Towards conformal invariance of 2-dimensional lattice models

Stanislav Smirnov
Université de Genève

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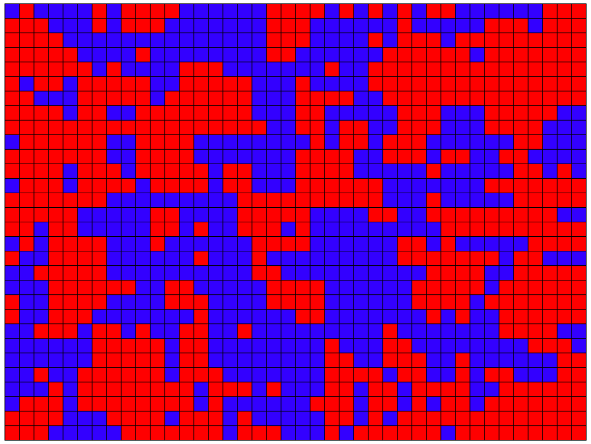
2-dim lattice models of natural phenomena:
Ising, percolation, self-avoiding polymers, . . .



- Realistic models from solid-state physics, ferromagnetism, polymer chemistry, . . .
- Connections to Conformal Field Theory
- Unrigorous arguments describe phase transitions exactly [den Nijs, Nienhuis, Cardy, Duplantier, . . .]

  Average cardinality of a finite $p-$percolation cluster $\approx |p - p_c|^{-43/18}$

  Dimension(Ising random cluster at T_c) $= 187/96$

  Number of length N simple curves on a lattice $\approx \mu(\text{lattice})^N \, N^{11/32}$

For a long time poor mathematical understanding

Example: critical percolation

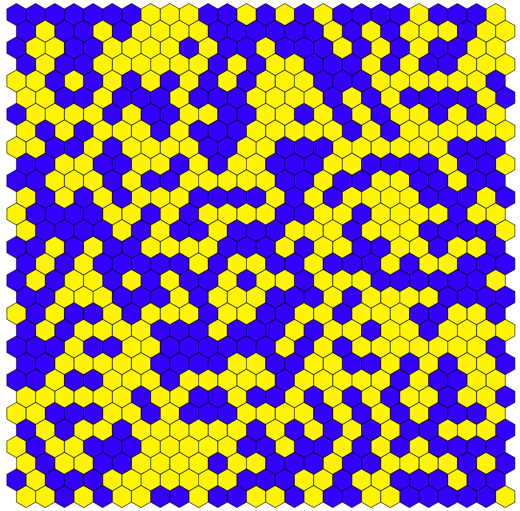
to color every hexagon we toss a coin:

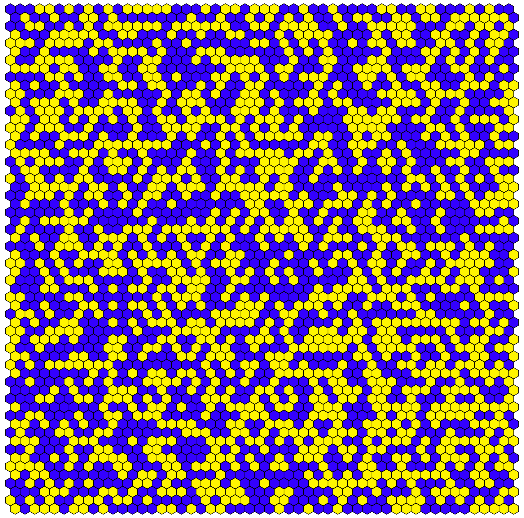
tails $\Rightarrow$ blue, heads $\Rightarrow$ yellow

Blue hexagons are holes in a yellow rock.

Can the water sip through? Hard to see!

The reason is that clusters (connected sets of blue hexagons) are complicated fractals of dimension $91/48$ (meaning that a cluster of diameter D on average has $\approx D^{91/48}$ hexagons).





Cardy's prediction: in the scaling limit

$$\mathbb{P}\,(\text{crossing}) = \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}\, m^{1/3}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; m\right)$$

Proved on hexagonal lattice [S 2001]

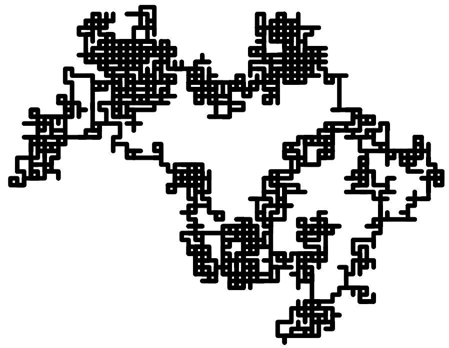
Underlying idea in physics arguments: at critical temperature
(i) the model has a continuum scaling limit (as mesh$\to 0$)
(ii) the limit is universal (independent of the lattice)
and conformally invariant (preserved by conformal maps)

Recently mathematical progress with new, rigorous approaches. Oded Schramm described possible conformally invariant scaling limits of cluster interfaces: one-parameter family of SLE(κ) curves. With their help many predictions were proved or explained.

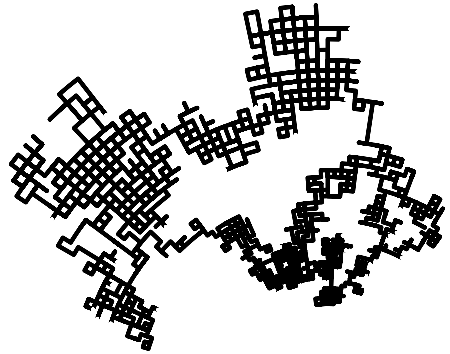
We will discuss the point which is still less understood both from mathematics and physics points of view: **why (i) and (ii) hold?**

Classical example: Random Walk $\longrightarrow$ Brownian Motion

As lattice mesh goes to zero, RW $\to$ BM: probability measure on broken lines converges weakly to probability measure on continuous curves. BM is conformally invariant [P. Lévy] and universal.

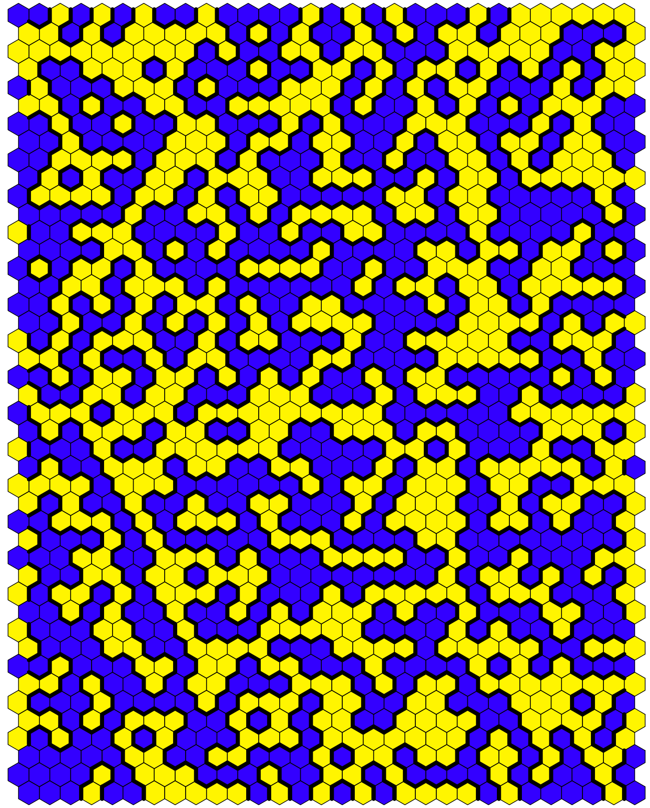


$\downarrow \phi$



Conjecturally: in most 2-dim models at critical temperatures, universal conformally invariant SLE curves arise as scaling limits of the interfaces (cluster boundaries).

We discuss available results and possible approaches to

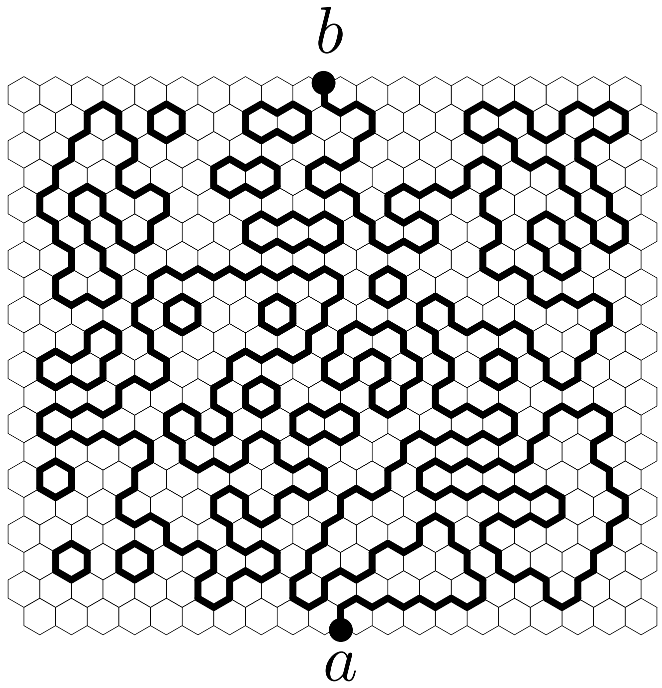


- percolation
- Ising
- Potts
- spherical $O(n)$
- Fortuin-Kasteleyn random cluster
- self-avoiding random walk
- uniform spanning tree

Those include most "classical models."
Fit into two families of **loop gases**
corresponding to high and low temperature expansions.

(I) $O(n)$ loop gas. Configurations of disjoint simple loops on hexagonal lattice. Loop-weight $n \in [0, 2]$, edge-weight $x > 0$.

$$\boldsymbol{Z = \sum_{\text{configs}} n^{\#\ \text{loops}}\, x^{\#\ \text{edges}}}$$



Dobrushin boundary conditions:

besides loops, an interface $\gamma : a \leftrightarrow b$.

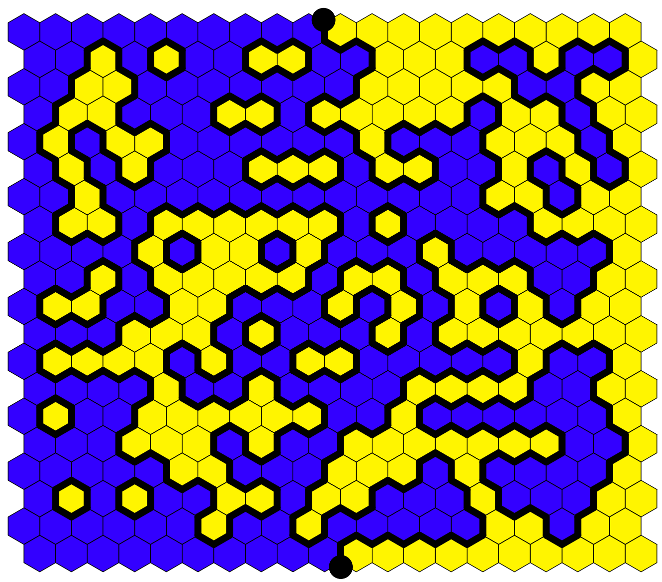
Conjecture [Kager-Nienhuis,...]. $\exists$ *conformally invariant scaling limits for* $x = x_c(n) := 1/\sqrt{2+\sqrt{2-n}}$ *and* $x \in (x_c(n), +\infty)$.

Two different limits correspond to dilute / dense phases

(limiting loops are simple / non-simple)

For $x \in (0, x_c(n))$ interfaces converge to lines – no conformal invariance

Hexagons of two colors (Ising spins ± 1), which change whenever a loop is crossed.



For $\boldsymbol{n = 1}$ the partition function becomes

$$\boldsymbol{Z = \sum x^{\#\ \text{edges}}}$$
$$\boldsymbol{= \sum x^{\#\ \text{pairs of neighbors of opposite spins}}}$$

$\boldsymbol{n = 1,\ x = 1/\sqrt{3}}$**: Ising model at** $\boldsymbol{T_c}$

Note: critical value of x is known [Wannier]

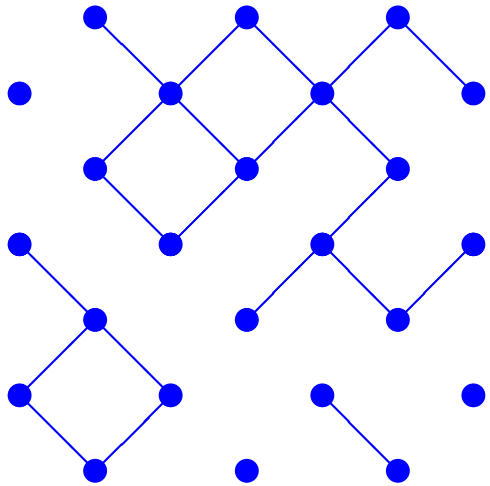
$\boldsymbol{n = 1,\ x = 1}$**: critical percolation** (on hexagons = sites of the dual triangular lattice) All configs are equally likely ($p_c = 1/2$ [Kesten, Wierman]).

$\boldsymbol{n = 0,\ x = 1/\sqrt{2+\sqrt{2}}}$**:** a version of **self-avoiding random walk**

(II) Fortuin-Kasteleyn random cluster model

Configuration: some edges of square lattice declared open.

$$\boldsymbol{Z = \sum \left(\frac{p}{1-p}\right)^{\#\ \text{open edges}} q^{\#\ \text{clusters}}}$$

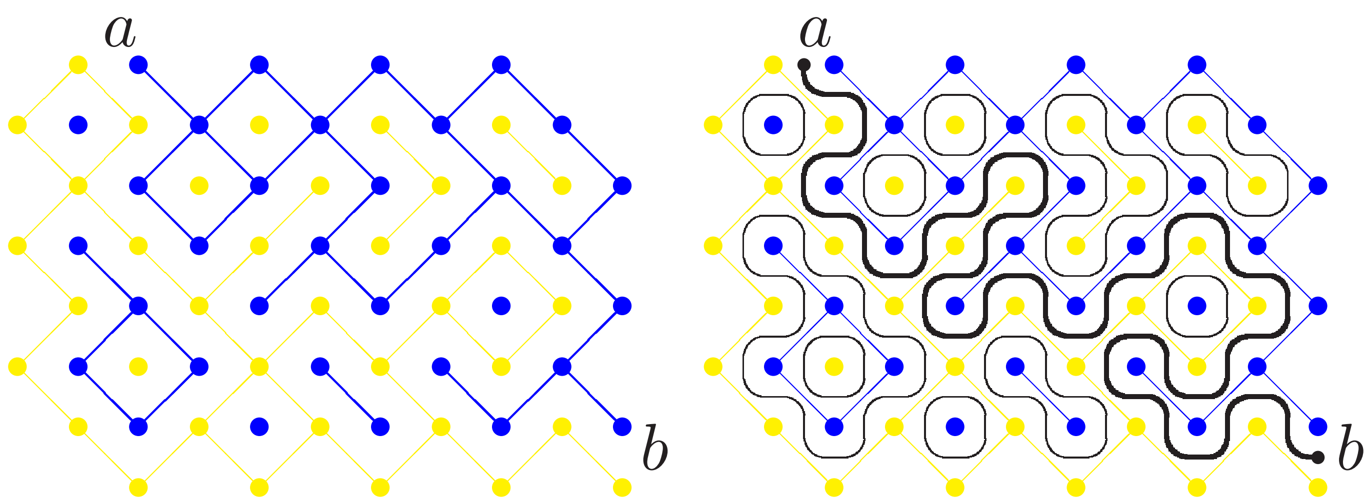


Edge-weight $p \in [0,1]$, cluster-weight $q \in [0,\infty)$.
Clusters are max graphs connected by open edges.

Conjecture [Rohde-Schramm,...]. *There is a conformally invariant scaling limit for $q \in [0,4]$ and $p = p_c(q) = \sqrt{q}/(\sqrt{q}+1)$.*

Random cluster representation of q-state Potts model: $\boldsymbol{q = 2}$ FK Ising model, $\boldsymbol{q = 1}$ bond percolation on the square lattice, $\boldsymbol{q = 0}$ uniform spanning tree.

Loop representation of the FK random cluster model



Configurations are dense loop collections on the medial lattice

Loops separate clusters from dual clusters

Dobrushin boundary conditions: besides loops an interface $\gamma : a \leftrightarrow b$

For $p = p_c(q)$ the partition function is $\boldsymbol{Z = \sum \left(\sqrt{q}\right)^{\#\ \mathrm{loops}}}$

Scaling limit (as lattice mesh $\longrightarrow 0$) of . . .

(i) observable: spin correlation, crossing probability, . . .
(ii) one interface
(iii) full loop or cluster collection, random height function, . . .

Rem 1 Scaling limit of (i) implies scaling limit of (ii) and (iii).

Rem 2 (ii) might be optimal: Oded Schramm classified all possible scaling limits – SLE(κ) curves, which are well adapted to calculations: scaling exponents deduced using Itô's stochastic calculus (on the real line).

Conformal invariance of the scaling limit was established for:

[1948, Lévy] Random Walk

[2000, Kenyon] some parameters of UST/LERW

[2001, S] critical percolation on hexagonal lattice

[2001, Lawler-Schramm-Werner] UST/LERW

[2003/6, Schramm-Sheffield] Harmonic Explorer/Discrete GFF

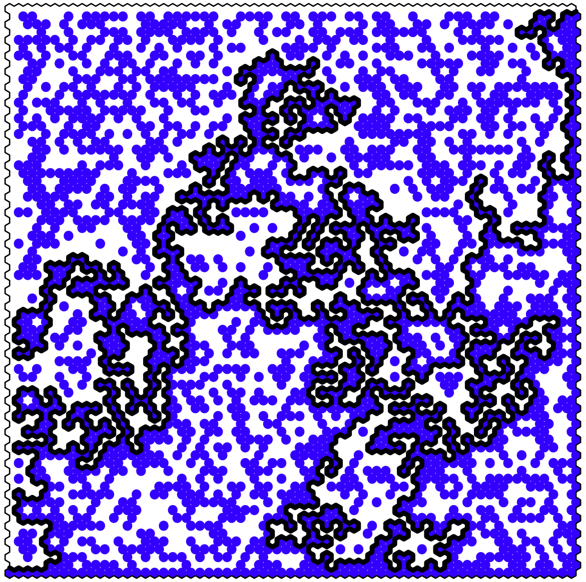
[2006, S] FK Ising at criticality

[2007, S] Ising at T_c

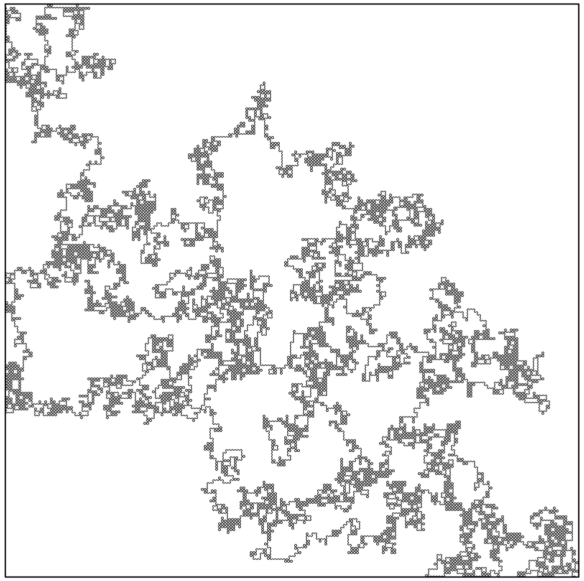
Interface converges to conformally invariant SLE(κ) curve for

c	κ	n	$O(n)$ loop gas dense/dilute	FK loops, $n = \sqrt{q}$
-2	8	0	$\cdots$	**uniform spanning tree, lerw [Lawler-Schramm-Werner 2001]**
0	6	1	**site percolation on the triangular lattice [S 2001]**	bond percolation on the square lattice
$\frac{1}{2}$	$\frac{16}{3}$	$\sqrt{2}$	$\cdots$	**FK Ising [S 2006]**
1	4	2	$\cdots$	FK 4-Potts
$\frac{1}{2}$	3	1	**Ising [S 2007]**	
0	$\frac{8}{3}$	0	Self Avoiding Random Walk	$\cos\left(\frac{4\pi}{\kappa}\right) = -\frac{n}{2}$

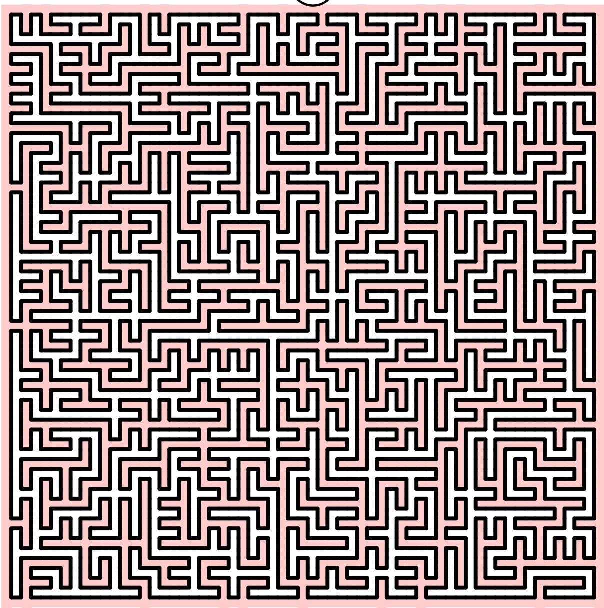
percolation $\kappa = 6$



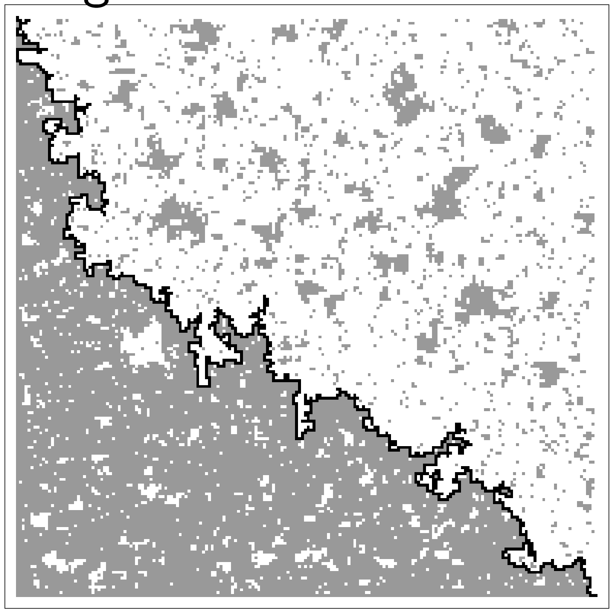
FK Ising $\kappa = 16/3$



ust $\kappa = 8$ © Schramm



Ising $\kappa = 3$



Using FK Ising model as an example we will discuss how to

(A) find a discrete conformal invariant and
show it has a (conformally invariant) scaling limit

(B) construct (conformally invariant) scaling limit of one interface
(constructing full picture is very similar)

Related topics:

- interfaces on Riemann surfaces, general boundary conditions
- other lattices
- $p \neq p_c$
- multi-point observables, spin pair corellations
- observables for $O(n)$ model
- Nienhuis prediction for the critical value $x_c(n)$
- other values of q and n

(A) How to find a conformally invariant observable?

We need *discrete conformal invariant*

Discrete harmonic or **dicrete analytic** (=preholomorphic) function solving prescribed boundary value problem

- more accessible in the discrete case than other invariants
- most other invariants can be reduced to it

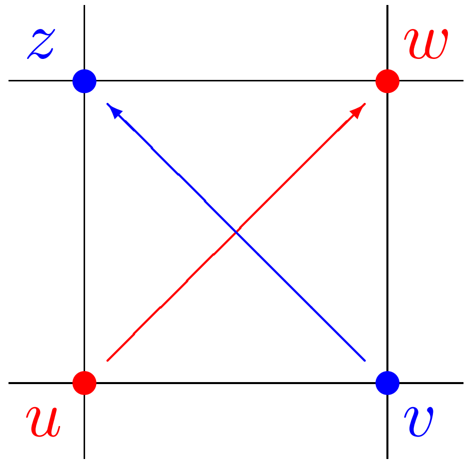
Boundary value problems

- **Dirichlet** or **Neumann**: clear discretization, scaling limit.
- **Riemann-Hilbert**: wider choice! discretization? scaling limit?

Leads to conformally covariant functions: $F(z)\,(dz)^{\alpha}(d\bar{z})^{\beta}$

Discrete analytic (preholomorphic): discrete version of the Cauchy-Riemann equations $\partial_{i\alpha}F = i\partial_{\alpha}F$:

$$\boldsymbol{F(z) - F(v) = i\,(F(w) - F(u))}$$



Discrete complex analysis starts like the usual one.
Easy to prove: if $F, G \in \mathrm{Hol}$, then

- $F \pm G \in \mathrm{Hol}$
- $F' \in \mathrm{Hol}$ (defined on the dual lattice)
- $\oint F = 0$
- $\int^z F$ is well-defined and $\int^z F \in \mathrm{Hol}$
- maximum principle
- $F = H + i\tilde{H} \Rightarrow H$ discrete harmonic (mean-value property)
- H discrete harmonic $\Rightarrow \exists\ \tilde{H}$ such that $H + i\tilde{H} \in \mathrm{Hol}$

Problem: $F, G \in \mathrm{Hol} \not\Rightarrow F \cdot G \in \mathrm{Hol}$

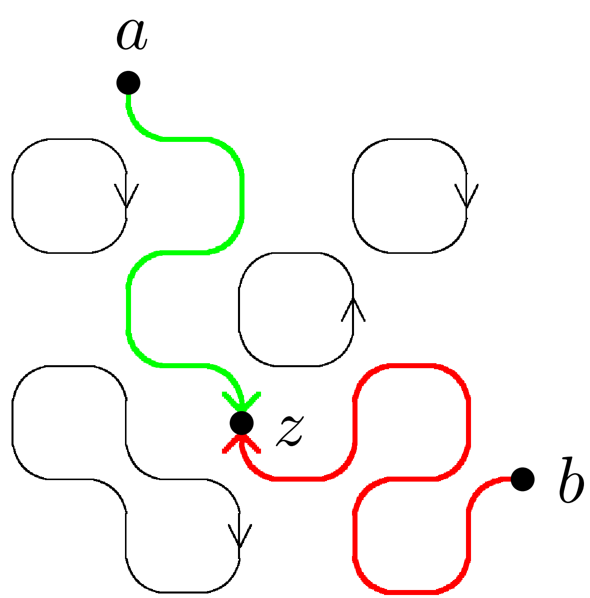
How to find a discrete analytic observable?

- physics intuition: Coulomb gas, order/disorder operators, . . .
- combinatorics: long-range order vs local rearrangements
- integrable structure: Yang-Baxter, . . .
- pfaffians: cf. observables for dimer models [Kenyon]
- reverse engineering: discretize invariants of CFTs or SLEs
- complex analysis: Riemann-Hilbert problem
  vs Riemann-Hilbert boundary value problem

Which observable is discrete analytic for the FK model?

$$\boldsymbol{F(z) := \mathbb{E}\, \chi_{z\in\gamma} \cdot \exp\left(-i\, \sigma\, \mathrm{winding}(\gamma, b \to z)\right)}$$

Parafermion with spin $\sigma = 1 - 4k$
where $\mathbf{2\cos(2\pi k) = \sqrt{q}}$
For FK Ising model spin $\sigma = 1/2$

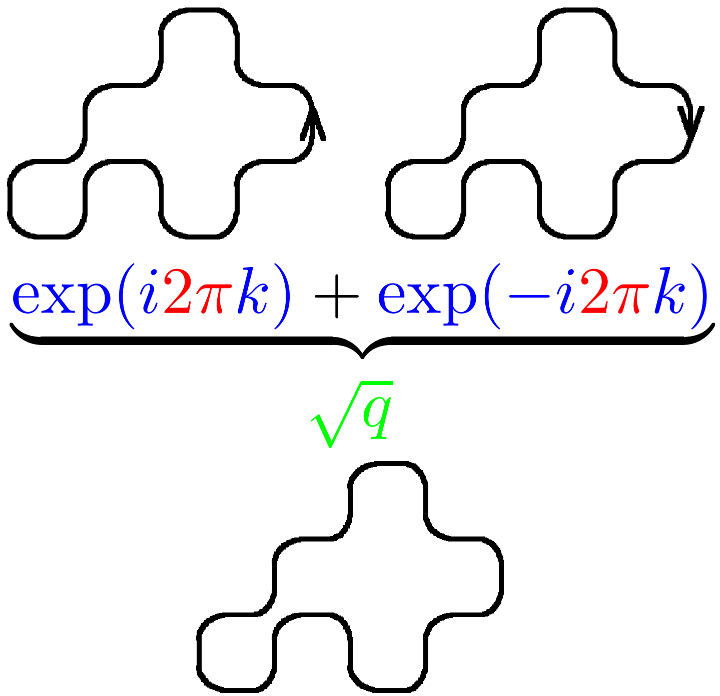


Motivation: orient loops randomly $\Leftrightarrow$ height function changing by ± 1 whenever crossing a loop
(geographic map with contour lines)
Orient interface $b \to z$ and $a \to z$ $\Leftrightarrow$ $+2$ monodromy at z
$$\boldsymbol{F(z) = Z_{+2\ \text{monodromy at z}}}$$

Where complex weights come from? [cf. Baxter]

Set $\mathbf{2\cos(2\pi k)} = \sqrt{q}$. Orient loops.
New $\mathbb{C}$ partition function (local!):

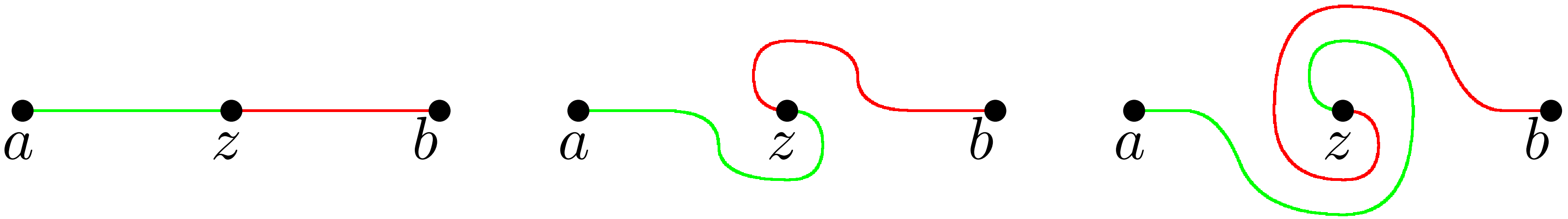
$$\boldsymbol{Z^{\mathbb{C}} = \sum \prod_{\text{sites}} \exp(i\ \textbf{winding} \cdot k)}$$



Forgetting orientation projects onto the original model: $\mathrm{Proj}\left(Z^{\mathbb{C}}\right) = Z$

Two halves of the interface γ
are attached to $\partial\Omega$
$\Rightarrow$ must weight γ differently,
by $\boldsymbol{\exp\left(-i\ \textbf{winding}\ \sigma/2\right)}$ with $\boldsymbol{\sigma = 1 - 4k}$

winding	0	π	2π
weight	1	$\exp(-i\sigma\pi)$	$\exp(-i\sigma 2\pi)$



weight	1	$-i$	-1

For FK Ising $q = 2 = (2\cos(2\pi k))^2$, $k = 1/8$, $\sigma = 1/2$: a **fermion**

Theorem. *For FK Ising when lattice mesh $\epsilon \to 0$*

$$\boldsymbol{\epsilon^{-\sigma} F(z) \rightrightarrows \Phi'(z)^{\sigma} \text{ inside } \Omega,}$$

where Φ maps conformally Ω to a horizontal strip, $a, b \mapsto \text{ends}$.

Proof: (works to some extent for all q)

- Local rearrangements $\Rightarrow F$ preholomorphic
- $\mathrm{Im}\,(F(z)\cdot(\text{tangent to }\partial\Omega)^{\sigma}) = 0$

$\Rightarrow \mathrm{Im}\left(F(z)^{1/\sigma}dz\right) = 0$ along $\partial\Omega$

(discrete Riemann boundary value problem)

Conclusion: F is a discrete version of $(\Phi')^{\sigma}$

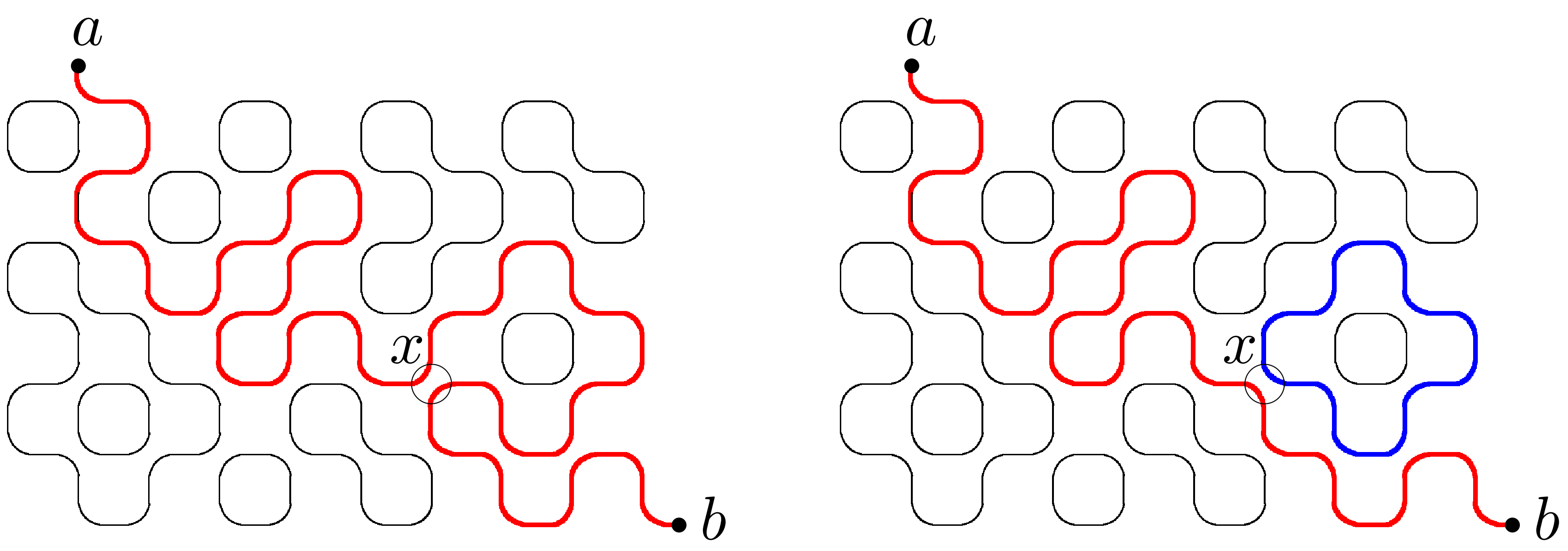
Surprisingly difficult: $F \rightrightarrows (\Phi')^{\sigma}$

For Ising $\sigma = 1/2$, work with $H(z) := \mathrm{Im}\int_{z_0}^{z} F^2(u)du$:

- well-defined
- approximately harmonic
- solves Dirichlet BVP

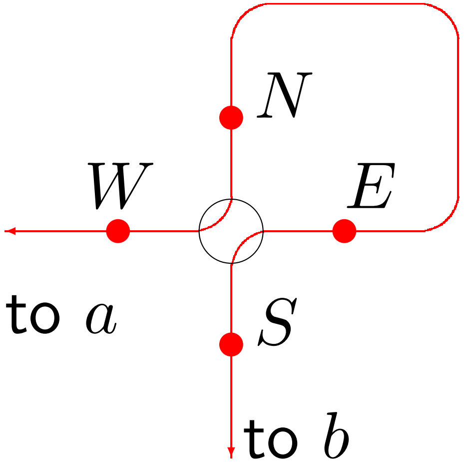
Conclusion: $H \rightrightarrows \mathrm{Im}\Phi$ therefore $F \rightrightarrows \sqrt{\Phi'}$ $\square$

Proof: Local rearrangement

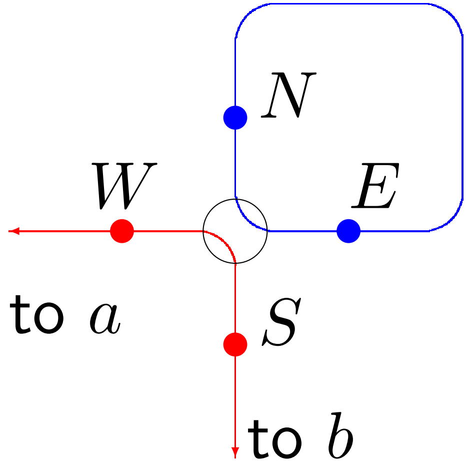


If a-b interface passes through x,
changing connections at x creates two configurations.
Additional loop on the right $\Rightarrow$ weights differ by a factor of $\sqrt{q} = \sqrt{2}$

Proof: discrete CR relation $F(N) + F(S) = F(E) + F(W)$



$X\lambda^2$	$F(N)$	0
X	$F(S)$	$X\sqrt{2}$
$X\lambda$	$F(W)$	$X\lambda\sqrt{2}$
$X\bar{\lambda}$	$F(E)$	0



$\lambda = \exp(-i\pi/4)$ is the weight per $\pi/2$ turn. Two configurations together contribute equally to both sides of the relation:

$$X\lambda^2 + X + X\sqrt{2} = X\bar{\lambda} + X\lambda + X\lambda\sqrt{2}$$

$$i + 1 + \sqrt{2} = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\sqrt{2} \qquad \square$$

Proof: Riemann-Hilbert boundary value problem

When z is on the boundary, winding of the interface $b \to z$ is uniquely determined, same as for $\partial\Omega \Rightarrow$ determine $\mathrm{Arg}(F)$ on $\partial\Omega$.

F solves the discrete version of the covariant Riemann BVP
$\mathrm{Im}\left(\boldsymbol{F(z)} \cdot (\text{tangent to } \boldsymbol{\partial\Omega})^{\boldsymbol{\sigma}}\right) = \mathbf{0}$ with $\sigma = 1/2$.

Continuum case: $F = (\Phi')^{\sigma}$, where $\Phi : \Omega \to \text{horizontal strip}.$

Proof: convergence Consider $\int_{z_0}^{z} F^2(u)du$ – solves Dirichlet BVP.

Save for later use: martingale property $F(z, \Omega) = \mathbb{E}_{\gamma'} F(z, \Omega \backslash \gamma')$

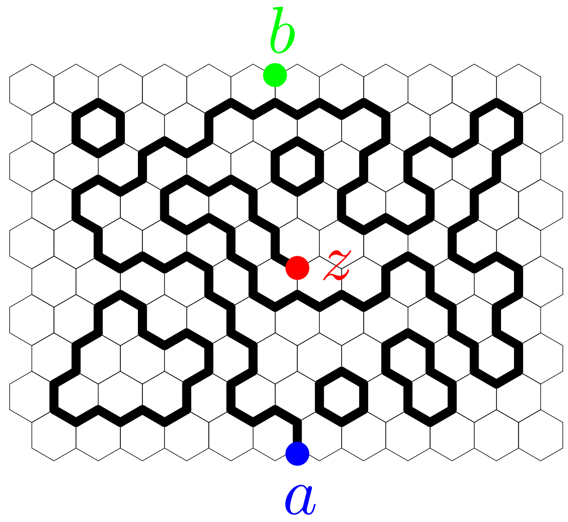
Which observable is discrete analytic for the $O(n)$ model?

Set $F(z) = Z_{+1 \text{ monodromy at z}}$

Curve stops at z before reaching b.

$2\cos(2\pi k) = n$, new spin $\sigma = 1/4 - 3k/2$

Ising: $n = 1$, $k = -1/6$, $\sigma = 1/2$, a fermion



Theorem. *For the Ising model at T_c*

$$\epsilon^{-\sigma} F(z) \rightrightarrows \Psi'(z)^{\sigma} \text{ inside } \Omega$$

as lattice mesh $\epsilon \to 0$. Here Ψ maps Ω to a halfplane, $a, b \mapsto \infty, 0$.

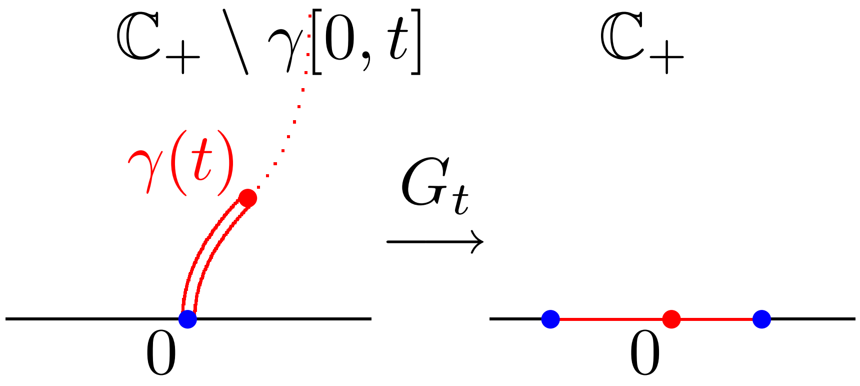
Proof: Similar yet different. Partially works for all values of n.

Explains Nienhuis predictions of critical temperature x_c! (sorry, no proof yet)

Relates to Kenyon's work (think dimer models on the Fisher lattice)

(B) Schramm-Loewner Evolution.

LE is a slit $\gamma(t)$ obtained by solving an ODE for the Riemann map G_t:



$$\boldsymbol{\partial_t \left(G_t(z) - w(t)\right) \; = \; 2/G_t(z)}$$

$\boldsymbol{G_t(z) = z - w(t) + 2t/z + \mathcal{O}\left(1/z^2\right)}$ — normalization at ∞.

SLE(κ) is a random curve obtained by taking $w(t) := \sqrt{\kappa} B_t$.

Schramm's Principle: *if an interface has a conformally invariant scaling limit, it is SLE(κ) for some $\kappa \in [0, \infty)$.*

Proof: Conformal invariance with Markov property (interface does not distinguish its past from the domain boundary) translates into $w(t)$ having i.i.d. increments.$\square$

To use the Principle one still has
(i) to show existence of the **scaling limit**
(ii) to prove its **conformal invariance**
(iii) calculate some observable to determine κ

For **(i)** in principle one needs $infinitely\ many\ observables$.
For percolation constructed from one observable using $locality$.

Fortunately **(iii)**$\Rightarrow$**(i-ii)**. A generalization of Schramm's Principle:
If there is a conformally covariant martingale observable, then the interface converges to SLE(κ) *with particular* $\kappa \in [0, \infty)$*.*

Used in UST$\rightarrow$SLE(8) convergence [Lawler-Schramm-Werner] with invariant observable. Can be used for percolation.

Theorem. *Interfaces in FK Ising and Ising model at T_c converge to conformally invariant scaling limits: SLE$(16/3)$ and SLE(3).*

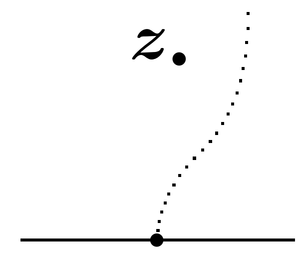
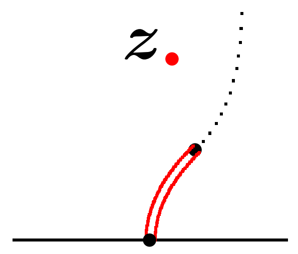
Proof for FK Ising: A priori estimates $\Rightarrow \{\gamma\}_{\text{mesh}}$ is precompact.

Enough to show: limit of any converging subsequence = SLE.

Pick a subsequential limit, map to $\mathbb{C}_+$, describe by
Loewner Evolution with unknown random driving force $w(t)$.

From the martingale property $\boldsymbol{F(z,\Omega) = \mathbb{E}_{\gamma'} F(z, \Omega \setminus \gamma')}$ of the observable extract expectation of increments of $w(t)$ and $w(t)^2$.

Lévy's characterization $\Rightarrow w(t)$ is the BM with speed κ. $\square$

F () $=$ $\mathbb{E}_{\gamma[0,t]}\, F$ ()

$$\| \qquad\qquad\qquad\qquad \|$$

$$(\log(z)')^{\sigma} \qquad\qquad \mathbb{E}_{G_t}\,(\log(G_t)')^{\sigma}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\frac{1}{z^{\sigma}}$$

use expansion of G_t at ∞

$$\|$$

$$\mathbb{E}_{G_t}\frac{1}{z^{\sigma}}\left(1+\frac{\sigma}{z}w(t)+\frac{\sigma(\sigma+1)}{2z^2}\left(w(t)^2-\frac{8t}{\sigma+1}\right)+\mathcal{O}\left(\frac{1}{z^3}\right)\right)^{\sigma}$$

Equating coefficients:

$$\mathbb{E}_{G_t}\left(w(t)\right) = 0, \quad \mathbb{E}_{G_t}\left(w(t)^2 - \frac{8t}{\sigma+1}\right) = 0$$

Stop at times t and s — same formulae for increments.
$w(t)$ is continuous by Loewner theorem.

By Lévy characterization theorem $w(t) = \sqrt{\dfrac{8}{\sigma+1}}B_t$.

So our curve is SLE$\left(\frac{8}{\sigma+1}\right)$, namely SLE$\left(\frac{16}{3}\right)$ when $\sigma = \frac{1}{2}$.

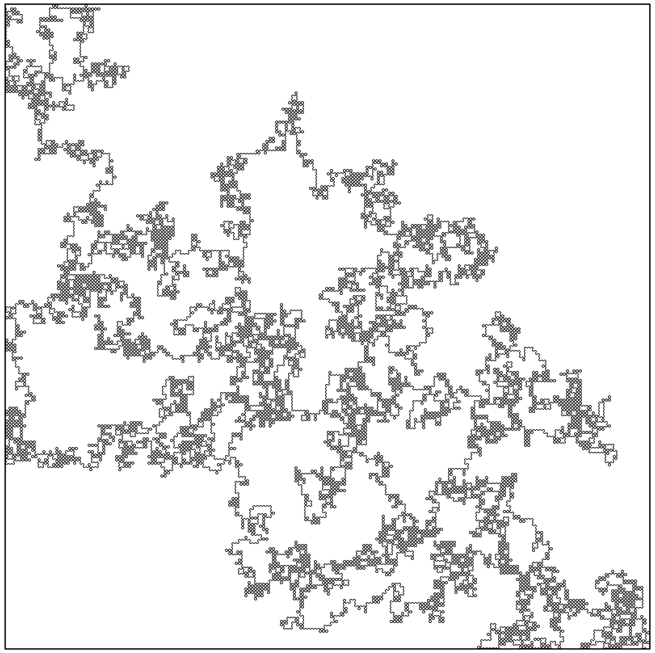
CONCLUSION

- In several cases proof of a conformally invariant scaling limit
- Some universality
- Good understanding in other cases
- Some things new for physicists

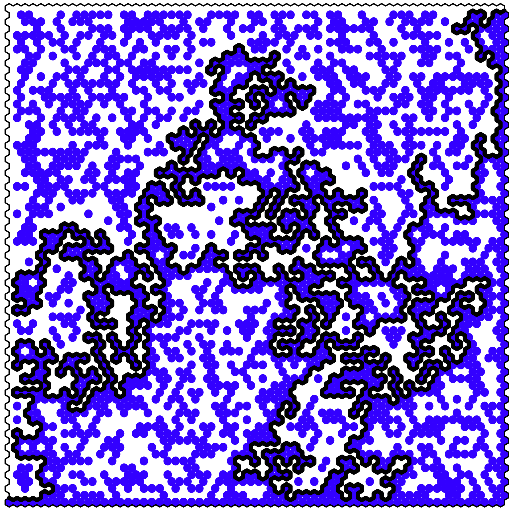
STILL A LOT DO!

- Remaining cases
- Universality
- Renormalization
- Perturbation theory $T \approx T_c$
- Connection to Yang-Baxter

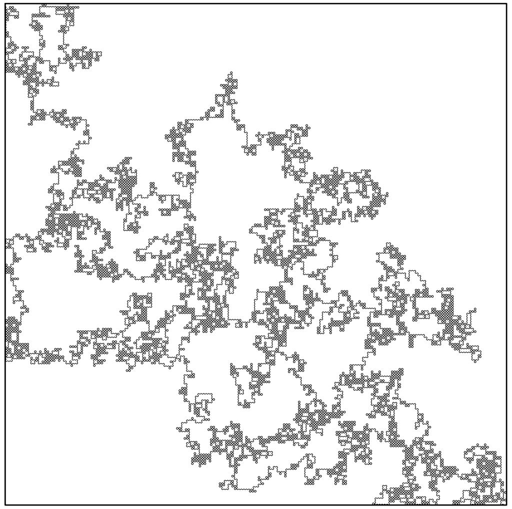
FK Ising $\rightarrow$ SLE$(16/3)$, $\text{Dimension} = 5/3$



percolation $\kappa = 6$

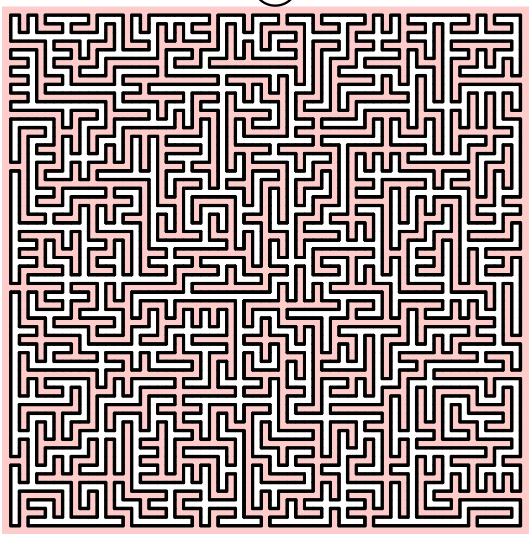


FK Ising $\kappa = 16/3$

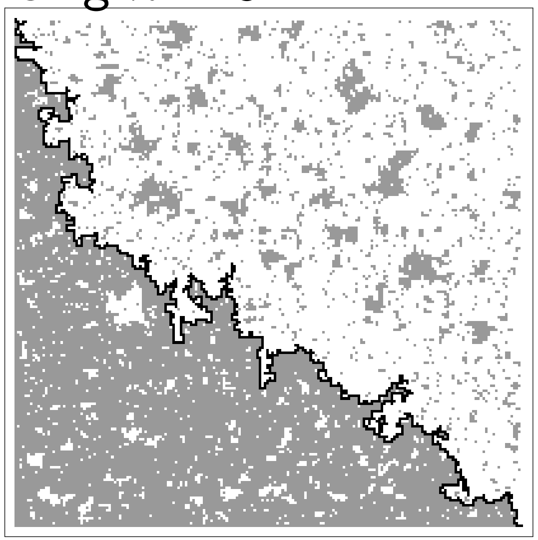


Square bond percolation?

ust $\kappa = 8$ ©O. Schramm



Ising $\kappa = 3$



Self-avoiding random walk?