The Uniqueness of Signature problem

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The problem

- $\triangle := \{(s,t) : 0 \leq s \leq t \leq 1\}$.

Theorem

(Lyons) $p \geq 1$. *If* $\mathbf{X}_{\cdot,\cdot} : \triangle \to T^{[p]}(\mathbb{R}^d)$ *finite p-variation, multiplicative, then* $\mathbf{X}$ *has unique multiplicative, finite p-variation extension* $S(\mathbf{X})_{\cdot,\cdot} : \triangle \to T(\mathbb{R}^d)$.

- **The Uniqueness of Signature problem:** Given $S(\mathbf{X})_{0,1}$, recover $\mathbf{X}_{s,t}$ **for all** s, t.

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The $p < 2$ case

- $p < 2$. $x : [0,1] \to \mathbb{R}^d$ finite p-variation. Explicit extension.

$$S(x)_{0,1} := 1 + \int_{0<s_1<1} \mathrm{d}x_{s_1} + \int_{0<s_1<s_2<1} \mathrm{d}x_{s_1} \otimes \mathrm{d}x_{s_2} + \ldots.$$

- Uniqueness of signature equivalent to finding **all** x such that

$$S(x)_{0,1} = (1, 0, 0, \ldots).$$

- If $x = \alpha\overleftarrow{\alpha}$,

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Review

Theorem

(K.T. Chen) x, x' irreducible, piecewise regular, continuous paths. $S(x)_{0,1} = S(x')_{0,1}$ iff x, x' are translation and reparametrisation of each other.

Theorem

(Hambly-Lyons) x bounded variation. $S(x)_{0,1} = 1$ if and only if exists $h(t)$ continuous, non-negative, $h(1) = h(0)$, such that for all $s < t$,

$$|x_t - x_s| \leq h(t) + h(s) - 2 \inf_{u \in [s,t]} h(u).$$

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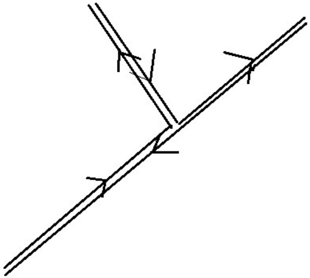
Height function



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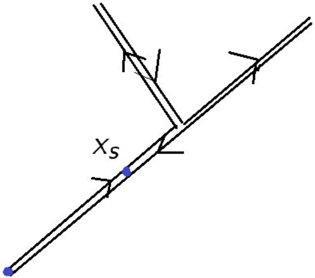

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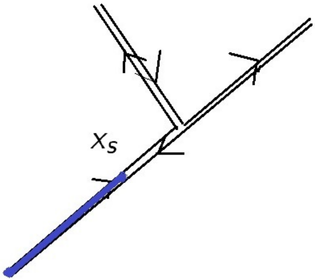
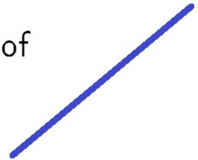


$h(s) =$ length of



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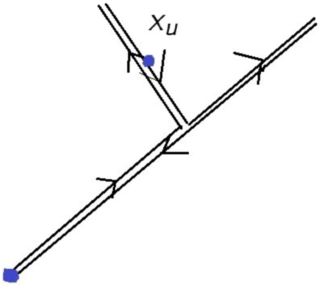

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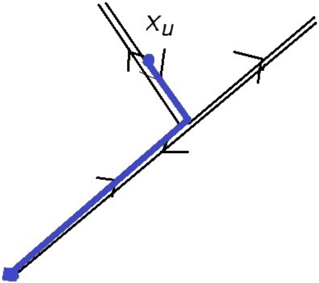
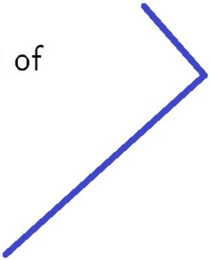


$h(u) =$ length of



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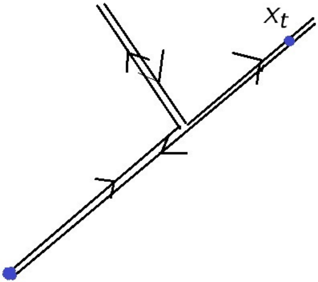

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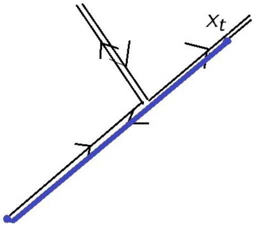
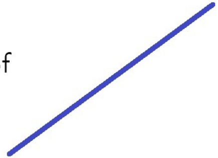


$h(t)=$ length of



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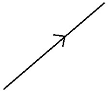

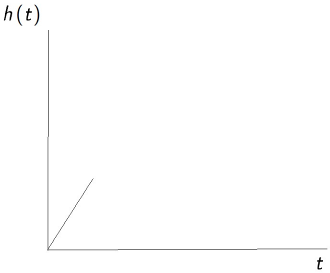

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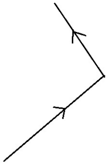

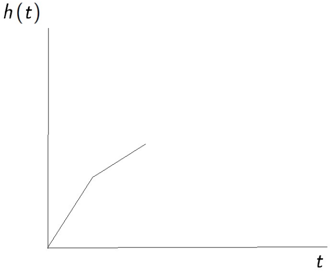

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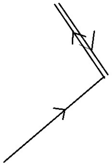

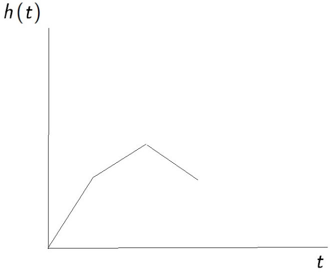

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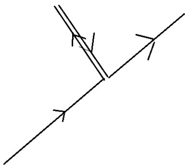

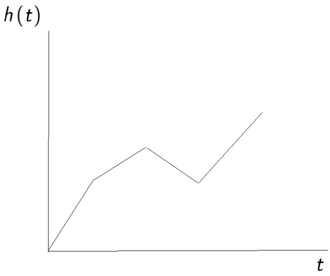

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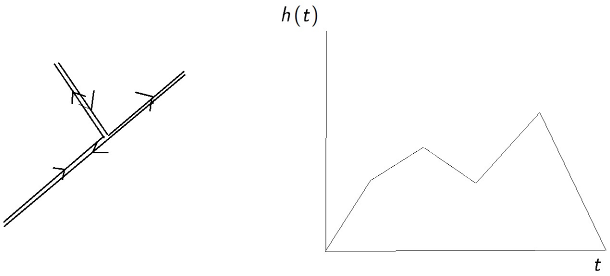

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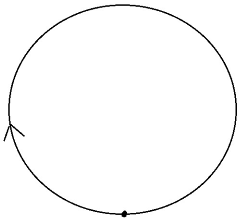

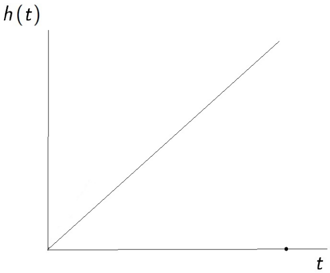

Inversion

- Explicit Inversion scheme: Weijun Xu (C^2 paths), Andrew Ursitti.

The $p > 1$ case

- Uniqueness of Signature for path with p variation, $1 < p$ is open.
- Key difficulty: extending **Coarea Formula**:

Theorem

(Ohtsuka) γ bounded variation. $n(x) = \left|\gamma^{-1}(\{x\})\right|$. $\Lambda_1(\mathrm{d}x)$ is one dimensional Hausdorff measure. Then

$$\int f(\gamma(t))\,\mathrm{d}t = \int f(x)\,n(x)\,\Lambda_1(\mathrm{d}x).$$

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Almost-sure uniqueness

Almost-sure uniqueness problem: μ probability measure. Prove there exists μ-null set $\mathcal{N}$: $x, x' \in \mathcal{N}^c$ and $S(x)_{0,1} = S(x')_{0,1} \Longrightarrow x = x'$ up to reparametrisation.

- Brownian motion—-Le Jan-Qian (2009) .
- Hypoelliptic diffusion—-Geng-Qian (2013).
- Fractional Brownian motion $H > \frac{1}{4}$—-Geng-B. (2014).

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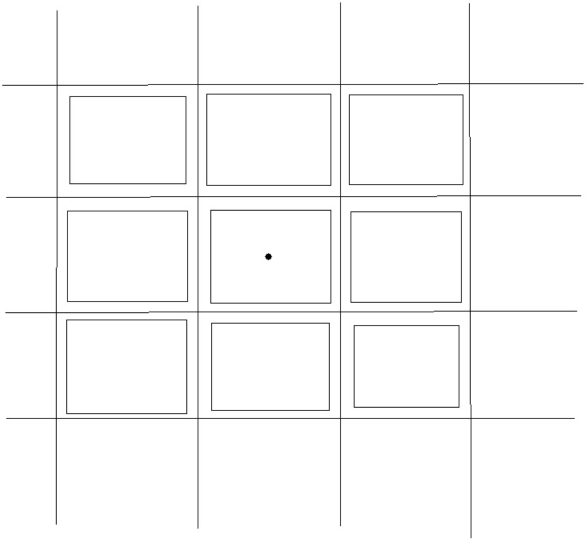

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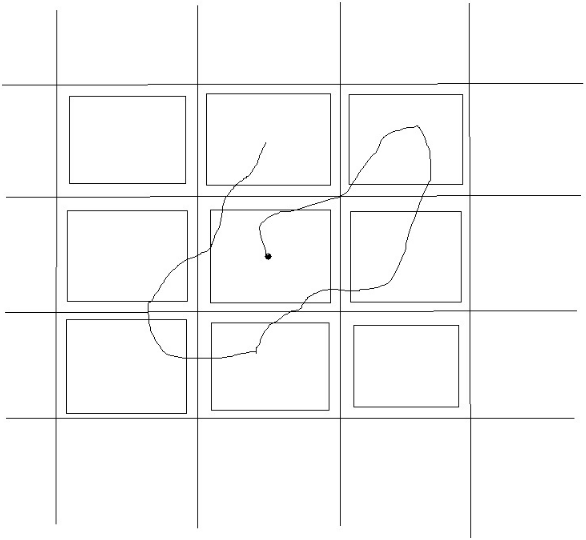

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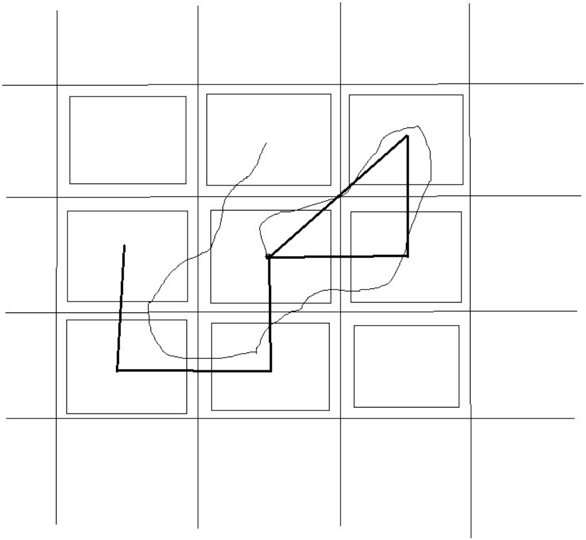
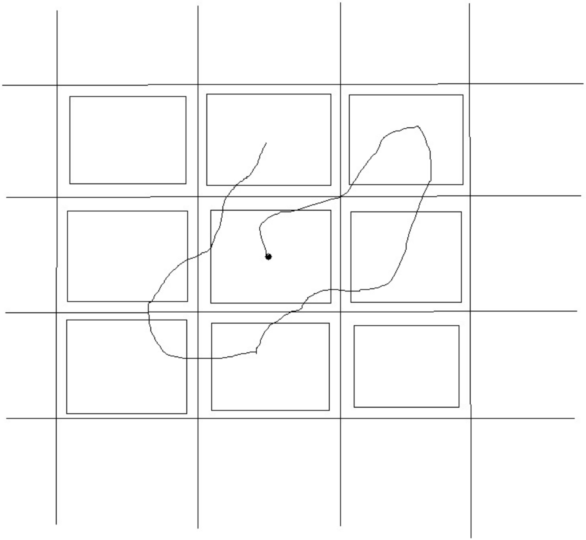


Details



Signature and integration against 1-forms

Fact

$\phi_1, \ldots, \phi_n$ smooth 1-forms. There exists a sequence of linear functionals F_n depending only on ϕs

$$\int_{0<s_1<\ldots<s_n<1} \phi_1 (\mathrm{d}x_{s_1}) \ldots \phi_n (\mathrm{d}x_{s_n}) = \lim_{n\to\infty} F_n \left(S(x)_{0,1} \right).$$

$\phi_{z_1}, \phi_{z_2}, \ldots$ non degenerate in interior of boxes. Compactly supported on boxes.

- Assume the total number of boxes visited by x in order is N. Let $z_1, \ldots, z_N$ be the boxes visited in order.

Fact

$\int_{0<s_1<\ldots<s_N<1} \phi_{z_1}(\mathrm{d}x_{s_1}) \ldots \phi_{z_N}(\mathrm{d}x_{s_N}) \neq 0.$

Fact

$\int_{0<s_1<\ldots<s_n<1} \phi_{w_1}(\mathrm{d}x_{s_1}) \ldots \phi_{w_n}(\mathrm{d}x_{s_n}) = 0$ *for all* $n > N$, *all* $w_1., \ldots, w_n$.

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For any $w_1, \ldots, w_N$, $\int_{0<s_1<\ldots<s_N<1} \phi_{w_1}(\mathrm{d}x_{s_1}) \ldots \phi_{w_N}(\mathrm{d}x_{s_N}) = 0$ *unless* $w_i = z_i$ *for all* i.

Reading the path

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Reading the path

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To prove $\int_{0<s_1<\ldots<s_n<1} \phi_{z_1}(\mathrm{d}x_{s_1})\ldots\phi_{z_n}(\mathrm{d}x_{s_n}) \neq 0$.

- Step 1:

$$\int_{0<s_1<\ldots<s_n<1} \phi_{z_1}(\mathrm{d}x_{s_1})\ldots\phi_{z_n}(\mathrm{d}x_{s_n})$$

$$= \int_0^{\text{entry time of box1}} \phi_{z_1}(\mathrm{d}x_{s_1}) \int_{\text{entry time of box2}}^{\text{entry time of box3}} \phi_{z_1}(\mathrm{d}x_{s_1})$$

$$\ldots \int_{\text{entry time of box } n}^{1} \phi_{z_n}(\mathrm{d}x_{s_n})$$

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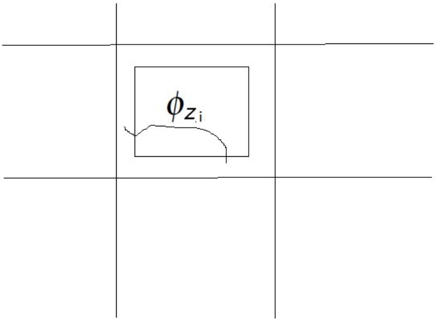
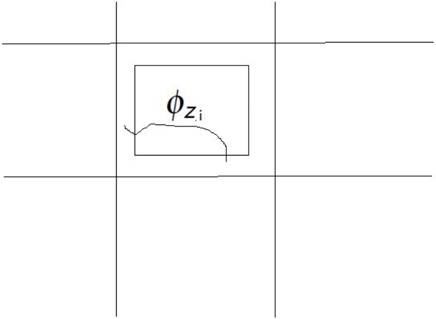
Step 2:

$$\int_{\text{entry time of box}i}^{\text{entry time of box}i+1} \phi_{z_i}(\mathrm{d}x_t) = 0 \implies \exists q_1, q_2 \in \mathbb{Q}, \int_{q_1}^{q_2} \phi_{z_i}(\mathrm{d}x_t) = 0.$$



Step 3: $A_{s,t} := \{x : x$ visited interior of box between time $s$and $t\}$.

$$\mathbb{P}\left(\left\{x : \int_s^t \phi_{z_i}(\mathrm{d}x_v) = 0\right\} \cap A_{s,t}\right) = 0.$$



Application

- **Time series:**

$$z_t = a_0 + a_1 x_t + a_2 x_t^2 + a_3 y_t^3 + \varepsilon$$

- **Rough path:**

$$\mathrm{d}y_t = By_t \mathrm{d}x_t$$

$$\Longrightarrow y_t = \sum_{n=0}^{\infty} B^{\otimes n}(y_0) \left(\int_{0<s_1<\cdots<s_n<1} \mathrm{d}x_{s_1} \cdots \mathrm{d}x_{s_n} \right).$$

- See work of H. Ni, Gyurko, Guy Flint, T. Lyons.

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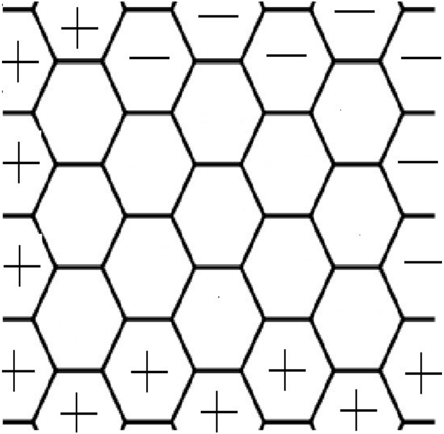
Almost-sure uniqueness for SLE curves

Theorem

(B.-Ni-Qian) $\kappa \leq 4$. *Almost-sure uniqueness of signature holds for SLE_κ curves.*

Scaling limit of 2D Ising model

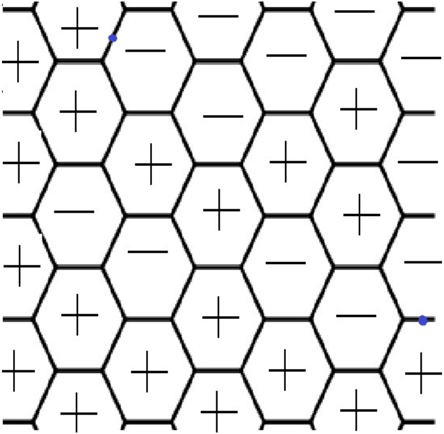


Scaling limit of critical 2D Ising model

Partition function: $\sigma(i) \in \{-1,1\}$,

$$\frac{e^{\beta J_{crit} \sum_{i\sim j} \sigma(i)\sigma(j)}}{\sum_{\text{config}} e^{\beta J_{crit} \sum_{i\sim j} \sigma(i)\sigma(j)}}$$

Scaling limit of 2D Ising model



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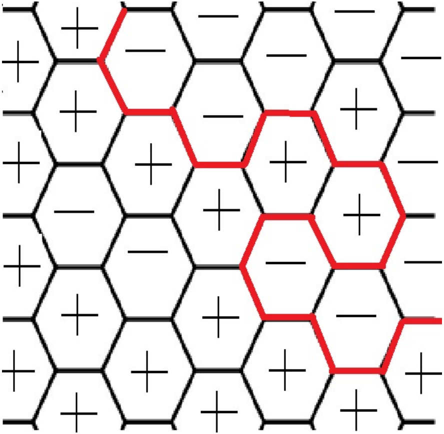
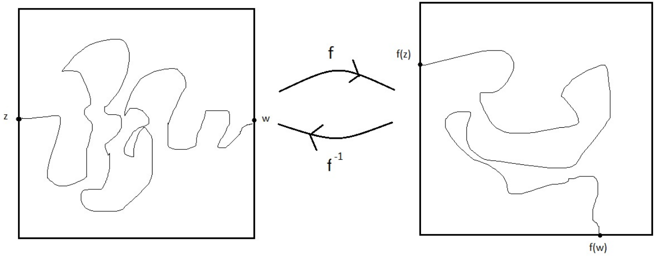


Conformal Invariance

Family of measures $\{\mu_D(z,w) : D \subset \mathbb{C}\}$ is **conformally invariant** if

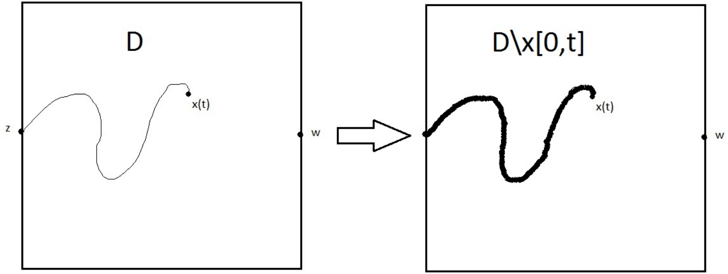
$$f \circ \mu_D(z,w) = \mu_{f(D)}(f(z), f(w)).$$



Domain Markov

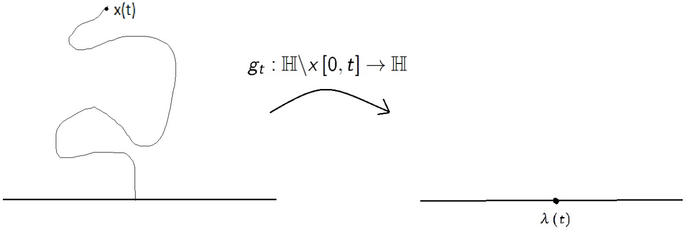
Family of measures $\{\mu_D(z,w) : D \subset \mathbb{C}\}$ is **Domain Markov** if

$$\mu_D(z,w|x[0,t]) = \mu_{D\backslash x[0,t]}(x(t),w).$$



Loewner's transform

$$\frac{\mathrm{d}g_t(z)}{\mathrm{d}t} = \frac{2}{g_t(z) - \lambda(t)}, \; g_0(z) = z.$$



A theorem of Schramm

Theorem

(Schramm)If the random curve x satisfies conformal invariance and domain Markov property, then $\lambda(t)$ is a Brownian motion $\sqrt{\kappa}B_t$.

Define g_t by

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If $g_t : \mathbb{H} \backslash x[0,t] \to \mathbb{H}$, x_t is called $\mathbf{SLE}_\kappa$ trace.

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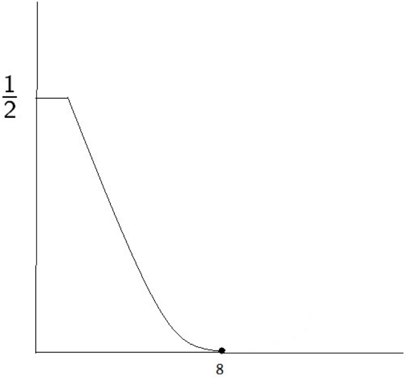
If $g_t : \mathbb{H} \backslash x[0,t] \to \mathbb{H}$, x_t is called $\mathbf{SLE}_\kappa$ trace.

Regularity of SLE curves

- (Viklund-Lawler) SLE_κ has Holder exponent

$$\min\left(1-\frac{\kappa}{24+2\kappa-8\sqrt{8+\kappa}},\frac{1}{2}\right)$$



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Regularity of SLE trace

- (Beffara) Hausdorff dimension of SLE_κ trace is
$$\min\left(1+\frac{\kappa}{8},2\right).$$
- (Lawler, Rezaei) "Away from the root", SLE_κ trace has Hölder exponent $\frac{1}{1+\frac{\kappa}{8}}-\varepsilon$ (parametrisation by Minkowski content).
- (Werness) $\kappa\leq 4$, SLE_κhas a $\frac{1}{1+\frac{\kappa}{8}}-\varepsilon$ Holder parametrisation. $\kappa>4$ **open problem**.

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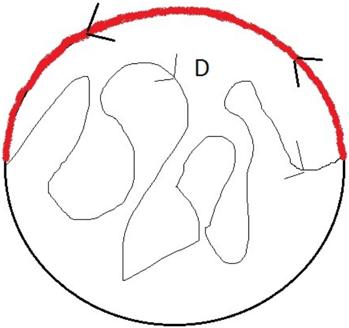
- (Werness)First 3 terms in expected signature of SLE_κ in $\mathbb{D}$.



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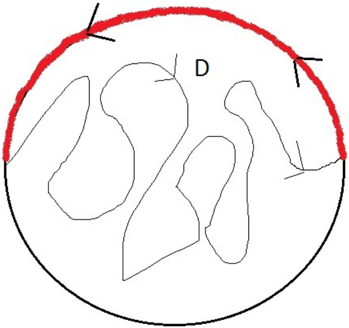

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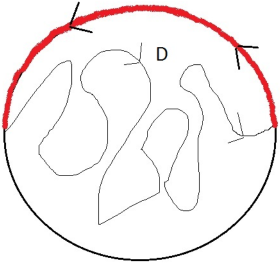
Expected signature of SLE$_\kappa$ curves

- $$\int_{0<s_1<s_2<s_3<1} \mathrm{d}x_{s_1}\mathrm{d}x_{s_2}\mathrm{d}y_{s_3} = \int_0^1 \frac{x_s^2}{2}\mathrm{d}y_s = \int x 1_D(x,y)\,\mathrm{d}x\mathrm{d}y.$$



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Uniqueness of Signature

- $$S(\gamma)_{0,1} = S(\gamma')_{0,1}$$
  $$\Longrightarrow \int \frac{x^n y^k}{n!k!} 1_D(x,y)\,\mathrm{d}x\mathrm{d}y = \int \frac{x^n y^k}{n!k!} 1_{D'}(x,y)\,\mathrm{d}x\mathrm{d}y.$$
- $D = D'$. $\gamma = \gamma'$ up to reparametrisation.

Corollary

Can get 4th term in expected signature of SLE_κ trace.

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Extensions

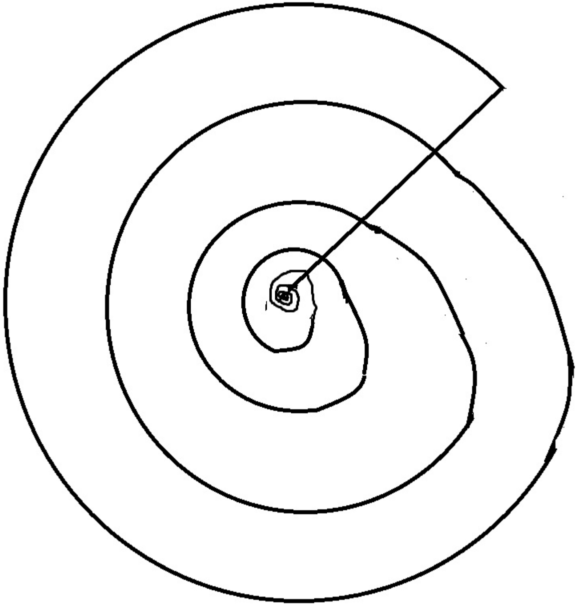
Theorem

γ finite $p < 2$ variation.

$$e_1^{*\otimes n+1} \otimes e_2^{*\otimes k+1}(S(\gamma)) = (-1)^k \int \frac{x^n y^k}{n!k!} \mathscr{W}(\gamma, (x,y)) \mathrm{d}x\mathrm{d}y$$

where $\mathscr{W}(\gamma, (x,y))$ is winding number of γ around (x,y).

Controlling the winding number



"Iso p-variation" inequality

Theorem

(B.)γ finite $p<2$ variation. $q<\frac{2}{p}$.

$$\|\mathscr{W}(\gamma,(x,y))\|_{L^q} \leq 4^{\frac{2}{p}}\zeta\left(\frac{2}{p}\right)\|\gamma\|_p^p.$$

"Iso p-variation" inequality

- Want $f \to \int_{\mathbb{R}^2} f(x,y)\mathscr{W}(\gamma,(x,y))\mathrm{d}x\mathrm{d}y$ bounded on $L^{q'}$, $q < \frac{2}{p}$.

Fact

$$f \to \int_{\mathbb{R}^2} \frac{f(x)}{w-x}\mathrm{d}A(x)$$

is bounded operator L^q to Lip $\left(1-\frac{2}{q}\right)$, $q > 2$.

Fact

$$g \to \int_0^1 g(\gamma_s)\mathrm{d}\gamma_s$$

in Lip$(r-1)^*$ for $r > p$.

Theorem

(Degree formula)Let Ω be an open domain in $\mathbb{R}^d$. $\partial\Omega$ has measure zero. $h \in C^1\left(\overline{\Omega}, \mathbb{R}^d\right)$. Then there exists $d(\cdot, \Omega, \mathbf{h}) \in L^1$ such that for all $g \in C\left(\mathbb{R}^d\right)$

$$\int_{\mathbb{R}^d} g(\mathbf{y})\, d(\mathbf{y}, \Omega, \mathbf{h})\, \mathrm{d}y = \int_\Omega \det(D\mathbf{h}(x))\, g(\mathbf{h}(x))\, \mathrm{d}x.$$

γ bounded variation. Let $f_1, \ldots, f_n \in \left(\mathbb{R}^d\right)^*$. Then

$$\int_{0<s_1<\ldots<s_n<1} g\left(f_1\left(\gamma_{s_1}\right), \ldots, f_n\left(\gamma_{s_n}\right)\right) f_1\left(\mathrm{d}\gamma_{s_1}\right) \ldots f_n\left(\mathrm{d}\gamma_{s_n}\right)$$
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$$\liminf_{n} \left\| \pi_n \left(\log S(\gamma)_{0,1} \right) \right\|_{\left(\mathbb{R}^d\right)^{\otimes n}}^{\frac{1}{n}} > 0$$

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Range of signature

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