Paracontrolled differential equations

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Joint work with Massimiliano Gubinelli and Peter Imkeller

Three motivating SPDEs

- Burgers equation:

$$\mathcal{L}u(t,x) = \partial_x(u(t,x)^2) + \partial_x\xi(t,x)$$

  $u\colon [0,T] \times \mathbb{T} \to \mathbb{R}^n$, $\mathcal{L} = \partial_t - \Delta$ heat operator, ξ space-time white noise;

- solution is (formally) given by the derivative of the KPZ equation: $u = \partial_x h$, where

$$\mathcal{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x);$$

- solution to KPZ (formally) given by Cole-Hopf transform of the stochastic heat equation: $h = -\log w$, where w solves

$$\mathcal{L}w(t,x) = -w(t,x)\xi(t,x).$$

Problem: nonlinearity

Burgers SPDE is ill-posed with classical methods:

$$\mathcal{L}u(t,x) = \partial_x(u(t,x)^2) + \partial_x \xi(t,x)$$

- Expect u to have at best the regularity of solution v to linear equation $\mathcal{L}v(t,x) = \partial_x \xi(t,x)$;
- thus $u(t,\cdot) \in C^{-1/2-} = B^{-1/2-}_{\infty,\infty}$. Square of a distribution?
- Hairer (2013) uses series expansion and rough path integral to define $(\partial_x h(t,x))^2$ ("$= u(t,x)^2$").
- Rough paths only work in one index dimension. Since x is spatial variable: extension to $x \in \mathbb{T}^d$?
- State 2012: no techniques available to *define* solutions on $\mathbb{T}^d$, let alone show existence and uniqueness.
- Now of course accessible with Hairer's theory of regularity structures.

Aims

- Extend rough path approach to allow for multidimensional spatial index variables;
- solve Burgers, KPZ, and heat equation pathwise continuously;
- rigorously prove the formal links between them.

1 Paracontrolled calculus and products of distributions

2 Application: Burgers, KPZ and heat equation

Products of distributions

- Aim: define products of distributions directly, without detour via integrals.
- Many ways of defining integral $\int f \mathrm{d}g$: measure theory, Riemann sums, smooth approximation, algebraic arguments, ...
- Most easily adapted for defining product uv of distributions: smooth approximations, $uv = \lim_{n\to\infty} u^n v^n$.
- Convenient approximations given via Littlewood-Paley blocks:
  - Write $\mathcal{F}$ for Fourier transform;
  - $u = \mathcal{F}^{-1}(\mathcal{F}u) = \mathcal{F}^{-1}(\sum_j 1_{[2^j,2^{j+1})}(|\cdot|)\mathcal{F}u) =: \sum_j \Delta_j u$;
  - $\Delta_j u$ is projection of u on Fourier modes of order 2^j;
  - $\Delta_j u$ has Fourier transform of compact support; thus $\Delta_j u \in C^\infty$.

Bony's paraproduct

u, v distributions, then formally:

$$uv = \lim_{n\to\infty} \Big(\sum_{j\le n} \Delta_j u\Big)\Big(\sum_{k\le n} \Delta_k v\Big) = \sum_{j,k} \Delta_j u \Delta_k v$$

Bony (1981): decompose into components with different behavior.

$$uv = \pi_<(u,v) + \pi_>(u,v) + \pi_\circ(u,v),$$

where

$$\pi_<(u,v) = \sum_{j<k-1} \Delta_j u \Delta_k v, \qquad \pi_>(u,v) = \sum_{k<j-1} \Delta_j u \Delta_k v = \pi_<(v,u)$$

$$\pi_\circ(u,v) = \sum_{|j-k|\le 1} \Delta_j u \Delta_k v.$$

Paraproduct II

Theorem (Bony (1981))

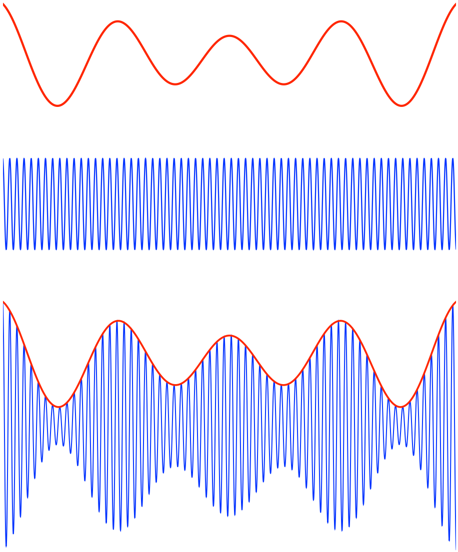
Let $\alpha, \beta \in \mathbb{R}$, $u \in C^\alpha$, $v \in C^\beta$

- $\pi_<(u, v)$ *always well-defined and in* $C^{(\alpha+\beta)\wedge\beta}$;
- $\pi_>(u, v)$ *always well-defined and in* $C^{(\alpha+\beta)\wedge\alpha}$;
- $\pi_\circ(u, v)$ *only defined if* $\alpha + \beta > 0$; *then in* $C^{\alpha+\beta}$.

Interpretation:

- resonance effect for $\pi_\circ(u, v)$;
- $\pi_<(u, v)$ and $\pi_>(u, v)$ are frequency modulations of v and u, respectively.

Paraproduct as frequency modulation



Paraproduct and controlled distributions

- Gubinelli (2004): For $\alpha \in (0,1)$, $g \in C^\alpha$, f is called controlled by g if

$$f(t) - f(s) = f'(s)(g(t) - g(s)) + f^\sharp(s,t), \qquad |f^\sharp(s,t)| \lesssim |t-s|^{2\alpha}.$$

  Easy to see: $f - \pi_<(f', g) \in C^{2\alpha}$.

- Hairer (2013): For $\gamma > 0$, $f : \mathbb{R}^d \to T$ is called modelled, $f \in \mathcal{D}^\gamma$, if

$$|f_x - \Gamma_{x,y} f_y|_\beta \lesssim |x-y|^{\gamma-\beta}.$$

  Easy to see: If $\mathcal{R}$ denotes reconstruction operator, then $\mathcal{R} f - \pi_<(f, \Pi) \in C^\gamma$, where

$$\pi_<(f,\Pi)(x) = \sum_{j<k-1} \int K_j(x-z) K_k(x-y) \Pi_z f_z(y) \mathrm{d}y \mathrm{d}z$$
$$= \sum_{j<k-1} \int K_{j,x}(z) \Pi_z f_z(K_{k,x}) \mathrm{d}z.$$

Product of paracontrolled distributions

Thus: call $f \in C^{\beta}$ paracontrolled by $g \in C^{\beta}$ if there exists $f' \in C^{\alpha}$ such that $f - \pi_{<}(f', g) \in C^{\alpha+\beta}$.

Lemma (Gubinelli, Imkeller, P. (2012))

If $\alpha + \beta + \gamma > 0$, and $\beta + \gamma < 0$, then

$$\|\pi_{\circ}(\pi_{<}(f, g), h) - f\pi_{\circ}(g, h)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_{\alpha}\|g\|_{\beta}\|h\|_{\gamma}.$$

Corollary

If $\alpha + \beta + \gamma > 0$, $h \in C^{\gamma}$, f is paracontrolled by g, and $\pi_{\circ}(g, h) \in C^{\gamma+\beta}$ is given, then fh can be constructed continuously. Moreover, fh is paracontrolled by h.

Burgers equation

- $$\mathcal{L}u(t,x) = \partial_x u^2(t,x) + \partial_x \xi(t,x), \qquad u(0) = u_0,$$
  where $u\colon [0,T]\times\mathbb{T}\to\mathbb{R}$; $\mathcal{L} = \partial_t - \Delta$ heat operator; $\xi(t,x)$ space-time white noise.
- Expect $u \in C([0,T], C^{-1/2-})$, so $u(t)^2$ not defined. But: expand
  $$u = X + X^{\vee} + 2X^{\vee\!\!\vee} + u^Q$$
  where $\mathcal{L}X = \partial_x\xi$, $\mathcal{L}X^{\vee} = \partial_x(XX)$, $\mathcal{L}X^{\vee\!\!\vee} = \partial_x(X^{\vee}X)$, $\mathcal{L}X^{(\tau_1,\tau_2)} = \partial_x(X^{\tau_1}X^{\tau_2})$. (Can take $X^\tau = \partial_x Y^\tau$, where Y^τ are tree data of Hairer's KPZ solution).
- Paracontrolled ansatz for u^Q:
  $$u^Q = \pi_<(u', Q) + u^\sharp,$$
  where $u' \in C([0,T], C^{1/2-})$, $\mathcal{L}Q = \partial_x X$, and $u^\sharp \in C([0,T], C^{1-})$.

Burgers equation and paracontrolled distributions

$$\mathcal{L}u(t,x) = \partial_x u^2(t,x) + \partial_x \xi(t,x), \qquad u(0) = u_0.$$

Paracontrolled ansatz: $u \in \mathcal{P}_{\mathrm{rbe}}$ if $u = X + X^{\vee} + 2X^{\Yright} + u^Q$ with

$$u^Q = \pi_<(u', Q) + u^\sharp.$$

- Only problematic term in u^2: $u^Q X$;
- paracontrolled structure: Can define u^2 continuously as long as $\pi_\circ(Q, X) \in C([0,T], C^{0-})$ is given (together with tree data $X, X^{\vee}, X^{\Yright}, X^{\Yright\!\Yright}, X^{\vee\vee}$).

Paracontrolled differential equation

$$\mathcal{L}u(t,x) = \partial_x u^2(t,x) + \partial_x \xi(t,x), \qquad u(0) = u_0.$$

Paracontrolled ansatz: $u \in \mathcal{P}_{\mathrm{rbe}}$ if $u = X + X^{\vee} + 2X^{\curlyvee} + u^Q$ with

$$u^Q = \pi_<(u', Q) + u^\sharp.$$

Can define u^2 continuously.

- Derive classical PDE for $u^\sharp$:

$$\mathcal{L}u^\sharp = \pi_\circ(u^\sharp, X) + F(u^Q, u', X, X^{\vee}, X^{\curlyvee}, X^{\Yup}, X^{\vee\vee}, Q, \pi_\circ(Q,X))$$

  for some concrete continuous function F;
- see that we should take $u' = u^Q + 4X^{\curlyvee}$ to have sufficiently regular RHS;
- get bound on $u^\sharp$; feed this back into $u^Q = \pi_<(u', Q) + u^\sharp$ to obtain bound on $u' = u^Q + 4X^{\curlyvee}$.
- Obtain local existence and uniqueness of paracontrolled solutions. Solution depends pathwise continuously on extended data $(u_0, \xi, X, X^{\vee}, X^{\curlyvee}, X^{\Yup}, X^{\vee\vee}, \pi_\circ(Q,X))$.

KPZ equation

- KPZ equation:

$$\mathcal{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x), \qquad h(0) = h_0.$$

  Expect $h(t) \in C^{1/2-}$, so $\partial_x h(t) \in C^{-1/2-}$ and $(\partial_x h(t))^2$ not defined.
- But: expand

$$u = Y + Y^{\vee} + 2Y^{\vee\!\!\vee} + h^P,$$

  where $\mathcal{L}Y = \xi$, $\mathcal{L}Y^{\vee} = \partial_x Y \partial_x Y$, ... In general: $\partial_x Y^\tau = X^\tau$.
- Make paracontrolled ansatz for h^P:

$$h^P = \pi_<(h', P) + h^\sharp$$

  with $h' \in C([0,T], C^{1/2-})$, $h^\sharp \in C([0,T], C^{2-})$, $\mathcal{L}P = X$. Write $h \in \mathcal{P}_{\mathrm{kpz}}$.
- Can define $(\partial_x h(t))^2$ for $h \in \mathcal{P}_{\mathrm{kpz}}$ and obtain local existence and uniqueness of solutions.

KPZ and Burgers equation

$h \in \mathcal{P}_{\mathrm{kpz}}$ if

$$h = Y + Y^{\vee} + 2Y^{\vee\!\vee} + h^P, \qquad h^P = \pi_<(h', P) + h^\sharp.$$

$u \in \mathcal{P}_{\mathrm{rbe}}$ if

$$u = X + X^{\vee} + 2X^{\vee\!\vee} + u^Q, \qquad u^Q = \pi_<(u', Q) + u^\sharp.$$

- If $h \in \mathcal{P}_{\mathrm{kpz}}$, then $\partial_x h \in \mathcal{P}_{\mathrm{rbe}}$.
- If h solves KPZ equation, then $u = \partial_x h$ solves Burgers equation with initial condition $u(0) = \partial_x h_0$.
- If $u \in \mathcal{P}_{\mathrm{rbe}}$, then any solution h of $\mathcal{L}h = u^2 + \xi$ is in $\mathcal{P}_{\mathrm{kpz}}$.
- If u solves Burgers equation with initial condition $u(0) = \partial_x h_0$, and h solves $\mathcal{L}h = u^2 + \xi$ with initial condition $h(0) = h_0$, then h solves KPZ equation.

KPZ and heat equation

Heat equation:

$$\mathcal{L}w(t,x) = -w(t,x)\xi(t,x), \qquad w(0) = w_0.$$

Paracontrolled ansatz: $w \in \mathcal{P}_{\mathrm{rhe}}$ if

$$w = e^{-Y - Y^{\vee} - 2Y^{\curlyvee}} w^P, \qquad w^P = \pi_<(w', P) + w^\sharp$$

(comes from Cole-Hopf transform).

- Slightly cheat to make sense of product $w\xi$ for $w \in \mathcal{P}_{\mathrm{rhe}}$:

$$w\xi = -\mathcal{L}w + e^{-Y - Y^{\vee} - 2Y^{\curlyvee}} \left[\mathcal{L}w^P - \mathcal{L}(Y^{\vee} + Y^{\curlyvee})w^P + (\partial_x(Y + Y^{\vee} + 2Y^{\curlyvee}))^2\right]$$
$$- 2e^{-Y - Y^{\vee} - 2Y^{\curlyvee}} \partial_x(Y + Y^{\vee} + 2Y^{\curlyvee})\partial_x w^P;$$

  (agrees with pointwise product $w\xi$ in the smooth case, continuous in the extended data).
- Obtain global existence and uniqueness of solutions.
- One-to-one correspondence between $\mathcal{P}_{\mathrm{kpz}}$ and strictly positive elements of $\mathcal{P}_{\mathrm{rhe}}$.
- Any solution of KPZ gives solution of heat equation. Any strictly positive solution of heat equation gives solution of KPZ equation.

Conclusion

- Products of distributions in general not defined;
- Itô/Stratonovich/rough path integral: work well for functions of a one-dimensional index.
- Paracontrolled distributions: work for general index sets. Pathwise theory. Allow us to solve Burgers, KPZ, and stochastic heat equation.

Crucial ingredients for paracontrolled distributions:

- identification of different components ("paraproduct");
- existence of $\pi_\circ(f, g)$ for reference distributions f, g; $\pi_\circ(f, g)$ must be constructed using probabilistic arguments.

Thank You