Smoothing property for stochastic differential equations driven by fractional Brownian motions

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Based on a joint work with C. Ouyang and X. Zhang

Motivation

The motivation of the talk is to study regularization properties of rough differential equations on $\mathbb{R}^n$

$$X_t^x = x + \sum_{i=1}^{d} \int_0^t V_i(X_s^x) dB_s^i \tag{1}$$

where the V_i's are C^∞-bounded vector fields on $\mathbb{R}^n$ and B is a d-dimensional fractional Brownian motion with parameter $H > \frac{1}{4}$.

Fractional Brownian motion

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Rough differential equations driven by fractional Brownian motions provide toy models for the study of non Markov random dynamical systems.

Young's differential equation driven by fBm

If $H > 1/2$, the equation

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Rough differential equations driven by fBm

Several properties of the probability distribution of the solution X_t have been discussed:

- Existence and smoothness of a density (B.-Hairer, Cass-Friz, Hairer-Pilai, Cass-Hairer-Litterer-Tindel)

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- Concentration properties (Cass-Litterer-Lyons)
- Upper and lower bounds for the density (B.-Ouyang-Tindel, B.-Nualart-Ouyang-Tindel, Besalu-Kohatsu-Tindel)

Smoothing property

We consider the functional operator

$$P_t f(x) = \mathbb{E}\left(f(X_t^x)\right).$$

under Kusuoka's condition:

Assumption

There exists an integer $l \geq 1$ and $\omega_I^J \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ such that for any $x \in \mathbb{R}^n$ and word I,

$$V_{[I]}(x) = \sum_{J \in \mathcal{A}(l)} \omega_I^J(x) V_{[J]}(x).$$

Smoothing property

We prove the following regularisation bound,

Theorem

If Kusuoka's condition is satisfied,

$$|V_{i_1} \cdots V_{i_k} P_t f(x)| \leq \frac{C_{k,q}(x)}{t^{kH}} \left(P_t |f|^q\right)^{1/q}(x), \quad 0 < t < 1.$$

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Observe that it implies

$$|V_{i_1} \cdots V_{i_k} P_t f(x)| \leq \frac{C_k(x)}{t^{kH}} \|f\|_\infty, \quad 0 < t < 1.$$

Integration by parts on the path space

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Lemma

$$V_i P_t f(x) = \mathbb{E}\left(\sum_{k=1}^{n} \alpha_i^k(t,x) V_k f(X_t^x) \right),$$

where α solves the following system of SDEs:

$$d\alpha_i^j(t,x) = \sum_{k,l=1}^{n} \alpha_i^k(t,x)\omega_{kl}^j(X_t^x)dB_t^l, \quad \alpha_i^j(0,x) = \delta_i^j.$$

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Then by ellipticity, we can find $\alpha_i^j(t,x)$ such that

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The change of variable formula shows that α solves the above system of SDEs. □

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$$\langle (\mathbf{1}_{[0,t_1]}, \cdots, \mathbf{1}_{[0,t_n]}), (\mathbf{1}_{[0,s_1]}, \cdots, \mathbf{1}_{[0,s_n]}) \rangle_{\mathcal{H}} = \sum_{i=1}^{n} R(t_i, s_i).$$

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- Asume $H > 1/2$. Let $\gamma > H - 1/2$. There exist constants $c_1, c_2 > 0$ such that for every continuous $f \in \mathcal{H}$, and $t \in (0,1]$,

$$c_1 t^{2H} \frac{\min_{[0,1]} |f|^4}{\|f\|_\infty^2 + \|f\|_\gamma^2} \leq \|f \mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 \leq c_2 t^{2H} \|f\|_\infty^2.$$

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- Assume $H\le 1/2$ and let $\gamma>1/2-H$. There exist constants $c_1,c_2>0$ such that for every $f\in C^\gamma$, and $t\in(0,1]$,

$$c_1 t^{2H}\min_{[0,1]}|f|^2\le\|f\mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2\le c_2 t^{2H}(\|f\|_\gamma^2+\|f\|_\infty^2).$$

A random variable F is said to be cylindrical ($F \in \mathcal{S}$) if it can be written as

$$F = f\left(\int_0^1 \langle h_s^1, dB_s\rangle, \dots, \int_0^1 \langle h_s^m, dB_s\rangle\right),$$

where $h^i \in \mathcal{H}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ bounded function with bounded derivatives. The Malliavin derivative of $F \in \mathcal{S}$ is the $\mathbb{R}^n$ valued stochastic process $(\mathbf{D}_t F)_{0 \le t \le 1}$ given by

$$\mathbf{D}_t F = \sum_{i=1}^m h^i(t) \frac{\partial f}{\partial x_i}\left(\int_0^1 \langle h_s^1, dB_s\rangle, \dots, \int_0^1 \langle h_s^m, dB_s\rangle\right).$$

Integration by parts on the path space

The keypoint of using Malliavin calculus in our problem is the following result:

Theorem

Let $F = (F_1, , \cdots , F_n)$ be a non degenerate vector in $\mathbb{D}^\infty$ with Malliavin matrix γ_F. Let $G \in \mathbb{D}^\infty$ and $\Phi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$. For $i = 1, \cdots , n$,

$$\mathbb{E}\left(\partial_i \Phi(F) G\right) = \mathbb{E}\left(\Phi(F) H_i(F, G)\right),$$

where

$$H_i(F, G) = \sum_{j=1}^{n} \delta\left(G(\gamma_F^{-1})_{ij} DF^i\right).$$

Integration by parts on the path space

Coming back to our problem, we can thus write

$$\mathbb{E}\left(\sum_{k=1}^{n}\alpha_i^k(t,x)V_k f(X_t^x)\right)=\mathbb{E}\left(\sum_{l=1}^{n}G_i^l(t,x)\partial_l f(X_t^x)\right),$$

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and finally get

$$V_i P_t f(x) = \mathbb{E}\left(f(X_t^x) \sum_{l,j=1}^{n} \delta\left(G_i^j(t,x)\left(\gamma^{-1}\right)_{lj} DX_t^l\right)\right).$$

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and finally get

$$V_iP_tf(x)=\mathbb{E}\left(f(X_t^x)\sum_{l,j=1}^{n}\delta\left(G_i^j(t,x)\left(\gamma^{-1}\right)_{lj}DX_t^l\right)\right).$$

By continuity of $\delta:\mathbb{D}^{k,p}\to\mathbb{D}^{k-1,p}$, we are let with the problem of estimating DX and γ^{-1}.

Estimates of the Malliavin derivatives

It is possible to prove the following estimates:

Theorem (B., Ouyang, E. Nualart, Tindel)

For $1/4 < H$, $m, p > 1$ *and* $t \in [0,1]$,

$$\|\mathbf{D}X_t^x\|_{m,p} \leq c_1 t^H$$

and

$$\|\gamma_t^{-1}\|_{m,p} \leq c_2 t^{-2H}.$$

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Putting back the pieces together finishes the proof of the regularization bound.

General case

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General case

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General case

Let us introduce $\beta_I^J(t,x)$ that satisfies the following linear equations:

$$\begin{cases} d\beta_I^J(t,x) = \sum_{j=1}^{d} \left(\sum_{K \in \mathcal{A}_1(I)} -\omega_{I*j}^{K}(X_t^x)\beta_K^J(t,x) \right) dB_t^j \\ \beta_I^J(0,x) = \delta_I^J \end{cases}$$

Now, let us introduce the following notations: for any $J \in \mathcal{A}_1(I)$,

$$D^{(J)} f(X_t^x) = \langle \mathbf{D}.f(X_t^x), \beta^J(\cdot,\ x) 1_{[0,t]} \rangle_{\mathcal{H}}.$$

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Also, for any I, $J \in \mathcal{A}_1(I)$, we define

$$M_{I,J}(t,x) = \langle \beta^I(\cdot,\ x)1_{[0,t]}(\cdot), \beta^J(\cdot,\ x)1_{[0,t]} \rangle_{\mathcal{H}}.$$

General case

The following result is the main technical difficulty

Theorem

For $I \in \mathcal{A}_1(I)$,

$$V_{[I]}[f(X_t^x)] = \sum_{J \in \mathcal{A}_1(I)} (M_{I,J}(t,x))^{-1} D^{(J)} f(X_t^x)$$

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For $I \in \mathcal{A}_1(l)$,

$$V_{[I]}[f(X_t^x)] = \sum_{J \in \mathcal{A}_1(l)} (M_{I,J}(t,x))^{-1} D^{(J)} f(X_t^x)$$

and for any $p \in (1, \infty)$,

$$\sup_{x \in \mathbb{R}^n} \mathbb{E}\left(\|(M_{I,J}(t,x))_{I,J \in \mathcal{A}_1(l)}\|^{-p} \right) < \infty$$

Estimate for the inverse of M

Step 1: A Taylor estimate shows that for $|I| \leq |J| \leq l$,

$$\beta_I^J(t,x) = \sum_{L \in \mathcal{A}} \delta_{I*L}^J (-1)^{|L|} B_t^L + \gamma_I^J(t,x)$$

where

$$\sup_{x \in \mathbb{R}^n} \mathbb{E}\left[\left(\sup_{t \in (0,1]} t^{-(l+1-|I|)H} |\gamma_I^J(t,x)|\right)^p\right] < \infty$$

holds for any $p \geq 1$.

Estimate for the inverse of M

Step 2: For $m \geq 0$ and $p \geq 1$, there exists a constant $C_{H,d,p} > 0$ such that for any $\epsilon > 0$

$$\sup_{\sum a_I^2 = 1} \mathbb{P}\left(\left\| \sum_{I \in \mathcal{A}(m)} a_I B_t^I \right\|_{\infty,[0,1]} < \epsilon \right) \leq C_{H,n,p} \epsilon^p$$

Step 3: Use of interpolation inequalities (B.-Hairer 2007 and Hairer-Pilai 2013)

IPP formula

With the formula in hands, we may then integrate by parts:

$$\mathbb{E}(\Phi V_{[I]}[f(X_t^x)]) = \mathbb{E}\left(f(X_t^x) T^*_{V_{[I]}} \Phi\right).$$

where

$$T^*_{V_{[I]}} \Phi = \delta \left(\Phi \sum_{J \in \mathcal{A}_1(I)} \beta^J(t,x) (M_{I,J}(t,x))^{-1} \right)$$

is a nice operator in Malliavin sense.

Extension

Along the same lines we can prove the following result:

Theorem

If Kusuoka's condition is satisfied,

$$\left| V_{I_1} \cdots V_{I_k} P_t V_{I_{k+1}} \cdots V_{I_{k+l}} f(x) \right| \leq \frac{C(x)}{t^{(|I_1|+\cdots+|I_{k+l}|)H}} \|f\|_\infty, 0 < t < 1.$$

Extension

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Global gradient bounds require geometric conditions. Assume that the V_i's form an elliptic system of vector fields and that

$$[V_i, V_j] = \sum_k \omega_{ij}^k V_k$$

with

$$\omega_{ij}^k = -\omega_{ik}^j,$$

then we have the following global bound

Theorem

$$\sum_{i=1}^n (V_i P_t f)^2(x) \leq P_t \left(\sum_{i=1}^n (V_i f)^2 \right)(x)$$