The generalized KPZ equation

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The generalized KPZ-equation...

...is the following generic equation in $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$

$$\partial_t u = \Delta u + f(u)\,(\partial_x u)^2 + k(u)\,\partial_x u + h(u) + g(u)\,\xi.$$

This reduces to KPZ if $f \equiv g \equiv 1$ and $k \equiv h \equiv 0$. One of the aims would be to justify rigorously the Cole-Hopf transform of the KPZ equation.

Moreover, we would like to have a class of processes containing the solution of the stochastic heat equation and invariant under composition with $C^\infty(\mathbb{R})$-functions (Ito formula).

The lack of regularity in space for u means that $\partial_x u$ is expected to be a distribution and therefore a notion of product of random distributions is needed.

The recent theory of Martin Hairer on Regularity Structures (RST) gives a general framework to solve these problems.

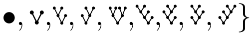
Trees

In [Hairer 13], Martin solves the KPZ equation using a regularization of the noise

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + (\partial_x h_\varepsilon)^2 - C_\varepsilon + \xi_\varepsilon$$

and an expansion

$$h_\varepsilon = \sum_{\tau \in \mathcal{T}} Y_\varepsilon^\tau + u_\varepsilon, \qquad \mathcal{T} = \{\bullet, \ldots\}$$



where $Y_\varepsilon^\bullet = G * \xi_\varepsilon$, for all $\tau = [\tau_1, \tau_2]$ with $\tau_1, \tau_2 \in \mathcal{T}$,

$$Y_\varepsilon^\tau = G * (\partial_x Y_\varepsilon^{\tau_1} \, \partial_x Y_\varepsilon^{\tau_2}) - C_\varepsilon^\tau,$$

and G is the heat kernel. Notice that the expansion of h_ε has constant coefficients and u_ε is a remainder.

Local expansions

For the generalized KPZ equations, one expects to use similar expansions, but with non-constant coefficients.

Perhaps the main idea of RST is to write Taylor expansions *in space* of the solution in terms of an enlarged family of (random) monomials.

The inspiration comes from RPT, in particular in the setting developed by Massimiliano Gubinelli of *controlled paths*.

RST has an important algebraic-combinatorial component. Every equation has an associated family of trees that must be computed.

Trees

Monomials are indexed by finite trees built in this way

- a tree can be 1, X, Ξ
- a tree can be a root to which we attach other trees (their product)
- a tree can become a new tree by formal integration with respect to a kernel depending on a integer parameter $k \geq 0$ (convolution with the heat kernel differentiated k times)

Examples: $\mathcal{I}(\Xi)$, $X^n \Xi \mathcal{I}_k(\Xi)$, $\mathcal{I}((\mathcal{I}_1(\Xi))^2)$

To a tree τ we associate a real number $|\tau|$ called its homogeneity: $|\Xi| = \alpha$, $|X| = 1$, $|1| = 0$

$$|\tau_1 \cdots \tau_n| = |\tau_1| + \cdots |\tau_n|, \quad |\mathcal{I}_k(\tau)| = |\tau| + 2 - k.$$

Homogeneity is related to (but different from) a notion of regularity. X^k has homogeneity k.

The Π_x operators

We fix a realization of the noise ξ. Each tree τ codes a (possibly random) monomial in the variable y centered at x with the following rules:

$$\Pi_x X(y) = (y - x), \qquad \Pi_x \Xi(y) = \xi(y),$$

$$\Pi_x(\tau_1 \cdots \tau_n)(y) = \prod_{i=1}^{n} \Pi_x \tau_i(y),$$

$$\Pi_x \mathcal{I}_k(\tau)(y) = (G^{(k)} * \Pi_x \tau)(y) - \sum_{i=0}^{|\mathcal{I}_k(\tau)|} (y - x)^i \, c_i(\mathcal{I}_k(\tau))$$

with the correct choice of $(c_j)_j$ such that for all τ

$$|\Pi_x \tau(y)| \leq |y - x|^{|\tau|}$$

Examples

If $\xi(t,x)$ is white noise in $\mathbb{R}_+ \times \mathbb{R}$, then $|\Xi| = -3/2 - \delta$, with $\delta > 0$ small and fixed. Then

$$\begin{aligned}\Pi_x \mathcal{I}(\Xi)(y) =& (G * \xi)(y) - (G * \xi)(x) \\ =& \int_{[0,t]\times\mathbb{R}} (G_{t-s}(y-z) - G_{t-s}(x-z))\,\xi(ds,dz) \\ =& \mathcal{I}(\xi)(y) - \mathcal{I}(\xi)(x).\end{aligned}$$

The homogeneity is $|\mathcal{I}(\Xi)| = 1/2 - \delta$.

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Notation in this talk:

- $\mathcal{I}(\Xi)$ denotes the abstract tree
- $\mathcal{I}(\xi)$ denotes the function $G * \xi$ (terrible abuse...)

Examples

For $\tau = \mathcal{I}(\Xi\mathcal{I}(\Xi))$

$$\begin{aligned}\Pi_x\tau(y) =& \mathcal{I}(\xi(\mathcal{I}\xi))(x) - \mathcal{I}(\xi(\mathcal{I}\xi))(y) \\ =& \int_{[0,t]\times\mathbb{R}} \left(G_{t-s}(y-z) - G_{t-s}(x-z)\right)\xi(ds,dz)\cdot \\ & \cdot \int_{[0,s]\times\mathbb{R}} G_{s-r}(z-w)\,\xi(dr,dw)\end{aligned}$$

and the homogeneity is $1-\delta$.

For $\tau = \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))$, setting $h(x) := \mathcal{I}(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(x)$,

$$\Pi_x\tau(y) = h(y) - h(x) - \partial_x h(x)\,(y-x)$$

and the homogeneity is $3/2-\delta$.

However one can also encounter more complex monomials. For instance for $\tau := \mathcal{I}(X\Xi)$

$$\Pi_x \mathcal{I}(X\Xi)(y) =$$

$$= \int_{[0,t]\times\mathbb{R}} (G_{t-s}(y-z) - G_{t-s}(x-z) - \partial_x G_{t-s}(x-z)\,(y-x)) \cdot$$

$$\cdot\,(z-x)\,\xi(ds,dz)$$

$$= \mathcal{I}((X-x)\xi)(y) - \mathcal{I}((X-x)\xi)(x) - (y-x)\mathcal{I}_1((X-x)\xi)(x)$$

and the homogeneity is $3/2 - \delta$.

Examples

For $\tau = \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))$ the homogeneity is $3/2 - \delta$ and

$$\begin{aligned}
&\Pi_x \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))(y) \\
&= \mathcal{I}(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(y) - \mathcal{I}(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(x) \\
&+ (y-x)(\mathcal{I}_1(\xi)(x)(\mathcal{I}(\xi\mathcal{I}(\xi))(x) - \mathcal{I}(\xi)(x)^2)) \\
&+ (y-x)(\mathcal{I}_1(\xi\mathcal{I}(\xi))(x)\mathcal{I}(\xi)(x) - \mathcal{I}_1(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(x)) \\
&- \mathcal{I}(\xi)(x)\mathcal{I}(\xi\mathcal{I}(\xi))(y) + \mathcal{I}(\xi)(x)^2\mathcal{I}(\xi)(y) \\
&- \mathcal{I}(\xi\mathcal{I}(\xi))(x)\mathcal{I}(\xi)(y) - \mathcal{I}(\xi)(x)^3 + 2\mathcal{I}(\xi)(x)\mathcal{I}(\xi\mathcal{I}(\xi))(x)
\end{aligned}$$

The Γ operators

The theory also needs a rule to transform a monomial centered at x into a polynomial centered at z, following the classical example:

$$(y-x)^k = (y-z+z-x)^k = \sum_{i=0}^{k} \binom{k}{i} (z-x)^i (y-z)^{k-i}$$

which becomes in the abstract setting

$$\Gamma_{zx} X^k = (X+z-x)^k = \sum_{i=0}^{k} \binom{k}{i} (z-x)^i X^{k-i}$$

and clearly satisfies the compatibility condition

$$\Pi_z \Gamma_{zx} = \Pi_x$$

The Γ operators

In general we have the recursive definition

$$\Gamma_{xy} X = X + (x - y), \qquad \Gamma_{xy} \Xi = \Xi, \qquad \Gamma_{xy} \prod_i \tau_i = \prod_i \Gamma_{xy} \tau_i$$

$$\Gamma_{xy} \mathcal{I}_k(\tau) = \mathcal{I}_k(\Gamma_{xy}\tau) - \sum_{j<|\tau|+2-k} (\Pi_x \mathcal{I}_{k+j}(\Gamma_{xy}\tau))(y) \frac{(X + x - y)^j}{j!}$$

On can check again the compatibility condition

$$\Pi_z \Gamma_{zx} = \Pi_x$$

Examples

For instance

$$\Gamma_{xz}\mathcal{I}(\Xi) = \mathcal{I}(\Xi) + (G * \xi)(x) - (G * \xi)(z)$$

and indeed setting $h := (G * \xi)$

$$\begin{aligned}\Pi_x \Gamma_{xz}\mathcal{I}(\Xi)(y) =& h(y) - h(x) + h(x) - h(z) = h(y) - h(z) \\ =& \Pi_z \mathcal{I}(\Xi)(y).\end{aligned}$$

Examples

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Another example:

$$\begin{aligned}\Gamma_{xz}\mathcal{I}(\Xi\mathcal{I}(\Xi)) =& (-\mathcal{I}(\xi)(x)\mathcal{I}(\xi)(z) + \mathcal{I}(\xi\mathcal{I}(\xi))(x) + \mathcal{I}(\xi)(z)^2 \\ & - \mathcal{I}(\xi\mathcal{I}(\xi))(z)) \\ & + (\mathcal{I}(\xi)(x) - \mathcal{I}(\xi)(z))\,\mathcal{I}(\Xi) \\ & + \mathcal{I}(\Xi\mathcal{I}(\Xi))\end{aligned}$$

Examples

And another:

$$\begin{aligned}\Gamma_{xz}\mathcal{I}(X\Xi) =&(-z\mathcal{I}(\xi)(x) + \mathcal{I}(\xi X)(x) + (xz - z^2)\mathcal{I}_1(\xi)(z)+\\ &z\mathcal{I}(\xi)(z) + (-x + z)\mathcal{I}_1(\xi X)(z) - \mathcal{I}(\xi X)(z))\\ &+ (x - z)\mathcal{I}(\Xi)\\ &+ (-x\mathcal{I}_1(\xi)(x) + \mathcal{I}_1(\xi X)(x) + z\mathcal{I}_1(\xi)(z) - \mathcal{I}_1(\xi X)(z))X\\ &+ \mathcal{I}(X\Xi)\end{aligned}$$

Examples

And another:

$$
\begin{aligned}
&\Gamma_{xz}\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))) = \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))) + \mathcal{I}(\xi)(x)\mathcal{I}(\xi)(y)^2 \\
&- \mathcal{I}(\xi)(x)\mathcal{I}(\xi\mathcal{I}(\xi))(y) - \mathcal{I}(\xi\mathcal{I}(\xi))(x)\mathcal{I}(\xi)(y) + \mathcal{I}(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(x) \\
&+ (y-x)(\mathcal{I}_1(\xi)(y)\mathcal{I}(\xi)(y)^2 - \mathcal{I}_1(\xi)(y)\mathcal{I}(\xi\mathcal{I}(\xi))(y) - \mathcal{I}_1(\xi\mathcal{I}(\xi))(y)\mathcal{I}(\xi \\
&+ \mathcal{I}_1(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(y)) - \mathcal{I}(\xi)(y)^3 + 2\mathcal{I}(\xi)(y)\mathcal{I}(\xi\mathcal{I}(\xi))(y) - \mathcal{I}(\xi\mathcal{I}(\xi\mathcal{I}(\xi)) \\
&+ \mathcal{I}(\Xi)(-\mathcal{I}(\xi)(x)\mathcal{I}(\xi)(y) + \mathcal{I}(\xi\mathcal{I}(\xi))(x) + \mathcal{I}(\xi)(y)^2 - \mathcal{I}(\xi\mathcal{I}(\xi))(y)) \\
&+ \mathcal{I}(\Xi\mathcal{I}(\Xi))(\mathcal{I}(\xi)(x) - \mathcal{I}(\xi)(y)) \\
&+ X(\mathcal{I}_1(\xi)(x)\mathcal{I}(\xi)(x)^2 - \mathcal{I}_1(\xi)(x)\mathcal{I}(\xi\mathcal{I}(\xi))(x) - \mathcal{I}_1(\xi\mathcal{I}(\xi))(x)\mathcal{I}(\xi)(x) \\
&+ \mathcal{I}_1(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(x) - \mathcal{I}_1(\xi)(y)\mathcal{I}(\xi)(y)^2 + \mathcal{I}_1(\xi)(y)\mathcal{I}(\xi\mathcal{I}(\xi))(y) \\
&+ \mathcal{I}_1(\xi\mathcal{I}(\xi))(y)\mathcal{I}(\xi)(y) - \mathcal{I}_1(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(y))
\end{aligned}
$$

Functional norm

We call T the space of finite linear combinations of trees from our class, and we consider a norm on functions $f : \mathbb{R}_+ \times \mathbb{R} \mapsto T$: for $\gamma > 0$ we say that $f \in \mathcal{D}^\gamma$ if f takes values in the linear span of the trees with homogeneity $< \gamma$ and for all $\beta < \gamma$

$$\|f(x) - \Gamma_{xy} f(y)\|_\beta \leq C_f \|x - y\|^{\gamma - \beta}$$

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$$\|f(x) - \Gamma_{xy} f(y)\|_\beta \leq C_f \|x - y\|^{\gamma - \beta}$$

This innocent-looking condition can be in practice very complicated to check, because of the presence of the Γ operators. It is a notion of Hölder regularity in this setting of random monomials.

If f takes values in sums of X^k, then the definition is equivalent to the classical C^γ-regularity (for $\gamma \notin \mathbb{N}$). The coefficient multiplying X^k is then the k-th derivative of f.

It turns out that this definition is

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CRUCIAL !!!!

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because the two main technical results of Martin's paper, the reconstruction theorem and the multi-level Schauder estimates, work on this function space.

In particular, it is in this space that the solution of the SPDE lives and the equation is solved (locally in time) by a fixed point.

One can reinterpret the theory of controlled rough paths in this setting (reminder: the remainder!).

Polynomials

What are the functions that belong to $\mathcal{D}^\gamma$ for all $\gamma \geq \gamma_0$? This means

$$f(x) - \Gamma_{xy} f(y) \equiv 0.$$

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$$x \mapsto \mathcal{I}(\xi X)(x) + x\mathcal{I}(\Xi) + \mathcal{I}_1(\xi(X - x))(x) X + \mathcal{I}(\Xi X)$$

$$x \mapsto \mathcal{I}(\xi \mathcal{I}(\xi))(x) + \mathcal{I}(\xi)(x)\mathcal{I}(\Xi) + \mathcal{I}(\Xi \mathcal{I}(\Xi))$$

$$x \mapsto \mathcal{I}(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(x) + \mathcal{I}(\xi\mathcal{I}(\xi))(x)\mathcal{I}(\Xi) + \mathcal{I}(\xi)(x)\mathcal{I}(\Xi\mathcal{I}(\Xi))$$
$$+ (\mathcal{I}_1(\xi)(x)\mathcal{I}(\xi)(x)^2 - \mathcal{I}_1(\xi)(x)\mathcal{I}(\xi\mathcal{I}(\xi))(x) - \mathcal{I}_1(\xi\mathcal{I}(\xi))(x)\mathcal{I}(\Xi)(x)$$
$$+ \mathcal{I}_1(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(x))X + \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))$$

Polynomials

What are the functions that belong to $\mathcal{D}^\gamma$ for all $\gamma \geq \gamma_0$? This means

$$f(x) - \Gamma_{xy} f(y) \equiv 0.$$

$$x \mapsto x + X, \quad x \to \Xi, \quad x \to f_1(x) f_2(x)$$

$$x \mapsto \mathcal{I}(\xi X)(x) + x\mathcal{I}(\Xi) + \mathcal{I}_1(\xi(X - x))(x)X + \mathcal{I}(\Xi X)$$

$$x \mapsto \mathcal{I}(\xi\mathcal{I}(\xi))(x) + \mathcal{I}(\xi)(x)\mathcal{I}(\Xi) + \mathcal{I}(\Xi\mathcal{I}(\Xi))$$

$$x \mapsto \mathcal{I}(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(x) + \mathcal{I}(\xi\mathcal{I}(\xi))(x)\mathcal{I}(\Xi) + \mathcal{I}(\xi)(x)\mathcal{I}(\Xi\mathcal{I}(\Xi))$$
$$+ (\mathcal{I}_1(\xi)(x)\mathcal{I}(\xi)(x)^2 - \mathcal{I}_1(\xi)(x)\mathcal{I}(\xi\mathcal{I}(\xi))(x) - \mathcal{I}_1(\xi\mathcal{I}(\xi))(x)\mathcal{I}(\Xi)(x)$$
$$+ \mathcal{I}_1(\xi\mathcal{I}(\xi\mathcal{I}(\xi)))(x))X + \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))$$

There is an integration operator $K_\gamma : \mathcal{D}^\gamma \mapsto \mathcal{D}^{\gamma+2}$.

The coefficients of the solution of generalized KPZ

$$\partial_t u = \Delta u + f(u)\,(\partial_x u)^2 + g(u)\,\xi.$$

At order $\gamma = 1/2$

$$u(x) + g(u(x))\mathcal{I}(\Xi)$$

The coefficients of the solution of generalized KPZ

At order $\gamma = 1$

$$u(x) + g(u(x))\mathcal{I}(\Xi) + g_1 g(u(x))\mathcal{I}(\Xi\mathcal{I}(\Xi)) + g^2 f(u(x))\mathcal{I}((\mathcal{I}_1(\Xi))^2)$$
$$+ (u_1(x) - g(u)\mathcal{I}_1(\xi)(x) + g_1(u)\mathcal{I}_1(\xi)(x)\mathcal{I}(\xi)(x)$$
$$- g_1 g(u)\mathcal{I}_1(\xi\mathcal{I}(\xi))(x) - g^2 f(u)\mathcal{I}_1((\mathcal{I}_1(\xi))^2)(x))X$$

The coefficients of the solution of generalized KPZ

At order $\gamma = 3/2$

$$
\begin{aligned}
& u(x) + g(u(x))\mathcal{I}(\Xi) + g_1 g(u(x))\mathcal{I}(\Xi\mathcal{I}(\Xi)) + g^2 f(u(x))\mathcal{I}((\mathcal{I}_1(\Xi))^2) \\
& + (u_1(x) - g(u)\mathcal{I}_1(\xi)(x) + g_1(u)\mathcal{I}_1(\xi)(x)\mathcal{I}(\xi)(x) \\
& - g_1 g(u)\mathcal{I}_1(\xi\mathcal{I}(\xi))(x) - g^2 f(u)\mathcal{I}_1((\mathcal{I}_1(\xi))^2)(x))X \\
& + 1/2 g_2 g(u(x))^2 \mathcal{I}(\Xi(\mathcal{I}(\Xi))^2) + g_1^2 g(u(x))\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))) \\
& + g_1 g^2 f(u(x))\mathcal{I}(\Xi\mathcal{I}((\mathcal{I}_1(\Xi))^2)) + 2g^3 f_1(u(x))\mathcal{I}((\mathcal{I}_1(\Xi))^2\mathcal{I}(\Xi)) \\
& + 2g_1 g^2 f(u(x))\mathcal{I}(\mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi\mathcal{I}(\Xi))) + 2g^3 f(u(x))^2 \mathcal{I}(\mathcal{I}_1(\Xi)\mathcal{I}_1((\mathcal{I}_1(\Xi))^2) \\
& + (u_1 g_1(u(x)) - g_1 g(u)\mathcal{I}_1(\Xi)(x) + g_1^2 g(u)\mathcal{I}_1(\xi)\mathcal{I}(\xi)(x) \\
& - g_1^2 g(u)\mathcal{I}_1(\xi\mathcal{I}(\xi))(x) - g_1 g^2 f(u)\mathcal{I}_1((\mathcal{I}_1(\xi))^2)(x))\mathcal{I}(\Xi X) \\
& + (-2g^2 f(u)\mathcal{I}_2(\xi)(x) + 2u_1 g f(u(x)) + 2g_1 g^2 f(u)\mathcal{I}_2(\xi)(x)\mathcal{I}(\xi)(x) \\
& - 2g_1 g^2 f(u)\mathcal{I}_2(\xi\mathcal{I}(\xi))(x) - 2g^3 f(u)^2 \mathcal{I}_2((\mathcal{I}_1(\xi))^2)(x))\mathcal{I}(\mathcal{I}_1(\Xi))
\end{aligned}
$$

Products of distributions and renormalization

An important message is that these monomials contain all *relevant* non-trivial products. For instance, in order to define $(\partial_x u)^2$, one writes a Taylor expansion of $\partial_x u$ and then squares it, thus obtaining another Taylor expansion of the same type, containing distributional derivatives of the monomials.

In this way, only the products of a finite number of monomials must be controlled. One of the main results of the theory is that this seemingly formal procedure can be made coherent and rigorous!

Renormalization

However some of the above monomials are ill-defined, in the sense that the stochastic integrals have infinite L^p norm for any p.

It turns out that they can be renormalized by hand, since they are explicit functions of ξ: one regularizes the noise replacing ξ by ξ_ε and subtracts some diverging quantity (like the constant C_ε seen in the Ito formula above) in order to produce a well-defined limit.

In the KPZ equation the number of additional random monomials is 5, and a previous paper by Hairer [2013] treats them one by one. In the generalized KPZ equation there are 43 additional monomials, including the 5 ones of KPZ, some simple and some complicated. A renormalization by hand becomes virtually impossible, a more general and efficient procedure is needed...

The tree of generalized KPZ

$$
\begin{aligned}
&\Xi + \Xi\mathcal{I}(\Xi) + (\mathcal{I}_1(\Xi))^2 + \Xi(\mathcal{I}(\Xi))^2 + \Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)) + \Xi\mathcal{I}((\mathcal{I}_1(\Xi))^2)\\
&+ (\mathcal{I}_1(\Xi))^2\mathcal{I}(\Xi) + \mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi\mathcal{I}(\Xi)) + \mathcal{I}_1(\Xi)\mathcal{I}_1((\mathcal{I}_1(\Xi))^2) + \Xi X\\
&+ \mathcal{I}_1(\Xi) + \Xi(\mathcal{I}(\Xi))^3 + \Xi\mathcal{I}(\Xi)\mathcal{I}(\Xi\mathcal{I}(\Xi)) + \Xi\mathcal{I}(\Xi)\mathcal{I}((\mathcal{I}_1(\Xi))^2)\\
&+ \Xi\mathcal{I}(\Xi(\mathcal{I}(\Xi))^2) + \Xi\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))) + \Xi\mathcal{I}(\Xi\mathcal{I}((\mathcal{I}_1(\Xi))^2))\\
&+ \Xi\mathcal{I}((\mathcal{I}_1(\Xi))^2\mathcal{I}(\Xi)) + \Xi\mathcal{I}(\mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi\mathcal{I}(\Xi))) + \Xi\mathcal{I}(\mathcal{I}_1(\Xi)\mathcal{I}_1((\mathcal{I}_1(\Xi))^2))\\
&+ (\mathcal{I}_1(\Xi))^2(\mathcal{I}(\Xi))^2 + (\mathcal{I}_1(\Xi))^2\mathcal{I}(\Xi\mathcal{I}(\Xi)) + (\mathcal{I}_1(\Xi))^2\mathcal{I}((\mathcal{I}_1(\Xi))^2)\\
&+ \mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi\mathcal{I}(\Xi))\mathcal{I}(\Xi) + \mathcal{I}_1(\Xi)\mathcal{I}_1((\mathcal{I}_1(\Xi))^2)\mathcal{I}(\Xi) + \mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi(\mathcal{I}(\Xi))^2)\\
&+ \mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))) + \mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi\mathcal{I}((\mathcal{I}_1(\Xi))^2))\\
&+ \mathcal{I}_1(\Xi)\mathcal{I}_1((\mathcal{I}_1(\Xi))^2\mathcal{I}(\Xi)) + \mathcal{I}_1(\Xi)\mathcal{I}_1(\mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi\mathcal{I}(\Xi)))\\
&+ \mathcal{I}_1(\Xi)\mathcal{I}_1(\mathcal{I}_1(\Xi)\mathcal{I}_1((\mathcal{I}_1(\Xi))^2)) + \mathcal{I}_1(\Xi\mathcal{I}(\Xi))\mathcal{I}_1((\mathcal{I}_1(\Xi))^2)\\
&+ (\mathcal{I}_1(\Xi\mathcal{I}(\Xi)))^2 + (\mathcal{I}_1((\mathcal{I}_1(\Xi))^2))^2 + \Xi\mathcal{I}(\Xi)X + \Xi\mathcal{I}(\Xi X)\\
&+ \Xi\mathcal{I}(\mathcal{I}_1(\Xi)) + (\mathcal{I}_1(\Xi))^2X + \mathcal{I}_1(\Xi)\mathcal{I}(\Xi) + \mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi X)\\
&+ \mathcal{I}_1(\Xi)\mathcal{I}_1(\mathcal{I}_1(\Xi)) + \mathcal{I}_1(\Xi\mathcal{I}(\Xi)) + \mathcal{I}_1((\mathcal{I}_1(\Xi))^2)
\end{aligned}
$$

The stochastic heat equation

$$\partial_t u = \Delta u + u\xi$$

$$u(x) + u(x)\mathcal{I}(\Xi) + u(x)\mathcal{I}(\Xi\mathcal{I}(\Xi))$$
$$+ X(u_1(x) - u(x)(\mathcal{I}_1(\Xi)(x) + \mathcal{I}_1(\Xi)(x)\mathcal{I}(\Xi)(x) - \mathcal{I}_1(\Xi\mathcal{I}(\Xi))(x)))$$
$$+ u(x)\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))) + (u_1(x) - u(x)\mathcal{I}_1(\Xi)(x) + u(x)\mathcal{I}_1(\Xi)(x)\mathcal{I}(\Xi)(x$$
$$- u(x)\mathcal{I}_1(\Xi\mathcal{I}(\Xi))(x))\mathcal{I}(\Xi X) + u(x)\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))))$$
$$+ (u_1(x) - u(x)\mathcal{I}_1(\Xi)(x) + u(x)\mathcal{I}_1(\Xi)(x)\mathcal{I}(\Xi)(x)$$
$$- u(x)\mathcal{I}_1(\Xi\mathcal{I}(\Xi))(x))\mathcal{I}(\Xi\mathcal{I}(\Xi X))$$
$$+ 1/2c(x)X^2$$

The stochastic heat equation

$$
\begin{aligned}
c = {} & u_2 + x^2\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi)(u_1 - u\mathcal{I}_1(\Xi)) + xu_1\mathcal{I}_2(\Xi) \\
& + x^2u\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi)^2\mathcal{I}(\Xi) - x^2u\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi\mathcal{I}(\Xi)) - xu_1\mathcal{I}_2(\Xi)\mathcal{I}(\Xi) \\
& - xu_1\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi X) + xu_1\mathcal{I}_2(\Xi\mathcal{I}(\Xi)) - xu_1\mathcal{I}_1(\Xi)\mathcal{I}_2(\Xi X) - xu\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi \\
& + xu\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi)\mathcal{I}(\Xi) + xu\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi X) - xu\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi\mathcal{I}(\Xi)) \\
& - xu\mathcal{I}_2(\Xi\mathcal{I}(\Xi))\mathcal{I}_1(\Xi) + xu\mathcal{I}_1(\Xi)^2\mathcal{I}_2(\Xi X) - u_1\mathcal{I}_2(\Xi X) - u\mathcal{I}_2(\Xi) \\
& - xu\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi)\mathcal{I}(\Xi)(\mathcal{I}(\Xi) + \mathcal{I}_1(\Xi X)) + xu\mathcal{I}_2(\Xi)(\mathcal{I}_1(\Xi)\mathcal{I}(\Xi\mathcal{I}(\Xi)) \\
& + \mathcal{I}_1(\Xi\mathcal{I}(\Xi))\mathcal{I}(\Xi)) + xu\mathcal{I}_2(\Xi)(\mathcal{I}_1(\Xi\mathcal{I}(\Xi))\mathcal{I}_1(\Xi X) - \mathcal{I}_1(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))) \\
& + xu\mathcal{I}_2(\Xi\mathcal{I}(\Xi))(\mathcal{I}_1(\Xi)\mathcal{I}(\Xi) - \mathcal{I}_1(\Xi\mathcal{I}(\Xi))) \\
& - xu\mathcal{I}_1(\Xi)\mathcal{I}_2(\Xi X)(\mathcal{I}_1(\Xi)\mathcal{I}(\Xi) - \mathcal{I}_1(\Xi\mathcal{I}(\Xi))) + u_1\mathcal{I}_2(\Xi)\mathcal{I}(\Xi X) \\
& + u_1\mathcal{I}_2(\Xi X)\mathcal{I}_1(\Xi X) - u_1\mathcal{I}_2(\Xi\mathcal{I}(\Xi X)) + u\mathcal{I}_2(\Xi)\mathcal{I}(\Xi) \\
& - u\mathcal{I}_2(\Xi\mathcal{I}(\Xi)) + u\mathcal{I}_1(\Xi)\mathcal{I}_2(\Xi X) - u\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi)\mathcal{I}(\Xi X) - u\mathcal{I}_2(\Xi)\mathcal{I}(\Xi)^2 \\
& + u\mathcal{I}_2(\Xi)\mathcal{I}(\Xi\mathcal{I}(\Xi)) + u\mathcal{I}_2(\Xi\mathcal{I}(\Xi))\mathcal{I}(\Xi) - u\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi)\mathcal{I}(\Xi X)
\end{aligned}
$$

The stochastic heat equation

$$
\begin{aligned}
&- u\mathcal{I}_2(\Xi)\mathcal{I}(\Xi)^2 + u\mathcal{I}_2(\Xi)\mathcal{I}(\Xi\mathcal{I}(\Xi)) + u\mathcal{I}_2(\Xi\mathcal{I}(\Xi))\mathcal{I}(\Xi) \\
&- u\mathcal{I}_2(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))) - u\mathcal{I}_1(\Xi)\mathcal{I}_2(\Xi X)\mathcal{I}(\Xi) - u\mathcal{I}_1(\Xi)\mathcal{I}_2(\Xi X)\mathcal{I}_1(\Xi X) \\
&+ u\mathcal{I}_1(\Xi)\mathcal{I}_2(\Xi\mathcal{I}(\Xi X)) + u(\mathcal{I}_2(\Xi X)\mathcal{I}_1(\Xi\mathcal{I}(\Xi)) + \mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi)\mathcal{I}(\Xi)\mathcal{I}(\Xi X) \\
&- u\mathcal{I}_2(\Xi)\mathcal{I}_1(\Xi\mathcal{I}(\Xi))\mathcal{I}(\Xi X) + u\mathcal{I}_2(\Xi)\mathcal{I}(\Xi)^3 - u\mathcal{I}_2(\Xi)\mathcal{I}(\Xi)\mathcal{I}(\Xi\mathcal{I}(\Xi)) \\
&+ u\mathcal{I}_2(\Xi)\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))) - u\mathcal{I}_2(\Xi\mathcal{I}(\Xi))\mathcal{I}(\Xi)^2 + u\mathcal{I}_2(\Xi\mathcal{I}(\Xi))\mathcal{I}(\Xi\mathcal{I}(\Xi)) \\
&+ u\mathcal{I}_2(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))\mathcal{I}(\Xi) + u\mathcal{I}_1(\Xi)\mathcal{I}_2(\Xi X)\mathcal{I}(\Xi)^2 \\
&+ u\mathcal{I}_1(\Xi)\mathcal{I}_2(\Xi X)\mathcal{I}(\Xi)\mathcal{I}_1(\Xi X) - u\mathcal{I}_1(\Xi)\mathcal{I}_2(\Xi X)\mathcal{I}(\Xi\mathcal{I}(\Xi)) \\
&- u\mathcal{I}_1(\Xi)\mathcal{I}_2(\Xi\mathcal{I}(\Xi X))\mathcal{I}(\Xi) - u\mathcal{I}_2(\Xi X)\mathcal{I}_1(\Xi\mathcal{I}(\Xi))\mathcal{I}(\Xi) \\
&- u\mathcal{I}_2(\Xi X)\mathcal{I}_1(\Xi\mathcal{I}(\Xi))\mathcal{I}_1(\Xi X) + u\mathcal{I}_2(\Xi X)\mathcal{I}_1(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))) \\
&- u\mathcal{I}_2(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))) + u\mathcal{I}_1(\Xi\mathcal{I}(\Xi))\mathcal{I}_2(\Xi\mathcal{I}(\Xi X))
\end{aligned}
$$