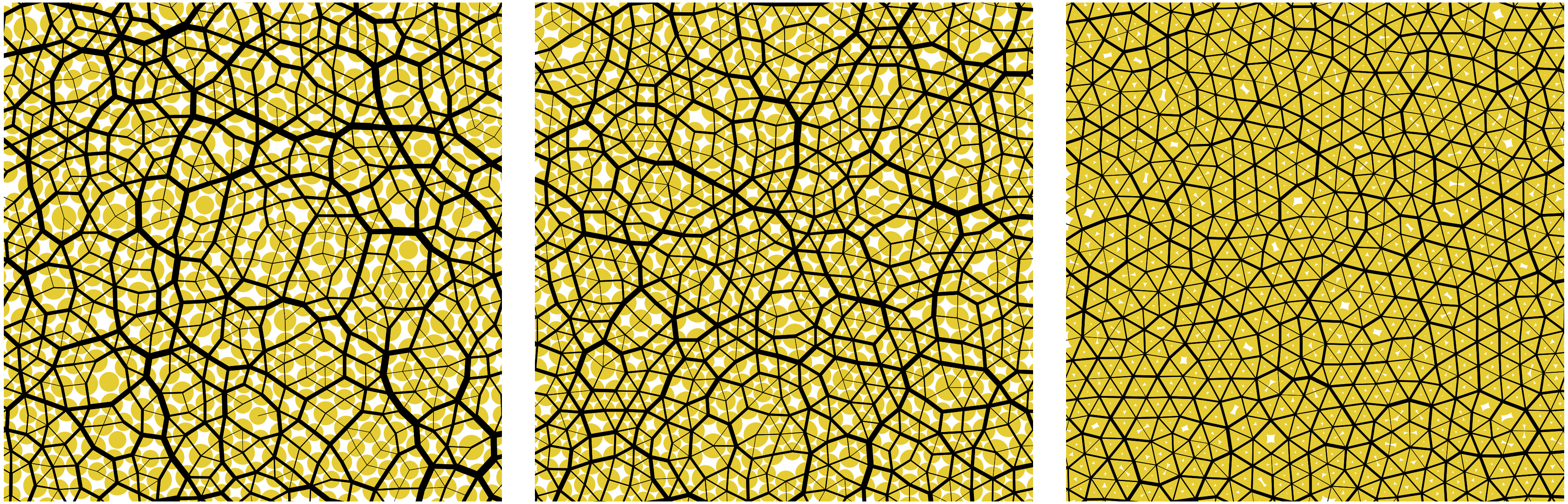
These curves can be made to collapse by scaling with the system size.




Fit parameters: $\phi$=0.89, $\nu$=1.70, $f_C$=1.25

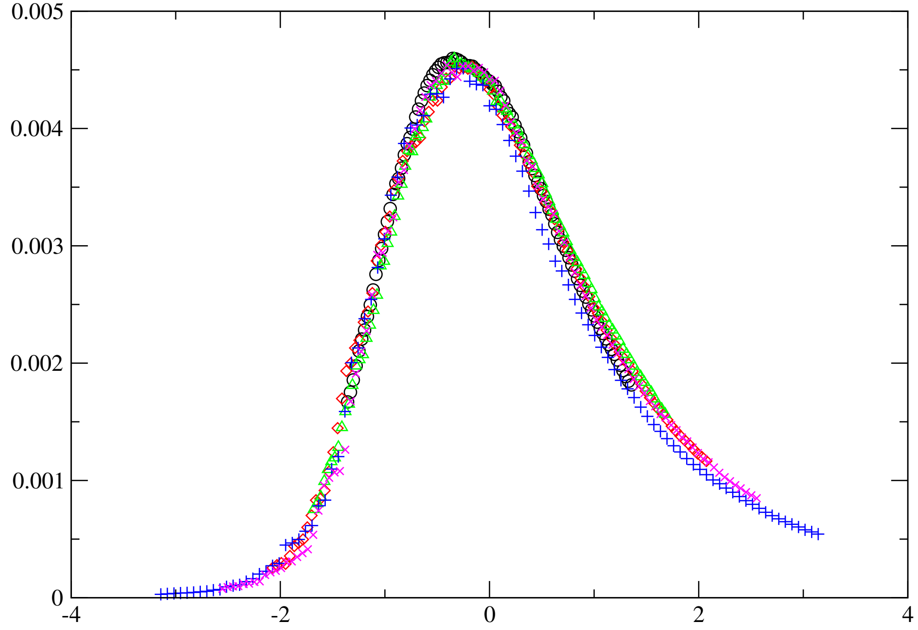
What happens with these parameters as the physical parameters vary. Take e.g. the pressure:

The force distribution depends strongly on the external pressure:

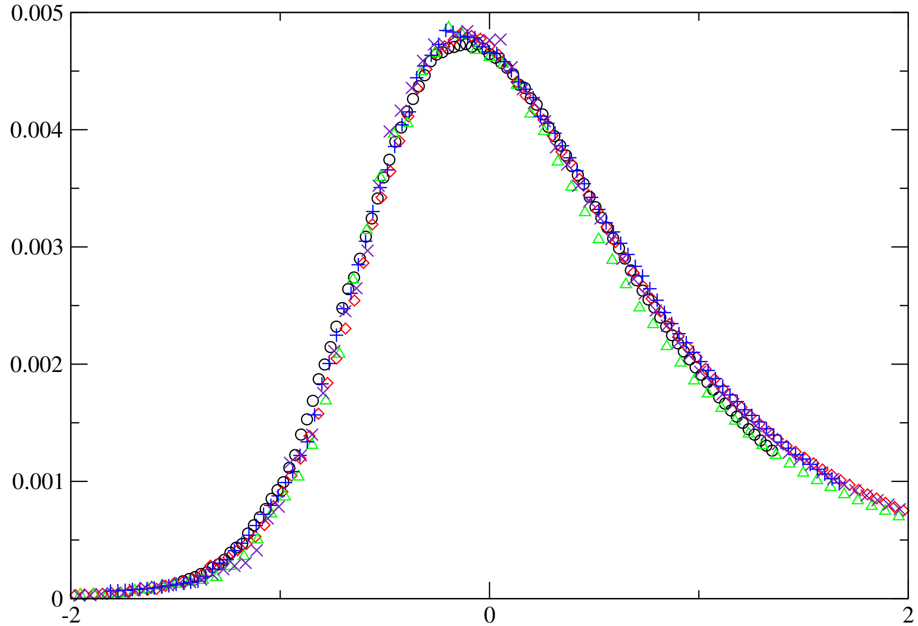


pressure = .001 pressure = .01 pressure = .1

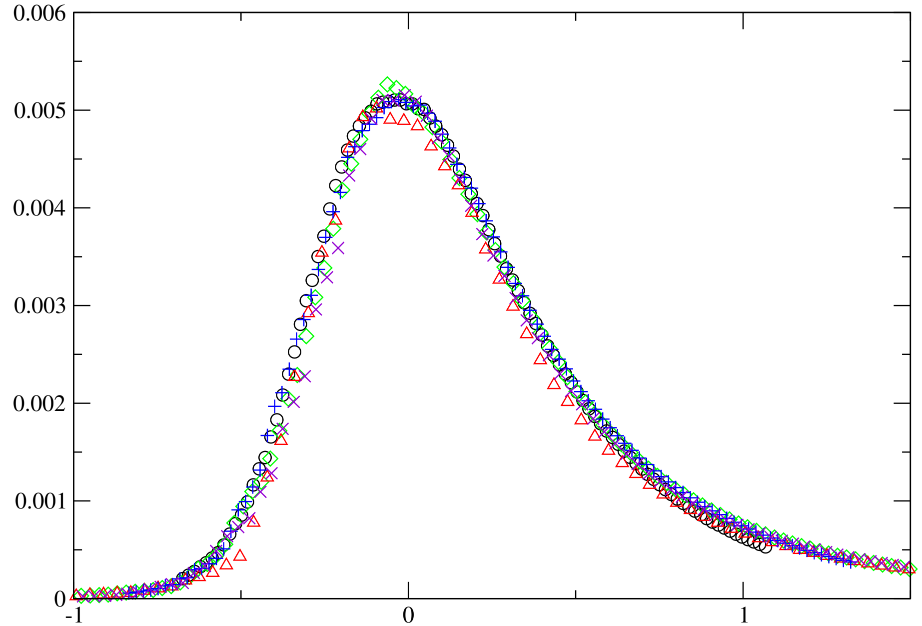
The data collapse with the same exponents $\nu = 1.7$ and $\phi = 0.89$, and, seemingly, with a very similar scaling function.




pressure = 0.001

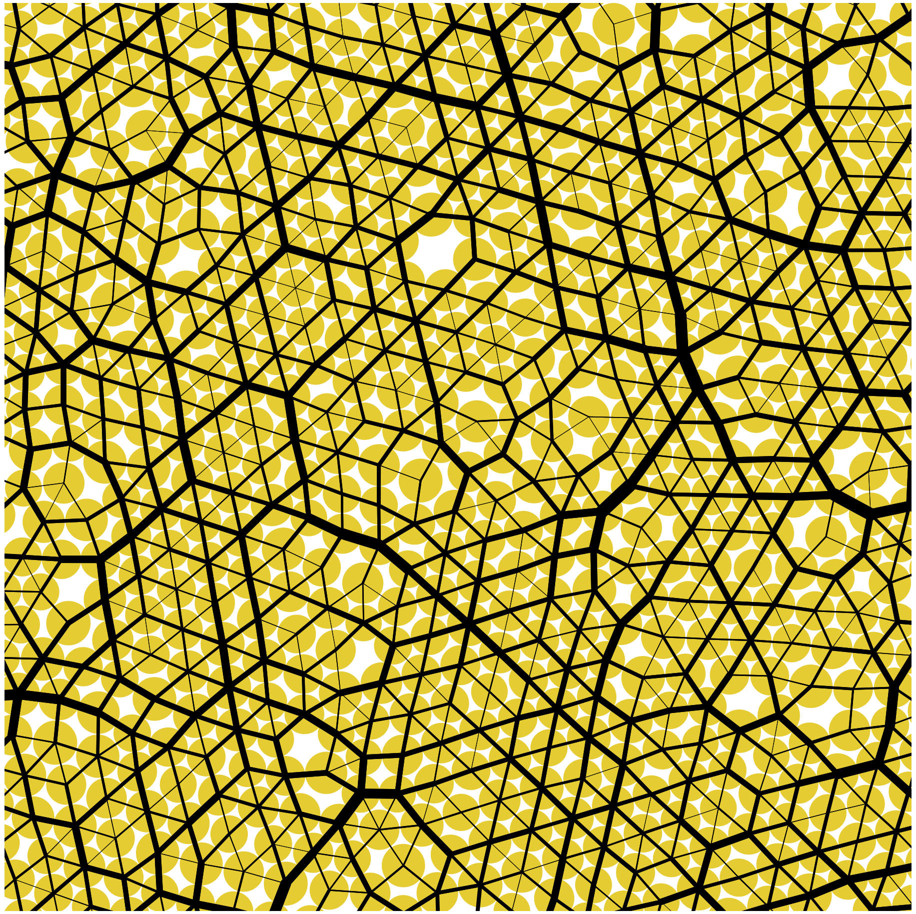



pressure = 0.01

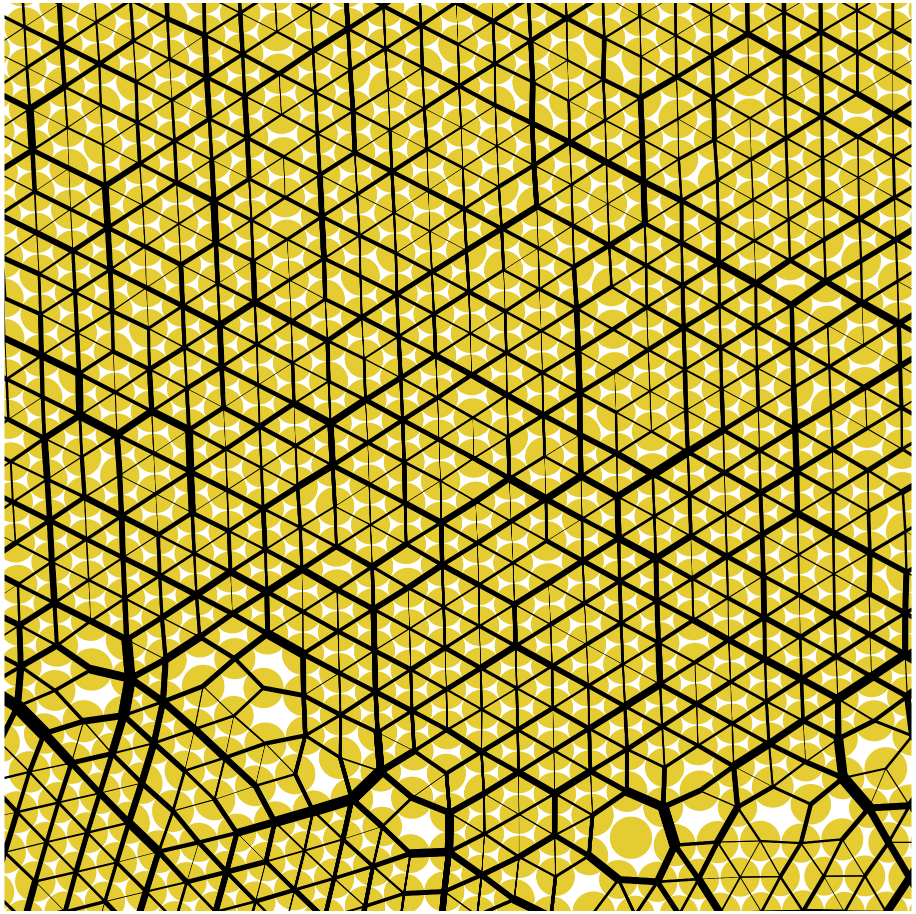



pressure = 0.1

## Another variable to play with is the polydispersity

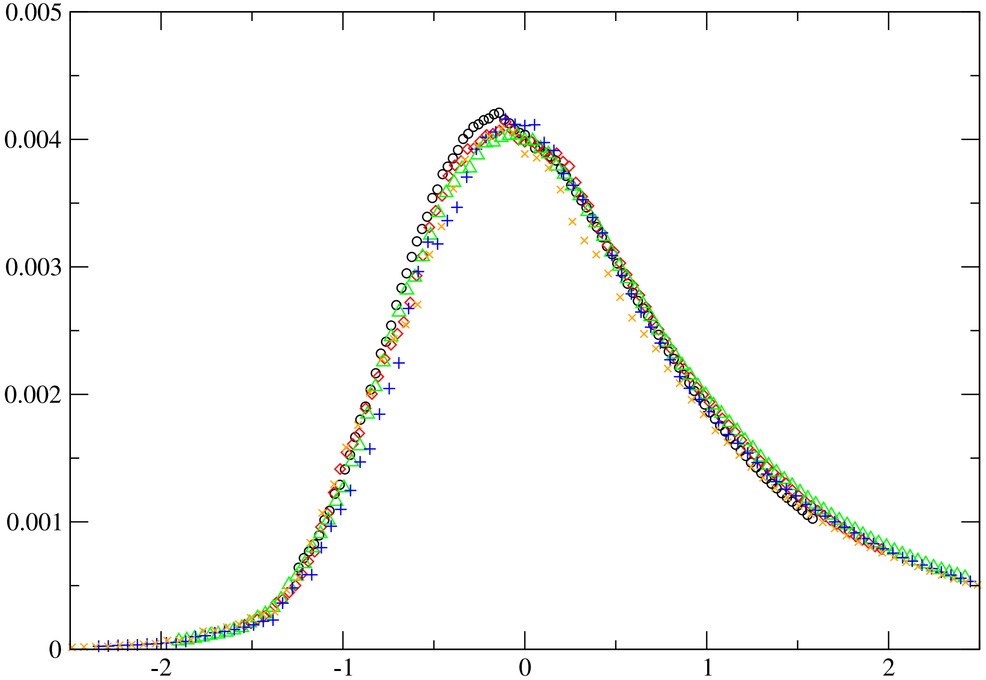


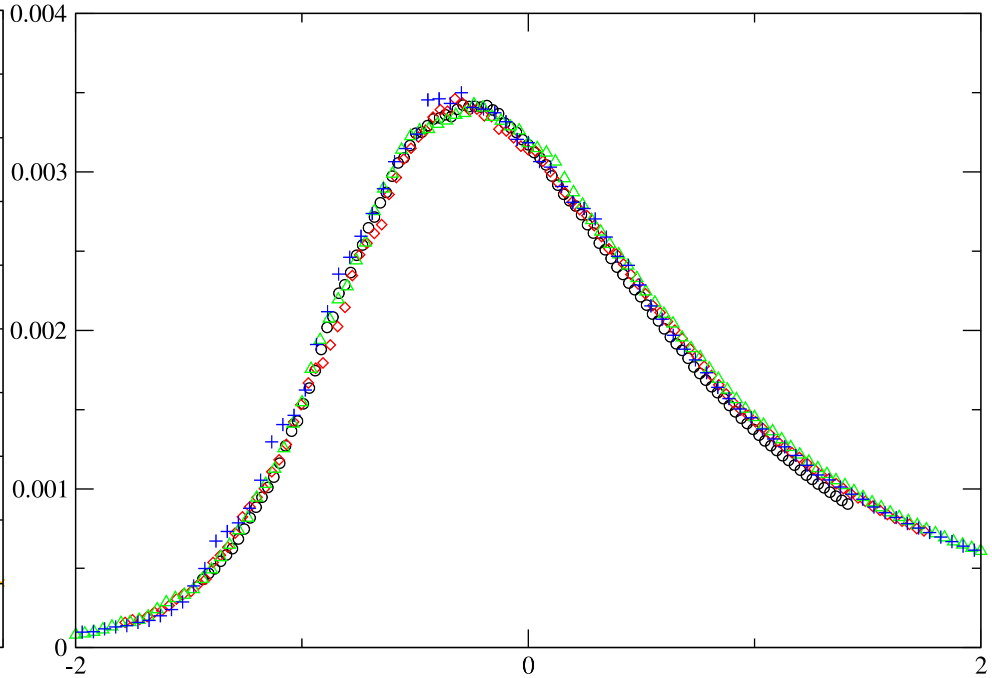
$\pm 10\%$



$\pm 5\%$

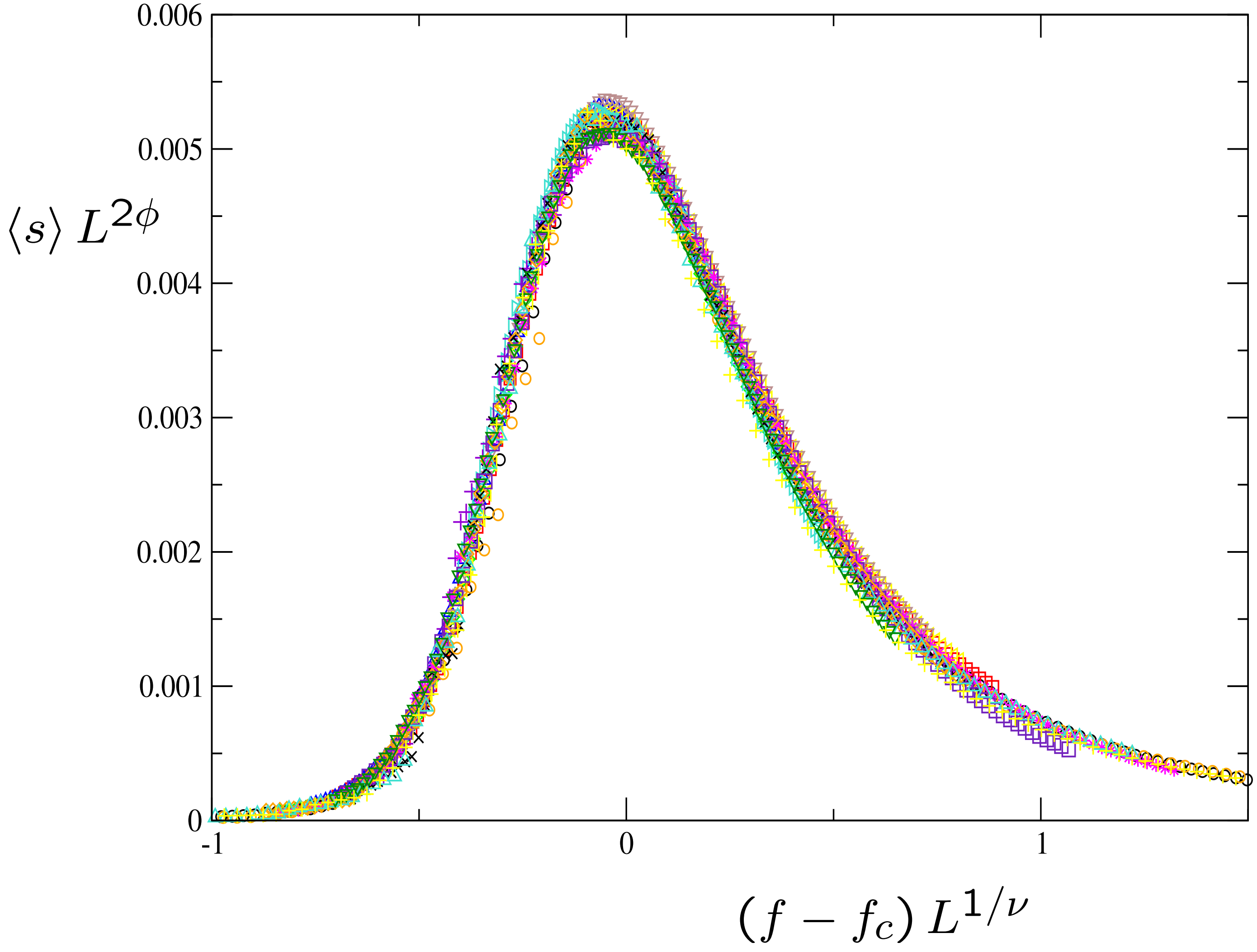
Indeed the data collapse just fine, with the same exponents:




We may conclude that pressure nor polydispersity affect the universality class.

To test the (in)dependence of the scaling function $\mathcal{S}$ we try a joint collapse of the data:




fit parameters:
global $\phi$, $\nu$,
and for each
physical condition:
(i) a vertical and
(ii) horizontal scale
and
(iii) critical
threshold $f_c$

What physical parameters do affect the universal numbers?
We explored the following candidates:

- **Force law.** Replacing Hertzian forces by harmonic springs does not affect $\nu$, $\phi$ or $\mathcal{S}$.

- **Friction between the particles.**
  This introduces torque balance in addition to force balance.
  There is no visible effect on the quantities $\nu$, $\phi$ and $\mathcal{S}$.

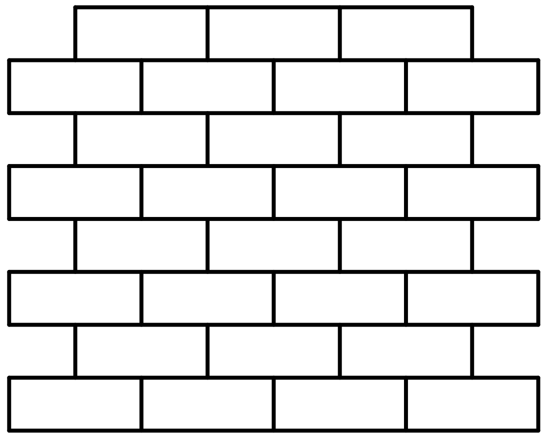
- **Shear stress** introduces anisotropy in force space.
  It has the effect that the force clusters are anisotropic in space, elongated in the direction of largest principal stress.
  This can be compensated by taking anisotropic samples.
  Again there is no effect on $\nu$, $\phi$ and $\mathcal{S}$: Weak anisotropic scaling.

- **Yield stress.** Even when the shear stress is so large that the stack is on the verge of flowing, the $\nu$, $\phi$ and $\mathcal{S}$ remain unchanged.

Another way to investigate the universality class is to try other models.

A very simple model for the distribution of vertical forces in static piles is the Q-model.

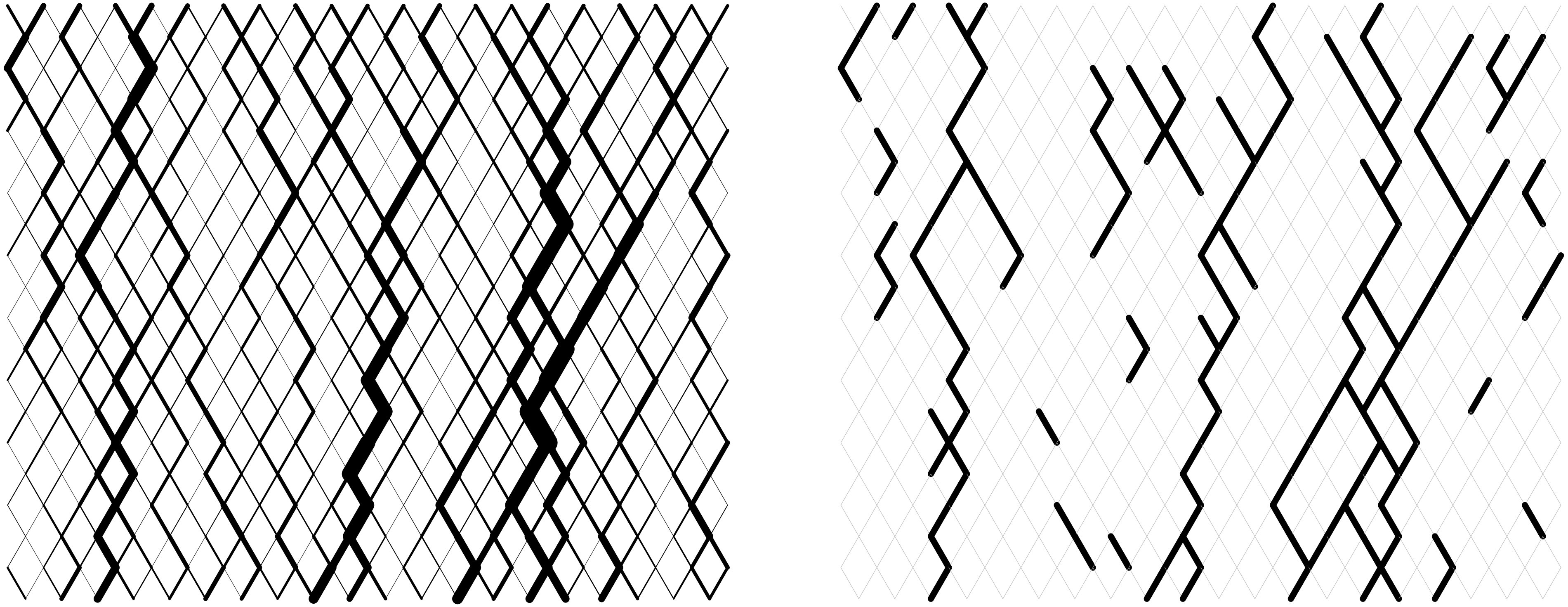
In the Q-model each grain (brick) rests on top of two others and randomly and independently of the rest of the stack, distributes its load over its two lower neighbors.



We also consider a variation, in which each overall force configuration is equally probable: the Force Ensemble, FE. This version has the same configuration space, but a different probability measure.

# The Q-model

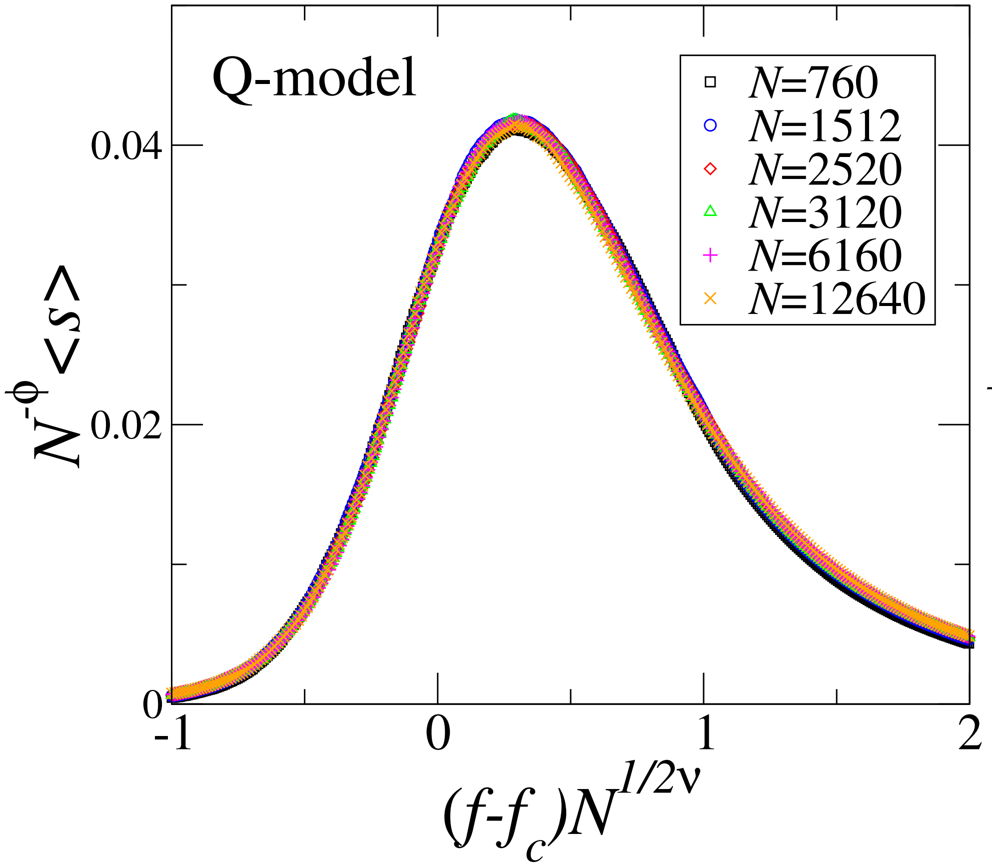
An example of a configuration, and of a clustering for some value of the threshold. NB all forces are vertical: the graph represents the topology, not the geometry.



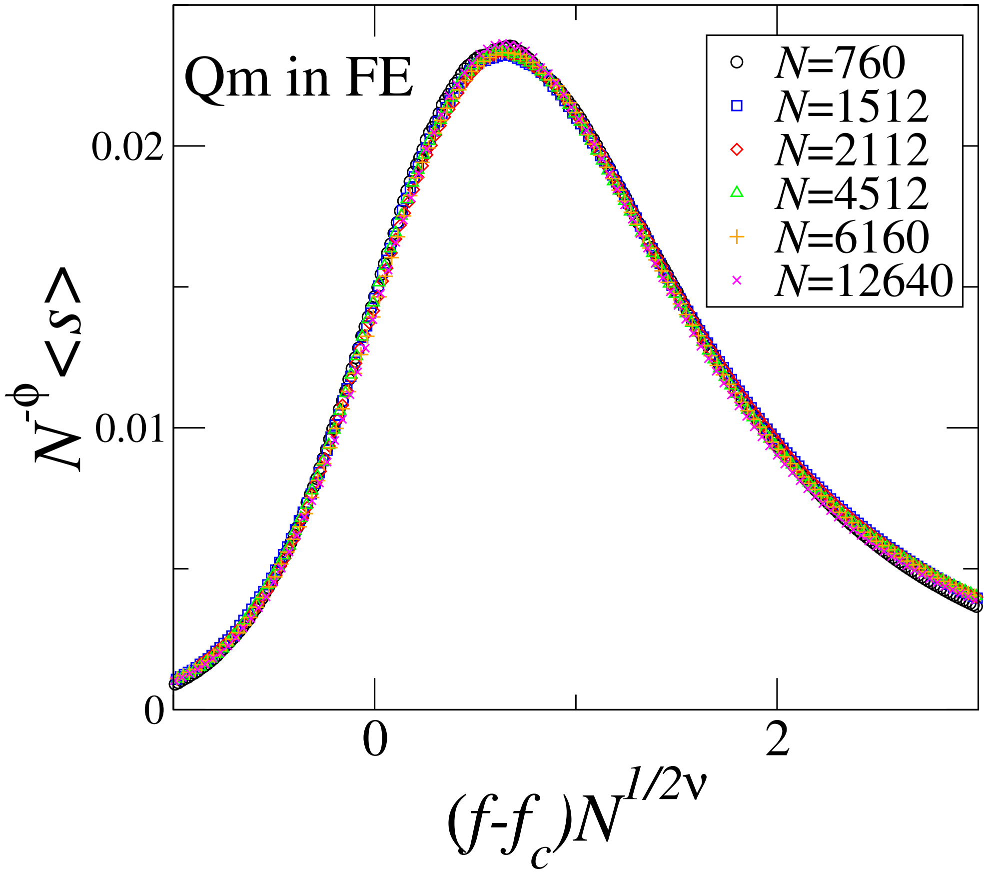
Data collapse works fine, with qualitatively the same results:

(Here number of particles $N$ is used in stead of area $L^2$)

The true Q-model.




Q-model topology with FE




However the exponents differ:

$\nu = 3.1(1)$, $\phi = 0.689(1)$ and $\nu = 1.65(5)$, $\phi = 0.816(1)$

Scaling seems to work. But the exponents differ from the more realistic MD simulations. In particular, the exponent $\nu$ is quite different. And the scaling function $\mathcal{S}$ has its maximum at positive argument.

We suspect that the uni-axiality of the force and the strong anisotropy in space, may be the source of the discrepancy.

Therefore we consider a more realistic, but still very simple model: Equal Hard Disks in the Force Ensemble (FE): all balanced force configurations with the same probability.

Equal disks packed in a triangular array.

All forces radial (no friction).

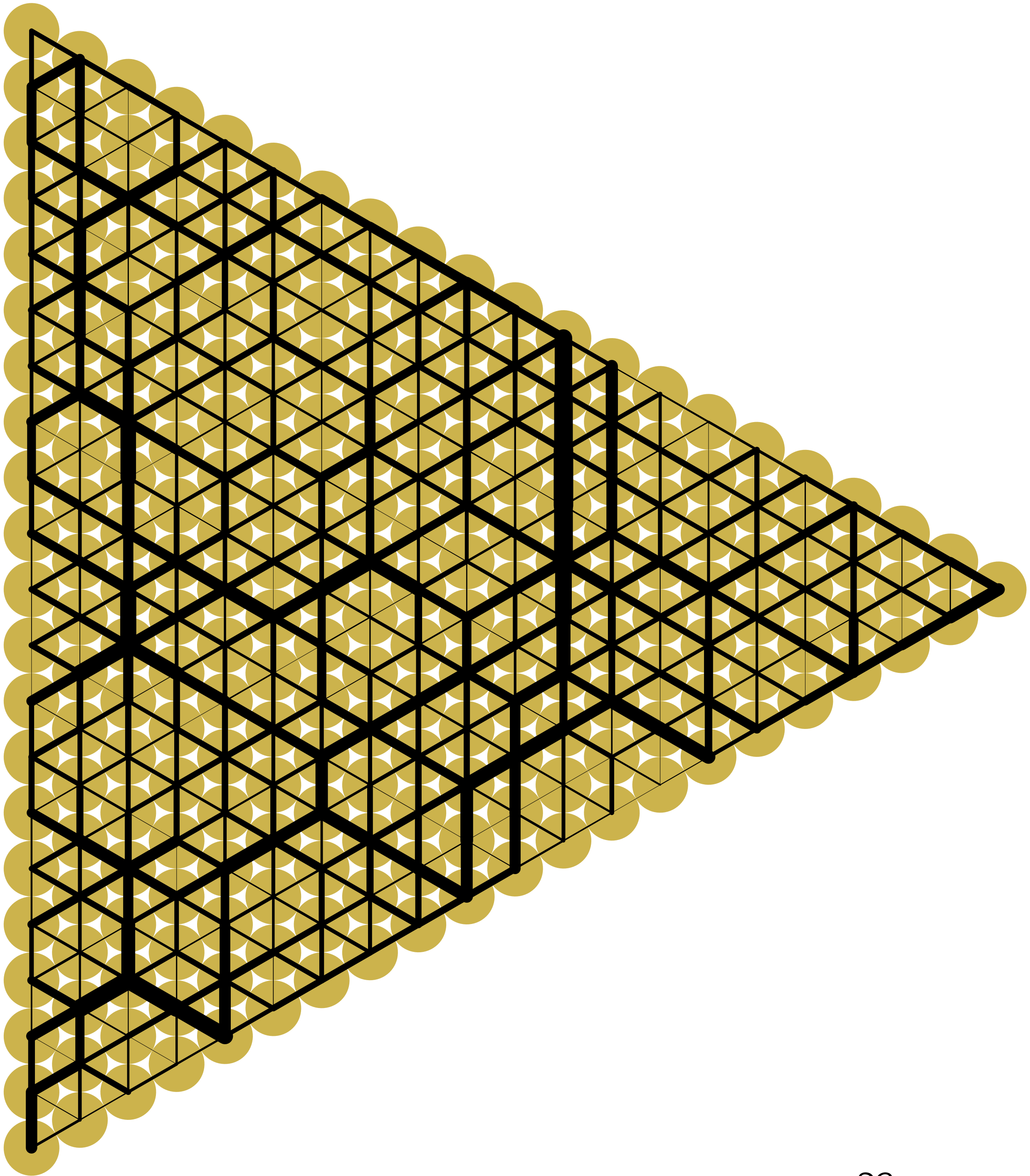
Three forces per grain, and two stability constraints, leaves one free variable per grain.

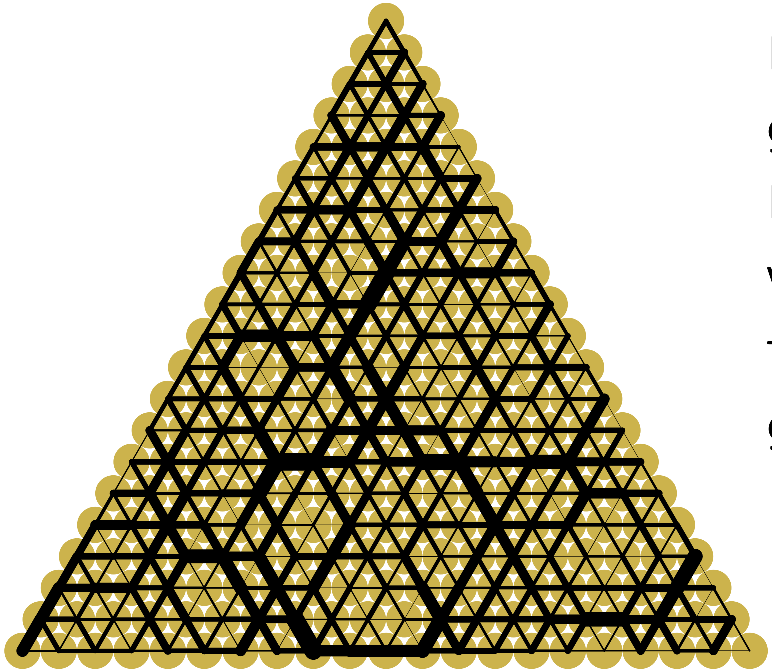
All forces are positive.

External pressure is fixed.

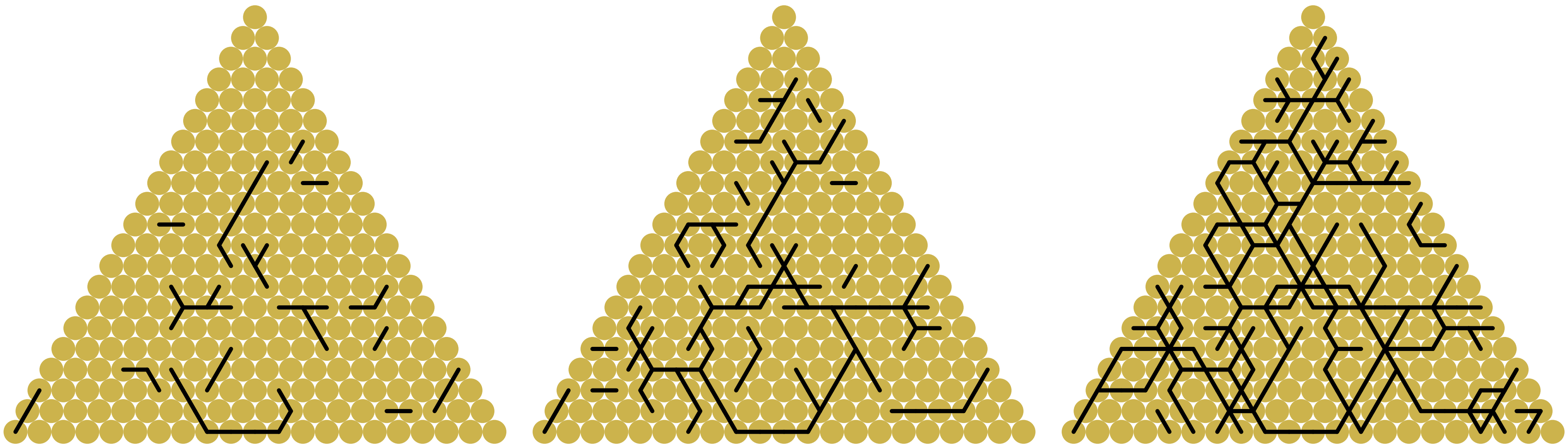
**FE** prescribes that all force configurations are equally probable.

The FE is sampled by means of Monte Carlo.

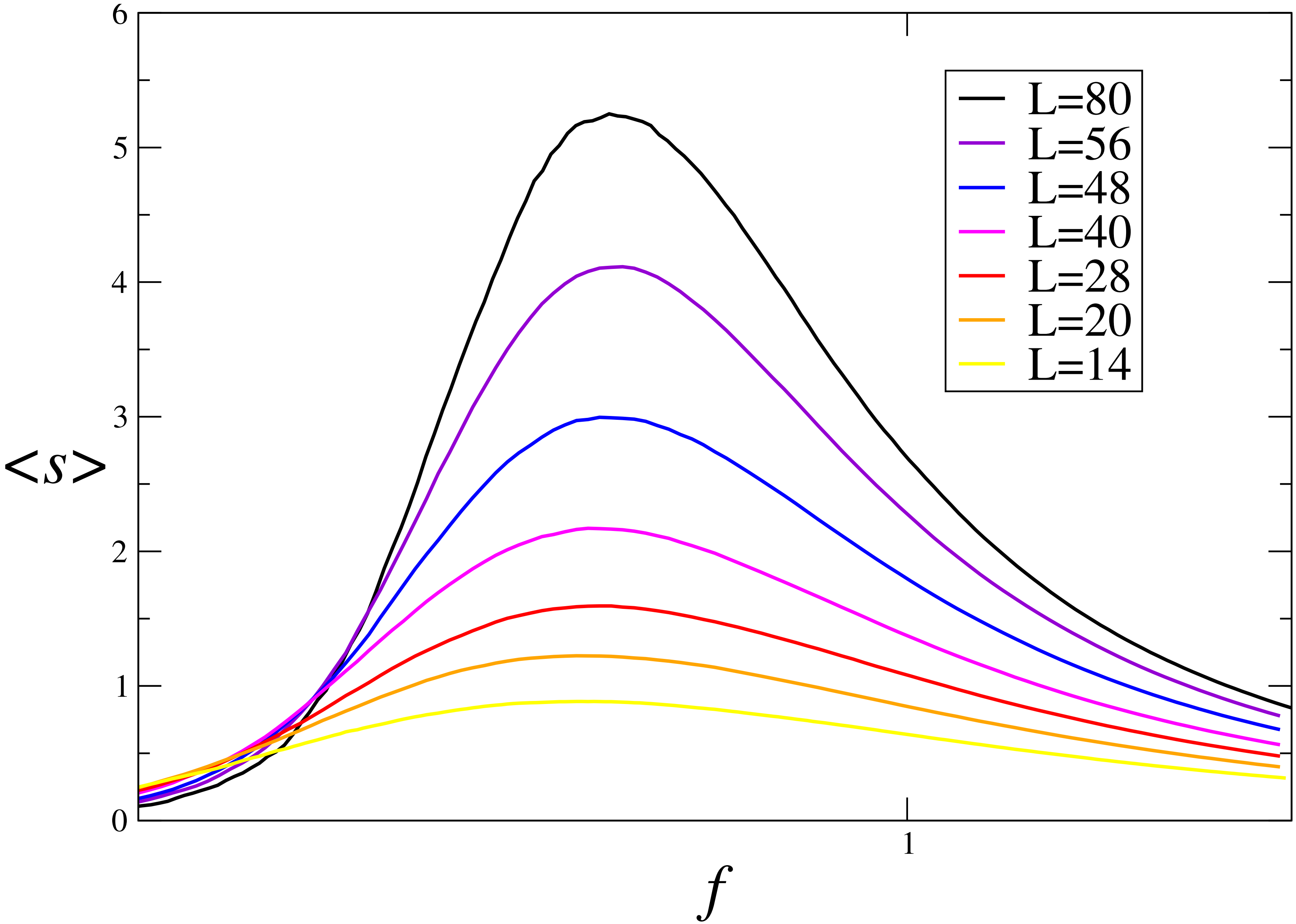




Force configurations are generated by Monte Carlo. For each force configuration we can vary the threshold to partition the contact graph into force clusters.



The mean square of the size of the clusters versus the threshold $f$.

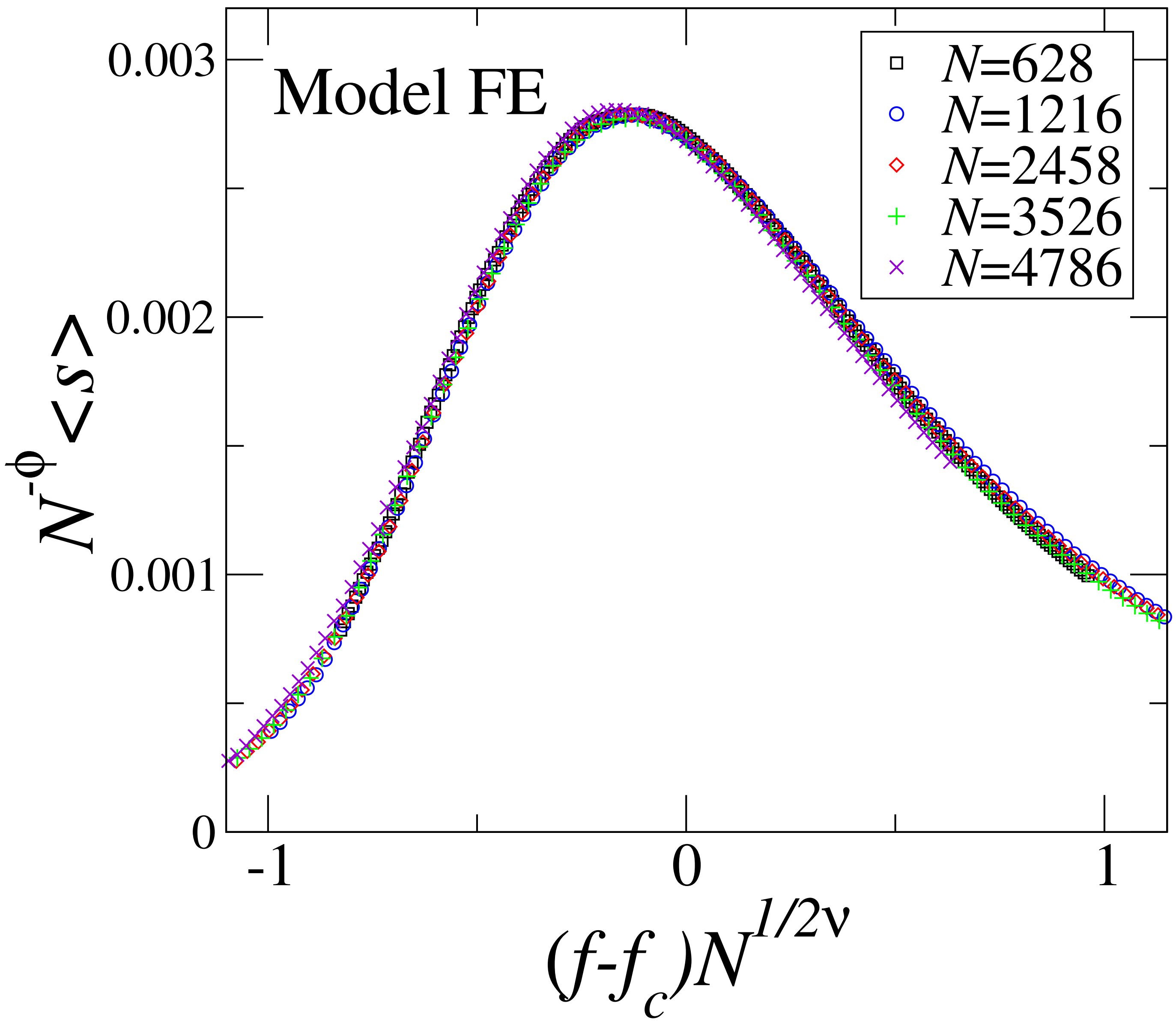


Again, scaling works quite well,

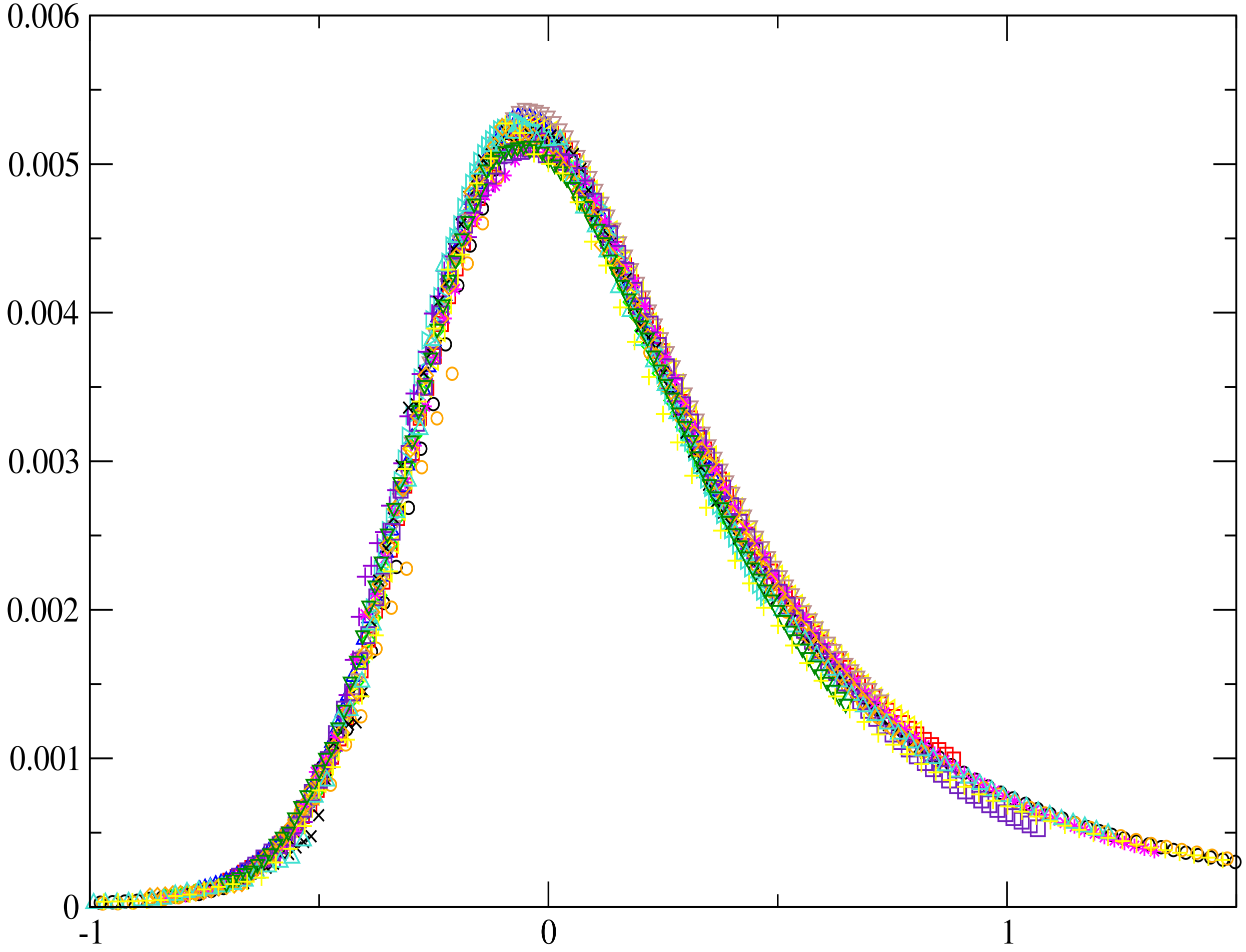
and now the exponents are within the uncertainty of the MD results.

$\phi = 0.894(3)$

$\nu = 1.80(5)$



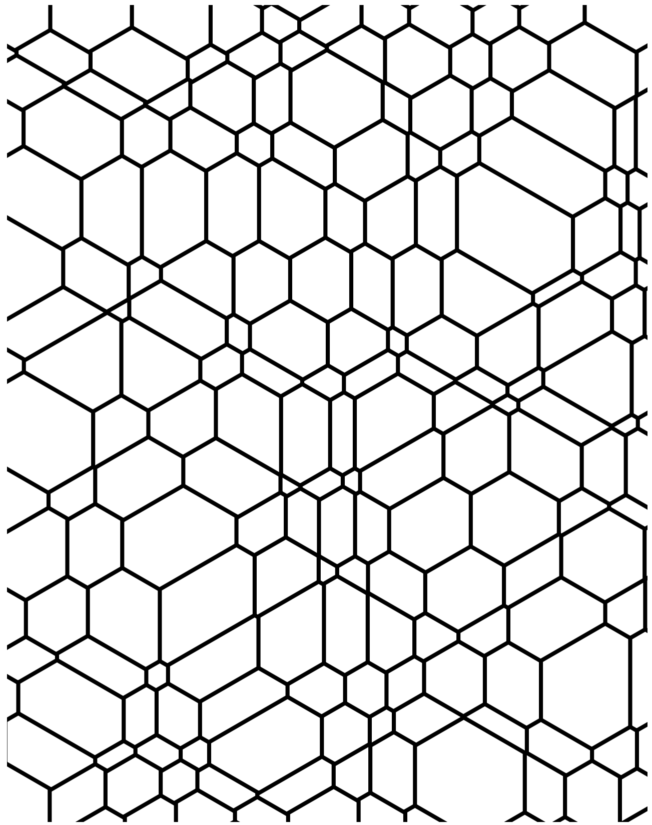
Indeed, a joint collapse of **FE** and **MD** data looks quite acceptable:

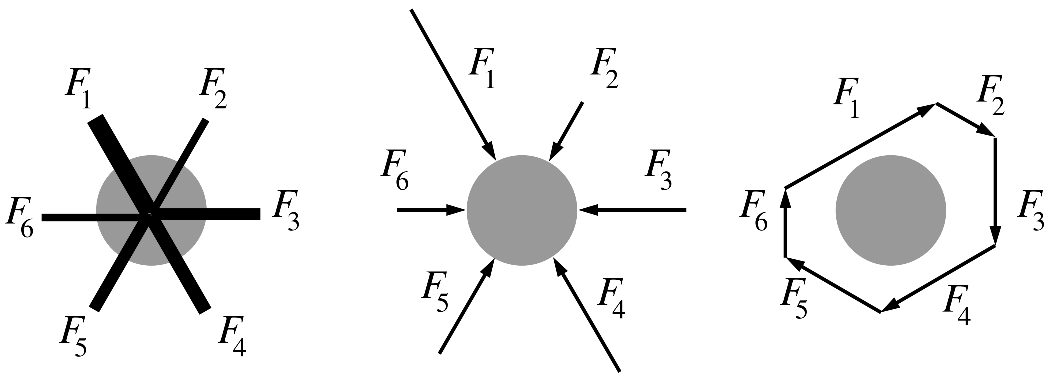



This confirms that the scaling behavior of force chains in the Force Ensemble and in the more realistic MD simulations is in the same universality class.

Having found (i) universal behavior, and (ii) a simple model in the same universality class with realistic simulations, it is noteworthy that the Force Ensemble for monodisperse hard disks is integrable.

The forces on each grain can be transformed into a hexagonal cell by turning them into the lengths of the sides of a hexagon. The balance condition on the forces translates into the condition that the hexagon is closed.





- The force networks in 2D appear to have universal fractal properties. NATURE 439 (7078): 828-830, Feb 16 (2006)

- Pressure, polydispersity, packing disorder, friction, force law, and also shear stress even up to yield stress are irrelevant.

- History with inelastic collision dynamics may be replaced by the trivial Force Ensemble

- The popular Q-model is in a different universality class

- Raises many questions:
  meaning of $f_C$, other exponents, 3 dimensions, grain shapes, jamming, solvable model, other universality classes,...