Gaussian rough paths, transportation–cost inequalities and multilevel Monte Carlo

Sebastian Riedel

Technische Universität Berlin

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Outline

Gaussian processes and rough paths

Applications

- Concentration of measure on paths spaces
- Multilevel and rough paths

Outline

Goal: Define, solve and analyze solution $Y = I_f(X, \xi)$ to

$$dY_t = f_0(Y_t)\,dt + \sum_{i=1}^{d} f_i(Y_t)\,dX_t^i; \quad Y_0 = \xi \in \mathbb{R}^m$$

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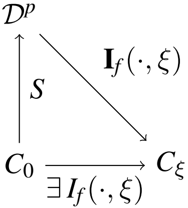
Motivation

- Models with correlations in the noise (eg. fractional Bm)
- nonlinear SPDE with 1d space variable (eg. Burger's equation, cf. [Hairer; CPAM '11])

One possible way: Use rough paths factorization



where

$$\mathcal{D}^p = \left\{ x, \int_{\Delta^2} dx \otimes dx, \ldots, \int_{\Delta^{[p]}} dx \otimes \cdots \otimes dx \right\}$$

$$S(x) = \left(x, \int_{\Delta^2} dx \otimes dx, \ldots, \int_{\Delta^{[p]}} dx \otimes \cdots \otimes dx \right)$$

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- Can show: If B^ε "nice" pathwise smooth approximation of Bm, then iterated integrals converge in rough paths topology:

$$\int_{\Delta^n} dB^\varepsilon \otimes \cdots \otimes dB^\varepsilon \to \int_{\Delta^n} \circ dB \otimes \cdots \otimes \circ dB \qquad (1)$$

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  for $\varepsilon \to 0$, $\rightsquigarrow$ “natural” rough path.
- Generalization of (1) to other processes?

For **Gaussian processes** X, the following holds:

Theorem (Friz, Victoir; '10)

If covariance

$$(s,t) \mapsto R(s,t) = \mathbb{E}(X_s \otimes X_t) \in \mathbb{R}^{d \times d}$$

is sufficiently smooth, i.e. is diagonal and has finite ρ-variation for some $\rho < 2$, then pathwise approximations of iterated integrals converge in rough paths space.

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- Define:

$$V_\rho(R;[s,t]) := \sup_{D \in \Pi([s,t]^2)} \left(\sum_{A \in D} |R(A)|^\rho \right)^{\frac{1}{\rho}}$$

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- example: Brownian motion

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- typically, R “smooth” outside the diagonal $D = \{(s,t) \,:\, s = t\}$

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   - (i) μ_+ *has finite mass,*
   - (ii) $R([u,v] \times [s,t]) \geq 0$ *for all* $[u,v] \subseteq [s,t]$,
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- Condition (iii) first appeared in [Jain, Monrad; '83] in the context of sample path regularity

Cameron–Martin embedding

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Under the same assumptions, we have the continuous embedding

$$\mathcal{H} \hookrightarrow C^{q-var}; \qquad q = \frac{1}{\frac{1}{2\rho} + \frac{1}{2}} \quad (\leq \rho)$$

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Recall: For fBm,

$$\mathcal{H}^H \hookrightarrow C^{q-\text{var}}; \qquad q = \frac{1}{H + \frac{1}{2}}$$

If X has stationary increments, $\sigma^2(s,t) =: \sigma^2(t-s)$ and we obtain

Corollary

1. *If σ^2 is convex, R has finite (2d) 1-variation*
2. *If*
   (i) *σ^2 is concave,*
   (ii) *σ^2 is increasing on $[0,T]$,*
   (iii) *σ^2 has finite (1d) ρ-variation (or is $1/\rho$ Hölder continuous),*

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Example I: bifractional Bm

centered Gaussian process $B^{H,K} \colon [0,T] \to \mathbb{R}$ with covariance

$$R(s,t) = \frac{1}{2^K}\left((s^{2H} + t^{2H})^K - |t-s|^{2HK}\right),$$

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Can show: covariance of bfBm has finite ρ-variation for $\rho = \frac{1}{2HK}$, CM paths have finite q-variation, $q = (HK + 1/2)^{-1}$.

Example II: fractional SHE

Ψ solution of the fractional stochastic heat equation

$$d\Psi_t = (-(-\Delta)^\alpha)\Psi_t \, dt + dW_t; \quad \Psi_0 = 0$$

where

- dW space-time white noise
- spatial variable $x \in [0, 2\pi]$, time variable $t \in [0, T]$
- $(-(-\Delta)^\alpha)$ fractional Laplacian with periodic (**P**), Dirichlet (**D**) or Neumann (**N**) ∂-conditions ($\alpha = 1 \rightsquigarrow$ usual Laplacian Δ)

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Can show: Covariance of $x \mapsto \Psi_t(x)$ has finite ρ-variation for $\rho = \frac{1}{2\alpha - 1}$.

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- $\mu \in P(E)$ has Gaussian concentration property if

$$\mu(A^r) \geq 1 - be^{-a(r-r_0)^2}$$

for all $r \geq r_0$, $A \in \mathcal{B}(E)$ s.t. $\mu(A) \geq 1/2$.

Functional approach?

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- Recall: If $\mu, \nu \in P(E)$, $p \geq 1$,

$$\mathcal{W}_p(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{E \times E} d(x,y)^p \, d\pi(x,y) \right)^{\frac{1}{p}}$$

Wasserstein p-metric

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Wasserstein p-metric

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$$H(\nu \,|\, \mu) = \begin{cases} \int_E \log\left(\frac{d\nu}{d\mu}\right) d\nu & \text{if } \nu \ll \mu \\ +\infty & \text{otherwise} \end{cases}$$

relative entropy of ν w.r.t. μ

Transportation–cost inequalities

p-transportation–cost inequality holds for $\mu \in P(E)$ iff

$$\mathcal{W}_p(\nu, \mu) \leq \sqrt{CH(\nu \mid \mu)} \qquad \text{for all } \nu \in P(E) \qquad (T_p(C))$$

Relations to Gaussian concentration

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Theorem (Gozlan; '09)

$T_2(C)$ is equivalent to dimension free Gaussian concentration property.

Gaussian context

$(E, \mathcal{H}, \gamma)$ Gaussian Banach space,

$$d_{\mathcal{H}}(x,y) = \begin{cases} \sqrt{\langle x-y, x-y\rangle_{\mathcal{H}}} & \text{if } x-y \in \mathcal{H} \\ +\infty & \text{otherwise} \end{cases}$$

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Theorem

$$\inf_{\pi \in \Pi(\nu,\gamma)} \int_{E\times E} d_{\mathcal{H}}(x,y)^2 \, d\pi(x,y) \leq 2H(\nu \,|\, \gamma)$$

for all $\nu \in P(E)$.

Possible proofs

- finite dimensional case: [Talagrand; ’96] (via tensorization)
- infinite dimensional case: [Feyel, Üstünel; ’03] (via Girsanov transform)
- **our proof:** combines large deviation approach from Gozlan with the Borell-Sudakov–Tsirelson inequality

Transportation inequalities stable under Lipschitz maps

Key observation:

Lemma (Djellout, Guillin, Wu; '04)

- $\Psi\colon (E, d_E) \to (F, d_F)$ *Lipschitz with constant* L
- $T_p(C)$ *holds for some* $\mu \in P(E)$

Then $T_p(L^2 C)$ *holds for* $\tilde{\mu}$ *where* $\tilde{\mu} = \mu \circ \Psi^{-1}$.

The additive noise case I

Consider

$$dY_t = dX_t + b(Y_t)\,dt; \quad Y_0 = \xi \in \mathbb{R}^d,$$

$X \colon [0,T] \to \mathbb{R}^d$ continuous Gaussian process,

$$I_b(\cdot,\xi) \colon C_0([0,T],\mathbb{R}^d) \to C_\xi([0,T],\mathbb{R}^d)$$

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solution map.
Assume continuous embedding of $\mathcal{H}$ into fractional Sobolev space $W^{p,\delta}$:

$$\mathcal{H} \hookrightarrow W^{\delta,p}$$

for some $\delta \in (0,1]$ and $p \in (1/\delta, \infty)$.

The additive noise case II

Corollary

Let $\mu \in P(C_\xi)$ *be the law of* Y. *Then*

$$\inf_{\pi \in \Pi(\nu,\mu)} \int_{C_\xi \times C_\xi} d_{W^{\delta,p}}(x,y)^2 \, d\pi(x,y) \leq CH(\nu \,|\, \mu)$$

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- fractional Bm: $\mathcal{H} \hookrightarrow W^{\delta,2}$ where $\delta < H + 1/2$

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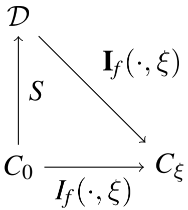
Solution defined via



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Lemma

$$\|I_f(x,\xi) - I_f(y,\xi)\|_\infty \leq L(x,y)(d_{\mathcal{H}}(x,y) \vee d_{\mathcal{H}}(x,y)^q)$$

where L has every moment for every $\pi \in \Pi(\nu,\gamma)$ with $\nu \ll \gamma$.

The multiplicative noise case III

Theorem

Let μ *be law of* Y *on* $C_\xi([0,T],\mathbb{R}^m)$*. Assume* $q=1$*. Then for every* $\varepsilon>0$*,*

$$\inf_{\pi\in\Pi(\nu,\mu)}\left(\int_{C_\xi\times C_\xi}\|x-y\|_\infty^{2-\varepsilon}\,d\pi(x,y)\right)^{\frac{1}{2-\varepsilon}}\leq\sqrt{CH(\nu\,|\,\mu)}$$

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The multiplicative noise case III

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for all $\nu\in P(C_\xi)$.

Examples:

- bifractional Bm $B^{H,K}$ with $HK=\frac{1}{2}$
- solution $x\mapsto\Psi_t(x)$ to the heat equation ($\alpha=1$)

Theorem

V linear polish, $\mathcal{B} \subseteq V$ normed space,

$$d_{\mathcal{B}}(x,y) = \begin{cases} \|x-y\|_{\mathcal{B}} & \text{if } x-y \in \mathcal{B} \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\mu \in P(V)$ and assume

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Let (E,d) metric space, $f \colon V \to (E,d)$ such that μ - a.s. for all $h \in \mathcal{B}$,

$$d(f(x+h), e) \leq c(x) \left(g(x) + \|h\|_{\mathcal{B}} \right)$$

for some $e \in E$ where $c \in L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\int cg \, d\mu < \infty$.

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Then $f : x \to d(f(x), e)$ has Gaussian tails.

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- $f = N_\alpha(S(\cdot), [0, T])$, $\rightsquigarrow$ integrabililty results from [Cass, Litterer, Lyons; '13] for "greedy partitions"

Multilevel and rough paths

Multilevel and rough paths

Goal: Find efficient way to evaluate

$$\mathbb{E}[g(Y_\cdot)]$$

where Y solves

$$dY_t = \sum_{i=1}^{d} f_i(Y_t)\, dX_t^i; \qquad Y_0 = \xi \in \mathbb{R}^m$$

and

$$g\colon C_\xi([0,T],\mathbb{R}^m) \to \mathbb{R}.$$

Numerical schemes

Simple step-3 Milstein scheme ([Deya, Neuenkirch, Tindel; '12]):

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$$\begin{aligned}\bar{Y}_0 &= Y_0 \\ \bar{Y}_{t_{m+1}} &= \bar{Y}_{t_m} + f_i(\bar{Y}_{t_m}) X^i_{t_m,t_{m+1}} \\ &\quad + \frac{1}{2} f_i f_j (\bar{Y}_{t_m}) X^i_{t_m,t_{m+1}} X^j_{t_m,t_{m+1}} \\ &\quad + \frac{1}{6} f_{i_1} f_{i_2} f_{i_3} (\bar{Y}_{t_m}) X^{i_1}_{t_m,t_{m+1}} X^{i_2}_{t_m,t_{m+1}} X^{i_3}_{t_m,t_{m+1}}\end{aligned}$$

where $t_m = \frac{mT}{k}$.

Assume $|R|_{\rho-\mathrm{var};[s,t]} \lesssim |t-s|^{\frac{1}{\rho}}$.

Theorem (Friz, R; ’14 and Bayer, Friz, R. Schoenmakers)

f smooth, then for all $\eta < \frac{1}{\rho} - \frac{1}{2}$ and $q \geq 1$ there exists a constant C s.t.

$$|\|Y - \bar{Y}\|_\infty|_{L^q} \leq C \left(\frac{1}{k}\right)^\eta \quad \forall\, k \in \mathbb{N}.$$

Multilevel approach 1/3

- For simplicity:
  - $P := g(Y)$
  - $\hat{P}_l :=$ some Milstein-approximation of P with step size $h_l = \frac{T}{M_0 M^l}$

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- For simplicity:
  - $P := g(Y)$
  - $\hat{P}_l :=$ some Milstein-approximation of P with step size $h_l = \frac{T}{M_0 M^l}$
- **Basic idea multilevel:**

$$\begin{aligned} \mathbb{E}[P] &\approx \mathbb{E}[\hat{P}_L] \quad \text{for } L \text{ large} \\ &= \mathbb{E}[\hat{P}_0] + \sum_{l=1}^{L} \mathbb{E}[\hat{P}_l - \hat{P}_{l-1}], \end{aligned}$$

now MC for every summand with N_l samples, $l = 0, \ldots, L$

Multilevel approach 2/3

- Set $\hat{P}_{-1} = 0$ and

$$\hat{Y}_l = \frac{1}{N_l} \sum_{i=1}^{N_l} \left(\hat{P}_l^{(i)} - \hat{P}_{l-1}^{(i)} \right)$$

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- Then, if $\mathbb{E}\left[\left| \hat{P}_l - P \right|^2 \right] \lesssim h_l^\beta$ for $l = 1, \ldots, L$,

$$\mathbb{V}(\hat{Y}_0) \lesssim \frac{\mathbb{V}(\hat{P}_0)}{N_0} \quad \text{and} \quad \mathbb{V}(\hat{Y}_l) \lesssim \frac{h_l^\beta}{N_l}$$

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- **Observe:**
  - **low levels:** computational cost for Milstein-approximation small, but variance large
  - **high levels:** computational cost for approximation large, but variance small

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- Set $\hat{Y} = \sum_{l=0}^{L} \hat{Y}_l$

Multilevel approach 3/3

- Set $\hat{Y} = \sum_{l=0}^{L} \hat{Y}_l$
- **Strategy:** for given

$$\mathrm{MSE} = \mathbb{E}\left[\left(\hat{Y} - \mathbb{E}[P]\right)^2\right] \leq \varepsilon^2,$$

optimize over $L, N_0, \dots, N_L$ in order to minimize computational cost C

Theorem (Bayer, Friz, R., Schoenmakers)

Assume $\exists\, \alpha < 1/2,\ \beta \leq 2\alpha < 1$ *s.t.*

(i) $\mathbb{E}[\hat{P}_l - P] \lesssim h_l^{\alpha}$,

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Then $\forall\, \varepsilon > 0$, *there are* $L, N_0, \ldots, N_L \in \mathbb{N}$ *s.t.*

$$MSE = \mathbb{E}\left[\left(\hat{Y} - \mathbb{E}[P]\right)^2\right] \leq \varepsilon^2,$$

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Recall: complexity for usual MC would be $C = \mathcal{O}\left(\varepsilon^{-(2+1/\alpha)}\right)$

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- All constants can be calculated explicitly

Theorem (Bayer, Friz, R., Schoenmakers)

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- *g Lipschitz*
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Then the MC evaluation of $\mathbb{E}[g(Y)]$ *under condition*

$$MSE \leq \varepsilon^2$$

can be achieved with computational complexity

$$\mathcal{O}(\varepsilon^{-\theta}) \quad \forall \theta > \frac{2\rho}{2-\rho}.$$

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Thank you for your attention.