

Uniqueness and non-uniqueness for stochastic heat equations with Hölder continuous coefficients

Leonid Mytnik

Rough Paths: Theory and Applications
January 27 - 31, 2014, IPAM

January 28, 2014

Stochastic heat equation

$$\frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + \sigma(X(t, x)) \dot{W}(x, t),$$

$$\begin{aligned} X(t, x) &= \int p_t(x - y) X(0, y) dy \\ &+ \int_0^t \int p_{t-s}(x - y) \sigma(X(s, y)) W(dy, ds). \end{aligned}$$

where \dot{W} is the Gaussian noise

$$E \left[\dot{W}(x, t) \dot{W}(y, s) \right] = \delta(t - s) k(x - y).$$

Main interest in the case of \dot{W} “white” noise in time and space in $d = 1$:

$$k(z) = \delta(z).$$

Uniqueness

We deal with the equation

$$\frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + \sigma(X(t, x)) \dot{W}(x, t).$$

► **Pathwise uniqueness (PU):**

$$X^1, X^2 \text{ — two solutions, } X^1(0, \cdot) = X^2(0, \cdot) \\ \implies X^1(t, \cdot) = X^2(t, \cdot), \forall t > 0.$$

► **Uniqueness in law (weak):**

$$X^1, X^2 \text{ — two solutions (even on different spaces),} \\ X^1(0, \cdot) = X^2(0, \cdot) \implies \{X^1(t, \cdot)\}_{t \geq 0} \stackrel{\text{law}}{=} \{X^2(t, \cdot)\}_{t \geq 0}.$$

Uniqueness

$$\frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + \sigma(X(t, x)) \dot{W}(x, t).$$

If \dot{W} is a space-time white noise, then function-valued solution exists if $d = 1$.

Uniqueness?

σ — Lipschitz \implies PU follows easily.

σ - non-Lipschitz ?

Super-Brownian motion

Branching Brownian motions in \mathbf{R}^d .

X^n :

$\sim n$ particles in \mathbf{R}^d at time 0.

$\frac{1}{n}, \frac{2}{n}, \dots$ — times of death or split,

$p_0 = p_2 = \frac{1}{2}$ — probabilities of death or split.

Critical branching: mean number of offspring = 1.

New particles move as independent Brownian motions.

$$X_t^n(A) = \frac{\# \text{ particles in } A \text{ at time } t}{n}, \quad A \subset \mathbf{R}^d.$$

$$X^n \Rightarrow X,$$

X is a super-Brownian motion — measure-valued process.

Properties of SBM

- ▶ Singular measure if $d > 1$.

Properties of SBM

- ▶ Singular measure if $d > 1$.
- ▶ Existence of density only in $d = 1$:
 $X_t(dx) = X_t(x)dx$

Properties of SBM

- ▶ Singular measure if $d > 1$.
- ▶ Existence of density only in $d = 1$:
 $X_t(dx) = X_t(x)dx$
- ▶ $d = 1$. $X_t(x)$ is jointly continuous in (t, x) . N. Konno, T. Shiga(88); M. Reimers (89):

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

\dot{W} — Gaussian space-time white noise.

Uniqueness for SBM

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sqrt{X} \dot{W}.$$

Weak uniqueness holds (by duality method)

Pathwise uniqueness (PU)?

\sqrt{X} — non-Lipschitz.

Is there a chance to get **PU**?

Pathwise uniqueness for SDEs

$$dX_t = \sigma(X_t)dB_t$$

B_t is a one-dimensional Brownian motion.

Theorem (Yamada, Watanabe (71))

If σ is Hölder continuous with exponent $1/2$, then PU holds.

Remark

There are counter examples for σ which is Hölder continuous with exponent less than $1/2$.

Proof of Yamada-Watanabe Theorem

Define (in a special way) function $\phi_n \in C_c^\infty(\mathbb{R})$ s.t.

$$\phi_n(x) \rightarrow |x|, \quad \phi_n'' \rightarrow \delta_0, \quad \text{as } n \rightarrow \infty.$$

Define $\tilde{X} = X^1 - X^2$. Then $\tilde{X}_0 = 0$ and

$$d\tilde{X}_t = (\sigma(X_t^1) - \sigma(X_t^2))dB_t.$$

Ito's formula:

$$\begin{aligned} \phi_n(\tilde{X}_t) &= \int_0^t \phi_n'(\tilde{X}_s)(\sigma(X_s^1) - \sigma(X_s^2)) dB_s \\ &\quad + \frac{1}{2} \int_0^t \phi_n''(\tilde{X}_s)(\sigma(X_s^1) - \sigma(X_s^2))^2 ds \end{aligned}$$

By the choice of ϕ_n and Hölder assumptions on σ one can show

$$\begin{aligned} E \left[\phi_n(\tilde{X}_t) \right] &\leq cE \left[\int_0^t \phi_n''(\tilde{X}_s) |\tilde{X}_s| ds \right] \\ &\rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Back to SPDEs

- ▶ **SPDE for super-Brownian motion density in $d = 1$**

$$\frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + \sqrt{X(t, x)} \dot{W}(x, t).$$

\dot{W} — space-time white noise.

Numerous attempts to prove **PU** failed.

PU question is still open.

- ▶ **General stochastic heat equation**

Let $\sigma(x)$ be Hölder continuous with exponent γ .

Our main interest: conditions on γ such that **PU** holds for

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sigma(X) \dot{W},$$

where \dot{W} is space-time white noise.

Main result

Theorem 1 (Perkins, M., 09)

Let $\sigma(x)$ be Hölder continuous with exponent γ .
For any $\gamma > 3/4$, **PU** holds for

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sigma(X) \dot{W},$$

where \dot{W} is space-time white noise.

SPDE driven by colored noise

We start with the equations that are close to the above.

Take less singular (spatially) noise.

Consider the problem of **PU** for

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sigma(X) \dot{W},$$

where the noise \dot{W} is “white” in time and “colored” in space:

$$E \left[\dot{W}(x, t) \dot{W}(y, s) \right] = \delta(t - s) k(x - y).$$

Assumptions

(H(α)) $k(z) \leq |z|^{-\alpha}$, $0 \leq \alpha < d$.

(H(γ)) $\sigma(x)$ is Hölder cont. with exponent γ .

Existence of function-valued solution:

$0 \leq \alpha < 2 \wedge d$, Peszat-Zabczyk(00), Dalang(99) (for Lipschitz case. Similar for non-Lipschitz).

Uniqueness for SPDE driven by colored noise

Theorem 2 (Sturm, Perkins, M., 05)

PU holds if

$$\gamma > \frac{1}{2} + \frac{\alpha}{2}.$$

Remark

For $d = 1$, $\alpha = 1$ (white noise case) we have

$$\gamma > 1 \dots$$

Proof of Theorem 2

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sigma(X) \dot{W},$$

X^1, X^2 — two solutions, $\tilde{X} = X^1 - X^2$.

$$\frac{\partial \tilde{X}_t(x)}{\partial t} = \frac{1}{2} \Delta \tilde{X}_t(x) + (\sigma(X_t^1(x)) - \sigma(X_t^2(x))) \dot{W}(t, x).$$

Choose the functions ϕ_n, f^n :

$$\begin{aligned} \phi_n(x) &\rightarrow |x|, \quad \text{as } n \rightarrow \infty, \\ f_x^n &\rightarrow \delta_x, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

$$\begin{aligned} E \left[\phi_n(\tilde{X}_t(f_x^n)) \right] &= \dots \\ &\quad \downarrow \qquad \qquad \downarrow \text{ (wish)} \\ E \left[|\tilde{X}_t(x)| \right] &= 0. \end{aligned}$$

One needs

$$I^n(t, x) = E \left[\frac{1}{2} \int_0^t \int_{R^{2d}} \phi_n''(|\tilde{X}_s(f_x^n)|) |\tilde{X}_s(z)|^\gamma |\tilde{X}_s(y)|^\gamma \right. \\ \left. \times f_x^n(z) f_x^n(y) k(z - y) dz dy ds \right] \\ \rightarrow 0.$$

One needs

$$I^n(t, x) = E \left[\frac{1}{2} \int_0^t \int_{R^{2d}} \phi_n''(|\tilde{X}_s(f_x^n)|) |\tilde{X}_s(z)|^\gamma |\tilde{X}_s(y)|^\gamma \right. \\ \left. \times f_x^n(z) f_x^n(y) k(z - y) dz dy ds \right] \\ \rightarrow 0.$$

Crucial: Hölder exponent of $\tilde{X}_s(x)$ in x .

Suppose $\tilde{X}_s(\cdot)$ is ξ -Hölder continuous. Then we can show PU if

$$\gamma > \frac{1}{2} + \frac{\alpha}{2\xi},$$

One needs

$$I^n(t, x) = E \left[\frac{1}{2} \int_0^t \int_{R^{2d}} \phi_n''(|\tilde{X}_s(f_x^n)|) |\tilde{X}_s(z)|^\gamma |\tilde{X}_s(y)|^\gamma \right. \\ \left. \times f_x^n(z) f_x^n(y) k(z - y) dz dy ds \right] \\ \rightarrow 0.$$

Crucial: Hölder exponent of $\tilde{X}_s(x)$ in x .

Suppose $\tilde{X}_s(\cdot)$ is ξ -Hölder continuous. Then we can show PU if

$$\gamma > \frac{1}{2} + \frac{\alpha}{2\xi},$$

Remark

If $\alpha = 0$ we get YW bound $\gamma > 1/2$. $\frac{\alpha}{2\xi}$ — "price" for spatial singularity of the noise.

We got condition for PU:

$$\gamma > \frac{1}{2} + \frac{\alpha}{2\xi}.$$

Proposition (Sanz-Solé, Sarrà)

For any $\xi < 1 - \frac{\alpha}{2}$, $\tilde{X}_s(\cdot)$ is Hölder continuous with exponent ξ .

By Theorem of Sanz-Solé, Sarrà we get

$$\gamma > \frac{1}{2} + \frac{\alpha}{2(1 - \alpha/2)},$$

Bad: $\alpha = 2/3 \implies \gamma > 1$.

Proposition (Sturm, Perkins, M.)

At the points x where $\tilde{X}_s(x) = 0$, $\tilde{X}_s(\cdot)$ is ξ -Hölder continuous

$$\forall \xi < \frac{1 - \alpha/2}{1 - \gamma} \wedge 1.$$

Corollary (Sturm, Perkins, M.)

Let $\gamma > \frac{1}{2} + \frac{\alpha}{2}$. At the points x where $\tilde{X}_s(x) = 0$, $\tilde{X}_s(\cdot)$ is ξ -Hölder continuous

$$\forall \xi < 1.$$

Remark Mueller-Tribe have the result similar to the above regularity result.

By condition on PU ($\gamma > \frac{1}{2} + \frac{\alpha}{2\xi}$) we get

$$\gamma > \frac{1}{2} + \frac{\alpha}{2}.$$

and this finishes the proof of Theorem 2.

Optimality of the bound

$$\gamma > \frac{1}{2} + \frac{\alpha}{2} ?$$

Note: $d = 1$, $\alpha = 1$ (**white noise** case) gives

$$\gamma > 1 \dots$$

Theorem 1

If one replaces the condition

$$\forall \xi < \frac{1 - \alpha/2}{1 - \gamma} \wedge 1$$

by

$$\forall \xi < \frac{1 - \alpha/2}{1 - \gamma}.$$

(allowing ξ to be > 1)

then by general condition on **PU**: $\gamma > \frac{1}{2} + \frac{\alpha}{2\xi}$,
one gets the following condition on **PU**:

$$\gamma > \frac{1}{2} + \frac{\alpha}{4},$$

That is for $\alpha = 1$ (**white noise** case in $d = 1$) we get

$$\gamma > 3/4$$

— **Theorem 1!!**

Remark

*T. Rippl and A. Sturm (2013) improved the condition on **PU** (from MPS06) for the colored noise case:*

$$\gamma > \frac{1}{2} + \frac{\alpha}{4}.$$

Regularity at zero points

Consider

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + |X|^\gamma \dot{W},$$

with \dot{W} — white noise. Take $x_0 : X_t(x_0) = 0$. Regularity at x_0 ?
(Similar to behavior of the difference of two solutions \tilde{X} we considered).

$$I_t(x) \equiv \int_0^t \int_R p_{t-s}(x-y) |X_s(y)|^\gamma W(ds, dy).$$

Regularity If X is $C^{p/2, p}$ at (t, x_0) , then $I(\cdot)$ is $C^{p'/2, p'}$ at (t, x_0) :

with

$$p' = p\gamma + 1/2$$

Regularity at zero points

Consider

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + |X|^\gamma \dot{W},$$

with \dot{W} — white noise. Take $x_0 : X_t(x_0) = 0$. Regularity at x_0 ?
(Similar to behavior of the difference of two solutions \tilde{X} we considered).

$$I_t(x) \equiv \int_0^t \int_R p_{t-s}(x-y) |X_s(y)|^\gamma W(ds, dy).$$

Regularity If X is $C^{p/2, p}$ at (t, x_0) , then $I(\cdot)$ is $C^{p'/2, p'}$ at (t, x_0) :

with

$$p' = p\gamma + 1/2$$

and $X_t(\cdot)$ is $C^{p'/2, p'}$ at (t, x_0) .

Iterate. . . $X_t(\cdot)$ is $C^{\xi/2, \xi}$ at (t, x_0) with $\xi = \frac{1}{2(1-\gamma)}$.

Adding Drift

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + b(X) + \sigma(X) \dot{W},$$

where \dot{W} is space-time white noise.

For $\sigma(\cdot) > \epsilon > 0$, Lip; b — bounded measurable, **PU** + \exists of strong solutions studied by Gyongy, Pardoux, Bally, and others. . .

If σ Hölder continuous with exponent $\gamma > 3/4$, b — Lipschitz, **PU** proved by Perkins, M.

Non-Lipschitz b ?

Adding Drift

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + b(X) + \sigma(X) \dot{W},$$

where \dot{W} is space-time white noise.

For $\sigma(\cdot) > \epsilon > 0$, Lip; b — bounded measurable, **PU** + \exists of strong solutions studied by Gyongy, Pardoux, Bally, and others. . .

If σ Hölder continuous with exponent $\gamma > 3/4$, b — Lipschitz, **PU** proved by Perkins, M.

Non-Lipschitz b ?

Theorem (Neuman, M., 14)

Let σ be Hölder continuous with exponent $\gamma > 3/4$, $\sigma(\cdot) \geq \epsilon > 0$.
Let b be bounded continuous. Then **PU** holds.

Non-uniqueness

- ▶ Is 3/4 sharp?

Counter example: for $\gamma < 3/4$ try to construct non-trivial solution to

$$\begin{cases} \frac{\partial}{\partial t} X(t, x) &= \frac{1}{2} \Delta X(t, x) + |X(t, x)|^\gamma \dot{W}(x, t), \\ X(0, \cdot) &= 0. \end{cases} \quad (1)$$

Non-uniqueness

- ▶ Is 3/4 sharp?

Counter example: for $\gamma < 3/4$ try to construct non-trivial solution to

$$\begin{cases} \frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + |X(t, x)|^\gamma \dot{W}(x, t), \\ X(0, \cdot) = 0. \end{cases} \quad (1)$$

- ▶ Theorem 3 (Burdzy, Mueller, Perkins(2010); M., Mueller, Perkins(2012))

If $0 < \gamma < 3/4$ there is solution $X(t, x)$ to (1) such that with positive probability, $X(t, x)$ is not identically zero.

Conclusion: Both uniqueness in law and pathwise uniqueness fail for (1).

Non-uniqueness

Proofs

Construct non-trivial solution to

$$\begin{cases} \frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + |X(t, x)|^\gamma \dot{W}(x, t), \\ X(0, \cdot) = 0. \end{cases} \quad (1)$$

Hard killing model (X^1, X^2) :

$$\begin{cases} \frac{\partial}{\partial t} X^i(t, x) = \frac{1}{2} \Delta X^i(t, x) + X^i(t, x)^\gamma \dot{W}^i(x, t) \\ \quad - \frac{\partial A}{\partial t} + \frac{\partial I^i}{\partial t}, \quad i = 1, 2, \\ X^i \geq 0, \quad X^1 X^2 = 0. \end{cases}$$

$I^i(dt, dx)$ is immigration of mass,

A is the killing term,

W^1, W^2 are independent.

If $I^1 = I^2$, then $X \equiv X^1 - X^2$ solves (1).

Aim: construct non-trivial (X^1, X^2) .

Approximation:

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} X^{i,\varepsilon}(t, x) = \frac{1}{2} \Delta X^{i,\varepsilon}(t, x) + X^{i,\varepsilon}(t, x)^\gamma \dot{W}^i(x, t) \\ \quad - \frac{\partial A^\varepsilon}{\partial t} + \frac{\partial I^{i,\varepsilon}}{\partial t}, \quad i = 1, 2, \\ \\ X^{i,\varepsilon} \geq 0, \quad X^{1,\varepsilon} X^{2,\varepsilon} = 0. \end{array} \right.$$

We construct $I^{1,\varepsilon} \neq I^{2,\varepsilon}$ in a special way, such that

$$I^{i,\varepsilon} \Rightarrow I, \quad i = 1, 2, \quad \text{as } \varepsilon \downarrow 0.$$

Hence

$$\begin{aligned} X^\varepsilon &= X^{1,\varepsilon} - X^{2,\varepsilon} \\ &\Rightarrow X, \quad \text{as } \varepsilon \downarrow 0. \end{aligned}$$

where X solves (1).

One can show (the most difficult part!) that X is non-trivial for

$$\gamma < 3/4.$$

Aim: construct non-trivial (X^1, X^2) .

Approximation: Let $\eta^{1,\varepsilon}, \eta^{2,\varepsilon}$ be independent Poisson random measures on $R_+ \times [-1, 1]$ with intensity $\varepsilon^{-1} dt dx$.

$$I^{i,\varepsilon}(t, A) \equiv \varepsilon \eta^{i,\varepsilon}([0, t] \times A), \quad i = 1, 2.$$

$(X^{1,\varepsilon}, X^{2,\varepsilon})$ is the corresponding hard killing process.

Clearly as $\varepsilon \downarrow 0$,

$$I^{i,\varepsilon} \Rightarrow dt dx 1(x \in [-1, 1]), \quad i = 1, 2.$$

Hence

$$\begin{aligned} X^\varepsilon &= X^{1,\varepsilon} - X^{2,\varepsilon} \\ &\Rightarrow X, \text{ as } \varepsilon \downarrow 0. \end{aligned}$$

where X solves (1).

We would like to show that X is non-trivial for

$$\gamma < 3/4.$$

Representation

$$X_t^{i,\varepsilon} = \sum_{k:t_k \leq t} X_t^{i,\varepsilon,k}$$

where $X_t^{i,\varepsilon,k}$ is the "cluster" starting at the atom $\varepsilon \delta_{x_k, t_k}$ of the immigration measure $I^{i,\varepsilon}$.

Consider $X^{1,\varepsilon}$. In the absence of the killing ($A = 0$),

$$\inf_{\varepsilon} P(\exists \text{ a cluster starting at some } t_k \leq 1/2 \text{ and surviving until } t = 1) > 0.$$

Let $Y^{1,\varepsilon}$ be one of such clusters. Shift time and space so that it starts at $\delta_{0,0}$.

It could be checked that for small t the total mass of the surviving cluster evolves as

$$\langle Y_1^{1,\varepsilon}, 1 \rangle \sim t^{\frac{1}{4(1-\gamma)} + \frac{1}{2}} \gg t^{3/2}, \quad \text{for } \gamma < 3/4.$$

Most of the mass of $\{Y_s^{1,\varepsilon}, s \leq t\}$ is inside the parabola

$$B_t = \{(s, x) : |x| \leq \sqrt{s}, s \leq t\}$$

How $Y^{1,\varepsilon}$ could be killed by $X^{2,\varepsilon}$ before (small) time t with probability 1?

- ▶ By clusters of $X^{2,\varepsilon}$ born before time 0 at $[-t, 0]$ for t small.
- ▶ By clusters of $X^{2,\varepsilon}$ born after time 0 inside B_t .

- ▶ Killing by clusters of $X^{2,\varepsilon}$ born at $[-t, 0]$ for t small.

For t small, to touch $(0, 0)$ any cluster of $X^{2,\varepsilon}$ born at $[-t, 0]$ should be born inside

$$\tilde{B}_t = \{(s, x) : |x| \leq \sqrt{|s|}, s \geq -t\}$$

One can show that by "branching processes" argument that to survive t units of time the immigration of mass to $X^{2,\varepsilon}$ inside \tilde{B}_t should be at least of order $t^{\frac{1}{4(1-\gamma)} + \frac{1}{2}}$. However the immigration is just of order

$$t^{3/2} \ll t^{\frac{1}{4(1-\gamma)} + \frac{1}{2}}, \quad \gamma < 3/4,$$

and hence the mass of $X^{2,\varepsilon}$ that was born inside \tilde{B}_t dies out by time 0.

- ▶ Killing by clusters of $X^{2,\varepsilon}$ born after time 0 inside B_t .

The immigration of mass in $X^{2,\varepsilon}$ inside B_t is of order $t^{3/2}$.
The mass of $Y^{1,\varepsilon}$ at time t is of order

$$t^{\frac{1}{4(1-\gamma)} + \frac{1}{2}} \gg t^{3/2}$$

and hence $Y^{1,\varepsilon}$ "wins" the competition.

Open Problems

► Uniqueness/Non-uniqueness of non-negative solutions to

$$\begin{cases} \frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + \psi + X(t, x)^\gamma \dot{W}(x, t), \\ X(0, \cdot) \geq 0. \end{cases}$$

Known:

1. $\gamma < 1/2$. Non-uniqueness in presence of immigration $\psi \geq 0$:
Burdzy, Mueller, Perkins(2010).
2. $\gamma = 1/2$. Non-uniqueness in presence of immigration $\psi \geq 0$:
Yu-Ting Chen (2013).

Open: (a) $\gamma \in (1/2, 3/4)$?(b) The case of "no immigration": $\psi = 0$?

Open Problems

▶ Uniqueness/Non-uniqueness of non-negative solutions to

$$\begin{cases} \frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + \psi + X(t, x)^\gamma \dot{W}(x, t), \\ X(0, \cdot) \geq 0. \end{cases}$$

Known:

1. $\gamma < 1/2$. Non-uniqueness in presence of immigration $\psi \geq 0$: Burdzy, Mueller, Perkins(2010).
2. $\gamma = 1/2$. Non-uniqueness in presence of immigration $\psi \geq 0$: Yu-Ting Chen (2013).

Open: (a) $\gamma \in (1/2, 3/4)$?(b) The case of "no immigration": $\psi = 0$?

▶ SPDEs driven by Levy noise.

Thank You