Uniqueness and non-uniqueness for stochastic heat equations with Hölder continuous coefficients

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Stoshastic heat equation

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sigma(X(t,x))\dot{W}(x,t),$$

$$\begin{aligned} X(t,x) &= \int p_t(x-y)X(0,y)dy \\ &\quad + \int_0^t \int p_{t-s}(x-y)\sigma(X(s,y))\, W(dy,ds). \end{aligned}$$

where $\dot{W}$ is the Gaussian noise

$$E\left[\dot{W}(x,t)\dot{W}(y,s)\right] = \delta(t-s)k(x-y).$$

Main interest in the case of $\dot{W}$ “white” noise in time and space in $d=1$:

$$k(z) = \delta(z)\,.$$

Uniqueness

We deal with the equation

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sigma(X(t,x))\dot{W}(x,t).$$

- **Pathwise uniqueness (PU):**
  X^1, X^2 — two solutions, $X^1(0,\cdot) = X^2(0,\cdot)$
  $\Longrightarrow X^1(t,\cdot) = X^2(t,\cdot)\,, \forall t > 0.$

- **Uniqueness in law (weak):**
  X^1, X^2 — two solutions (even on different spaces),
  $X^1(0,\cdot) = X^2(0,\cdot) \Longrightarrow \{X^1(t,\cdot)\}_{t\geq 0} \overset{law}{=} \{X^2(t,\cdot)\}_{t\geq 0}\,.$

Uniqueness

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sigma(X(t,x))\dot{W}(x,t).$$

If $\dot{W}$ is a space-time white noise, then function-valued solution exists if $d = 1$.
Uniqueness?
σ — Lipschitz $\Longrightarrow$ PU follows easily.
σ - non-Lipschitz ?

Super-Brownian motion

Branching Brownian motions in $\mathbf{R}^d$.
X^n:
$\sim n$ particles in $\mathbf{R}^d$ at time 0.
$\frac{1}{n}, \frac{2}{n}, \ldots$ — times of death or split,
$p_0 = p_2 = \frac{1}{2}$ — probabilities of death or split.
Critical branching: mean number of offspring $= 1$.
New particles move as independent Brownian motions.

$$X^n_t(A) = \frac{\#\text{ particles in } A \text{ at time } t}{n}, \quad A \subset \mathbf{R}^d.$$
$$X^n \Rightarrow X,$$

X is a super-Brownian motion — measure-valued process.

Properties of SBM

- Singular measure if $d > 1$.

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- Existence of density only in $d = 1$:
  $X_t(dx) = X_t(x)dx$
- $\mathbf{d = 1}$. $X_t(x)$ is jointly continuous in (t, x). N. Konno, T. Shiga(88); M. Reimers (89):

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

$\dot{W}$ — Gaussian space-time white noise.

Uniqueness for SBM

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

Weak uniqueness holds (by duality method)

Pathwise uniqueness (PU)?
$\sqrt{X}$ — non-Lipschitz.
Is there a chance to get **PU**?

Pathwise uniqueness for SDEs

$$dX_t = \sigma(X_t)dB_t$$

B_t is a one-dimensional Brownian motion.

Theorem (Yamada, Watanabe (71))

If σ is Hölder continuous with exponent $1/2$, then PU holds.

Remark

There are counter examples for σ which is Hölder continuous with exponent less than $1/2$.

Proof of Yamada-Watanabe Theorem

Define (in a special way) function $\phi_n \in C_c^\infty(R)$ s.t.

$$\phi_n(x) \;\to\; |x|, \;\; \phi_n'' \to \delta_0\,, \;\; \text{as } n \to \infty.$$

Define $\tilde{X} = X^1 - X^2$. Then $\tilde{X}_0 = 0$ and

$$d\tilde{X}_t \;=\; (\sigma(X_t^1) - \sigma(X_t^1))dB_t\,.$$

Ito's formula:

$$\phi_n(\tilde{X}_t) \;=\; \int_0^t \phi_n'(\tilde{X}_s)(\sigma(X_s^1) - \sigma(X_s^2))\,dB_s + \frac{1}{2}\int_0^t \phi_n''(\tilde{X}_s)(\sigma(X_s^1) - \sigma(X_s^2))^2\,ds$$

By the choice of ϕ_n and Hölder assumptions on σ one can show

$$E\left[\phi_n(\tilde{X}_t)\right] \;\leq\; cE\left[\int_0^t \phi_n''(\tilde{X}_s)|\tilde{X}_s|\,ds\right] \;\to\; 0, \;\; \text{as } n \to \infty.$$

Back to SPDEs

- **SPDE for super-Brownian motion density in $d = 1$**

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sqrt{X(t,x)}\dot{W}(x,t).$$

$\dot{W}$ — space-time white noise.
Numerous attempts to prove **PU** failed.
PU question is still open.

- **General stochastic heat equation**
Let $\sigma(x)$ be Hölder continuous with exponent γ.
Our main interest: conditions on γ such that **PU** holds for

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sigma(X)\dot{W},$$

where $\dot{W}$ is space-time white noise.

Main result

Theorem 1 (Perkins, M., 09)

Let $\sigma(x)$ *be Hölder continuous with exponent* γ.
For any $\gamma > 3/4$, **PU** *holds for*

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sigma(X)\dot{W},$$

where $\dot{W}$ *is space-time white noise.*

SPDE driven by colored noise

We start with the equations that are close to the above.
Take less singular (spatially) noise.
Consider the problem of **PU** for

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sigma(X)\dot{W},$$

where the noise $\dot{W}$ is "white" in time and "colored" in space:

$$E\left[\dot{W}(x,t)\dot{W}(y,s)\right] = \delta(t-s)k(x-y).$$

Assumptions
(H(α)) $k(z) \leq |z|^{-\alpha}$, $0 \leq \alpha < d$.
(H(γ)) $\sigma(x)$ is Hölder cont. with exponent γ.

Existence of function-valued solution:
$0 \leq \alpha < 2 \wedge d$, Peszat-Zabczyk(00), Dalang(99) (for Lipschitz case. Similar for non-Lipschitz).

Uniqueness for SPDE driven by colored noise

Theorem 2 (Sturm, Perkins, M., 05)

PU holds if

$$\gamma > \frac{1}{2} + \frac{\alpha}{2}.$$

Remark

For $d = 1$, $\alpha = 1$ *(white noise case) we have*

$$\gamma > 1 \ldots$$

Proof of Theorem 2

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sigma(X)\dot{W},$$

X^1, X^2 — two solutions, $\tilde{X} = X^1 - X^2$.

$$\frac{\partial \tilde{X}_t(x)}{\partial t} = \frac{1}{2}\Delta \tilde{X}_t(x) + (\sigma(X_t^1(x)) - \sigma(X_t^2(x)))\dot{W}(t,x).$$

Choose the functions ϕ_n, f^n:

$$\begin{aligned} \phi_n(x) &\to |x|, \quad \text{as } n \to \infty, \\ f_x^n &\to \delta_x, \quad \text{as } n \to \infty. \end{aligned}$$

$$\begin{array}{ccc} E\left[\phi_n(\tilde{X}_t(f_x^n))\right] & = & \dots \\ \downarrow & & \downarrow \ \text{(wish)} \\ E\left[|\tilde{X}_t(x)|\right] & = & 0. \end{array}$$

One needs

$$
\begin{aligned}
I^n(t,x) &= E\left[\frac{1}{2}\int_0^t\int_{R^{2d}} \phi_n''(|\tilde{X}_s(f_x^n)|)|\tilde{X}_s(z)|^\gamma|\tilde{X}_s(y)|^\gamma\right. \\
&\qquad\qquad \times f_x^n(z)f_x^n(y)k(z-y)dz\,dy\,ds] \\
&\to 0.
\end{aligned}
$$

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Crucial: Hölder exponent of $\tilde{X}_s(x)$ in x.
Suppose $\tilde{X}_s(\cdot)$ is ξ-Hölder continuous. Then we can show PU if

$$\gamma > \frac{1}{2} + \frac{\alpha}{2\xi},$$

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Remark

If $\alpha = 0$ we get YW bound $\gamma > 1/2$. $\frac{\alpha}{2\xi}$ — "price" for spatial singularity of the noise.

We got condition for PU:

$$\gamma > \frac{1}{2} + \frac{\alpha}{2\xi}.$$

Proposition (Sanz-Solé, Sarrà)

For any $\xi < 1 - \frac{\alpha}{2}$, $\tilde{X}_s(\cdot)$ is Hölder continuous with exponent ξ.

By Theorem of Sanz-Solé, Sarrà we get

$$\gamma > \frac{1}{2} + \frac{\alpha}{2(1 - \alpha/2)},$$

Bad: $\alpha = 2/3 \Longrightarrow \gamma > 1.$

Proposition (Sturm, Perkins, M.)

At the points x where $\tilde{X}_s(x) = 0$, $\tilde{X}_s(\cdot)$ is ξ-Hölder continuous

$$\forall \xi < \frac{1-\alpha/2}{1-\gamma} \wedge 1.$$

Corollary (Sturm, Perkins, M.)

Let $\gamma > \frac{1}{2} + \frac{\alpha}{2}$. At the points x where $\tilde{X}_s(x) = 0$, $\tilde{X}_s(\cdot)$ is ξ-Hölder continuous

$$\forall \xi < 1.$$

Remark Mueller-Tribe have the result similar to the above regularity result.

By condition on PU ($\gamma > \frac{1}{2} + \frac{\alpha}{2\xi}$) we get

$$\gamma > \frac{1}{2} + \frac{\alpha}{2}.$$

and this finishes the proof of Theorem 2.

Optimality of the bound

$$\gamma > \frac{1}{2} + \frac{\alpha}{2} \quad ?$$

Note: $d = 1$, $\alpha = 1$ (**white noise** case) gives

$$\gamma > 1 \ldots$$

Theorem 1

If one replaces the condition

$$\forall \xi < \frac{1-\alpha/2}{1-\gamma} \wedge 1$$

by

$$\forall \xi < \frac{1-\alpha/2}{1-\gamma}.$$

(allowing ξ to be > 1)
then by general condition on **PU**: $\gamma > \frac{1}{2} + \frac{\alpha}{2\xi}$,
one gets the following condition on **PU**:

$$\gamma > \frac{1}{2} + \frac{\alpha}{4},$$

That is for $\alpha = 1$ (**white noise** case in $d = 1$) we get

$$\gamma > 3/4$$

— **Theorem 1**!!

Remark

T. Rippl and A. Sturm (2013) improved the condition on **PU** *(from MPS06) for the colored noise case:*

$$\gamma > \frac{1}{2} + \frac{\alpha}{4}.$$

Regularity at zero points

Consider

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + |X|^{\gamma}\dot{W},$$

with $\dot{W}$ — white noise. Take $x_0 : X_t(x_0) = 0$. Regularity at x_0? (Similar to behavior of the difference of two solutions $\tilde{X}$ we considered).

$$I_t(x) \equiv \int_0^t \int_R p_{t-s}(x-y)|X_s(y)|^{\gamma} W(ds, dy).$$

Regularity If X is $C^{p/2,p}$ at (t, x_0), then $I_\cdot(\cdot)$ is $C^{p'/2,p'}$ at (t, x_0):

with

$$p' = p\gamma + 1/2$$

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Regularity If X is $C^{p/2,p}$ at (t, x_0), then $I_\cdot(\cdot)$ is $C^{p'/2,p'}$ at (t, x_0):

with

$$p' = p\gamma + 1/2$$

and $X_t(\cdot)$ is $C^{p'/2,p'}$ at (t, x_0).
Iterate... $X_t(\cdot)$ is $C^{\xi/2,\xi}$ at (t, x_0) with $\xi = \frac{1}{2(1-\gamma)}$.

Adding Drift

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + b(X) + \sigma(X)\dot{W},$$

where $\dot{W}$ is space-time white noise.
For $\sigma(\cdot) > \epsilon > 0$, Lip; b — bounded measurable, **PU** $+ \exists$ of strong solutions studied by Gyongy,Pardoux, Bally, and others...

If σ Hölder continuous with exponent $\gamma > 3/4$, b — Lipschitz, **PU** proved by Perkins, M.

Non-Lipschitz b?

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Non-Lipschitz b?

Theorem (Neuman, M., 14)

Let σ be Hölder continuous with exponent $\gamma > 3/4$, $\sigma(\cdot) \geq \epsilon > 0$. Let b be bounded continuous. Then **PU** *holds.*

Non-uniqueness

- Is $3/4$ sharp?
  Counter example: for $\gamma < 3/4$ try to construct non-triviual solution to

$$\begin{cases} \frac{\partial}{\partial t} X(t,x) &= \frac{1}{2}\Delta X(t,x) + |X(t,x)|^{\gamma}\dot{W}(x,t), \\ X(0,\cdot) &= 0. \end{cases} \tag{1}$$

Non-uniqueness

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  Counter example: for $\gamma < 3/4$ try to construct non-trivial solution to

$$\begin{cases} \frac{\partial}{\partial t} X(t,x) &= \frac{1}{2}\Delta X(t,x) + |X(t,x)|^{\gamma} \dot{W}(x,t), \\ X(0,\cdot) &= 0. \end{cases} \quad (1)$$

- Theorem 3 (Burdzy, Mueller, Perkins(2010); M., Mueller, Perkins(2012))

If $0 < \gamma < 3/4$ *there is solution* $X(t,x)$ *to (1) such that with positive probability,* $X(t,x)$ *is not identically zero.*

Conclusion: Both uniqueness in law and pathwise uniqueness fail for (1).

Non-uniqueness

Proofs

Construct non-triviual solution to

$$\left\{ \begin{array}{rcl} \frac{\partial}{\partial t} X(t,x) & = & \frac{1}{2}\Delta X(t,x) + |X(t,x)|^{\gamma} \dot{W}(x,t), \\ X(0,\cdot) & = & 0. \end{array} \right. \tag{1}$$

Hard killing model (X^1, X^2) :

$$\left\{ \begin{array}{rcl} \frac{\partial}{\partial t} X^i(t,x) & = & \frac{1}{2}\Delta X^i(t,x) + X^i(t,x)^{\gamma} \dot{W}^i(x,t) \\ & & -\frac{\partial A}{\partial t} + \frac{\partial I^i}{\partial t}, \quad i = 1,2, \\ X^i \geq 0, \;\; X^1 X^2 = 0. & & \end{array} \right.$$

$I^i(dt,dx)$ is immigration of mass,
A is the killing term,
W^1, W^2 are independent.
If $I^1 = I^2$, then $X \equiv X^1 - X^2$ solves (1).

Aim: cosntruct non-trivial (X^1, X^2).
Approximation:

$$\begin{cases} \frac{\partial}{\partial t} X^{i,\varepsilon}(t,x) = \frac{1}{2}\Delta X^{i,\varepsilon}(t,x) + X^{i,\varepsilon}(t,x)^{\gamma}\dot{W}^i(x,t) \\ \qquad\qquad - \frac{\partial A^{\varepsilon}}{\partial t} + \frac{\partial I^{i,\varepsilon}}{\partial t}, \quad i = 1,2, \\ X^{i,\varepsilon} \geq 0, \;\; X^{1,\varepsilon}X^{2,\varepsilon} = 0. \end{cases}$$

We construct $I^{1,\varepsilon} \neq I^{2,\varepsilon}$ in a special way, such that

$$I^{i,\varepsilon} \Rightarrow I, \quad i = 1,2, \;\; \text{as } \varepsilon \downarrow 0.$$

Hence

$$\begin{aligned} X^{\varepsilon} &= X^{1,\varepsilon} - X^{2,\varepsilon} \\ &\Rightarrow X, \;\; \text{as } \varepsilon \downarrow 0. \end{aligned}$$

where X solves (1).
One can show (the most difficult part!) that X is non-trivial for $\gamma < 3/4$.

Aim: cosntruct non-trivial (X^1, X^2).
Approximation: Let $\eta^{1,\varepsilon}, \eta^{2,\varepsilon}$ be independent Poisson random measures on $R_+ \times [-1,1]$ with intensity $\varepsilon^{-1} dt\, dx$.

$$I^{i,\varepsilon}(t, A) \equiv \varepsilon \eta^{i,\varepsilon}([0,t] \times A), \quad i = 1, 2.$$

$(X^{1,\varepsilon}, X^{2,\varepsilon})$ is the corresponding hard killing process.

Clearly as $\varepsilon \downarrow 0$,

$$I^{i,\varepsilon} \Rightarrow dt\, dx 1(x \in [-1,1]), \quad i = 1, 2.$$

Hence

$$\begin{aligned} X^\varepsilon &= X^{1,\varepsilon} - X^{2,\varepsilon} \\ &\Rightarrow X, \ \text{as } \varepsilon \downarrow 0. \end{aligned}$$

where X solves (1).
We would like to show that X is non-trivial for

$$\gamma < 3/4.$$

Representation

$$X_t^{i,\varepsilon} = \sum_{k:t_k\le t} X_t^{i,\varepsilon,k}$$

where $X_t^{i,\varepsilon,k}$ is the "cluster" starting at the atom $\varepsilon\delta_{x_k,t_k}$ of the immigration measure $I^{i,\varepsilon}$.

Consider $X^{1,\varepsilon}$. In the absence of the killing ($A = 0$),

$$\inf_{\varepsilon} P(\exists \text{ a cluster starting at some } t_k \le 1/2 \text{ and surviving until } t = 1) > 0.$$

Let $Y^{1,\varepsilon}$ be one of such clusters. Shift time and space so that it starts at $\delta_{0,0}$.

It could be checked that for small t the total mass of the surviving cluster evolves as

$$\left\langle Y_1^{1,\varepsilon}, 1 \right\rangle \sim t^{\frac{1}{4(1-\gamma)}+\frac{1}{2}} \gg t^{3/2}, \text{ for } \gamma < 3/4.$$

Most of the mass of $\{Y_s^{1,\varepsilon}, s \leq t\}$ is inside the parabola

$$B_t = \{(s,x) : |x| \leq \sqrt{s}, s \leq t\}$$

How $Y^{1,\varepsilon}$ could be killed by $X^{2,\varepsilon}$ before (small) time t with probability 1?

- By clusters of $X^{2,\varepsilon}$ born before time 0 at $[-t, 0]$ for t small.
- By clusters of $X^{2,\varepsilon}$ born after time 0 inside B_t.

- Killing by clusters of $X^{2,\varepsilon}$ born at $[-t,0]$ for t small.

  For t small, to touch $(0,0)$ any cluster of $X^{2,\varepsilon}$ born at $[-t,0]$ should be born inside

  $$\tilde{B}_t = \{(s,x) : |x| \leq \sqrt{|s|}, s \geq -t\}$$

  One can show that by "branching processes" argument that to survive t units of time the immigration of mass to $X^{2,\varepsilon}$ inside $\tilde{B}_t$ should be at least of order $t^{\frac{1}{4(1-\gamma)}+\frac{1}{2}}$.
  However the immigration is just of order

  $$t^{3/2} \ll t^{\frac{1}{4(1-\gamma)}+\frac{1}{2}}, \;\; \gamma < 3/4,$$

  and hence the mass of $X^{2,\varepsilon}$ that was born inside $\tilde{B}_t$ dies out by time 0.

- Killing by clusters of $X^{2,\varepsilon}$ born after time 0 inside B_t.

  The immigration of mass in $X^{2,\varepsilon}$ inside B_t is of order $t^{3/2}$.
  The mass of $Y^{1,\varepsilon}$ at time t is of order

  $$t^{\frac{1}{4(1-\gamma)}+\frac{1}{2}} \gg t^{3/2}$$

  and hence $Y^{1,\varepsilon}$ "wins" the competition.

Open Problems

- **Uniqueness/Non-uniqueness of non-negative solutions to**

$$\begin{cases} \frac{\partial}{\partial t}X(t,x) & = & \frac{1}{2}\Delta X(t,x) + \psi + X(t,x)^{\gamma}\dot{W}(x,t), \\ X(0,\cdot) & \geq & 0. \end{cases}$$

Known:

1. $\gamma < 1/2$. Non-uniqueness in presence of immigration $\psi \geq 0$:
   Burdzy, Mueller, Perkins(2010).
2. $\gamma = 1/2$. Non-uniqueness in presence of immigration $\psi \geq 0$:
   Yu-Ting Chen (2013).

Open: (a) $\gamma \in (1/2, 3/4)$?
(b) The case of "no immigration": $\psi = 0$?

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- **SPDEs driven by Levy noise.**

Thank You