Rough flows

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IPAM: Rough Paths: Theory and Applications.

It is easy to construct flows!

$$\text{approximate flow} \xrightarrow[\textit{continuous}]{\textit{simple}} \text{flow}$$

Deal with

- **classical RDEs** with **infinite dimensional** state space/**signal**: choose an approximate flow with the awaited "*Taylor expansion*"
- **stochastic mean field RDEs**
- **path-dependent RDEs**
- analogue of Le Jan-Watanabe-Kunita... **stochastic flows**

Layout of the talk

1. Approximate flows and flows

2. From controlled ODEs to RDEs

3. Rough flows

1.1 Feyel-de la Pradelle' sewing lemma

▶ Definition. *E a Banach space. An E-valued map* $(\mu_{ts})_{0\leqslant s\leqslant t\leqslant 1}$ *is said to be* **approximately additive** *if we have*

$$\left|\left(\mu_{tu}+\mu_{us}\right)-\mu_{ts}\right| \leqslant c_1|t-s|^a,$$

for all $0\leqslant s\leqslant u\leqslant t\leqslant 1$*, for some positive constants* c_1 *and* $a>1$*.*

For a partition $\pi_{ts}=\{s<t_1<\cdots<t_n<t\}$ of $[s,t]$, set

$$\mu_{\pi_{ts}} := \sum_{i=0}^{n-1} \mu_{t_{i+1}t_i}.$$

1.1 Feyel-de la Pradelle' sewing lemma

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▶ Lemma [FdlP, 06']. **An approximately additive continuous map defines a unique additive map** φ *st.* $\left|\varphi_{ts}-\mu_{ts}\right|\lesssim|t-s|^a$*; moreover*

$$\left|\varphi_{ts}-\mu_{\pi_{ts}}\right|\lesssim c_1^2\left|\pi_{ts}\right|^{a-1}.$$

1.2 Approximate flows and flows

▶ Definition. *A* $\mathcal{C}^1$**-approximate flow** *is a family of* $\mathcal{C}^2$ *maps* $\mu_{ts} : E \to E$, *continuous wrt* (s,t) *for uniform convergence, with* $\|\mu_{ts} - I\|_{\mathcal{C}^2} \leqslant o_{t-s}(1)$, *and*

$$\|\mu_{tu} \circ \mu_{us} - \mu_{ts}\|_{\mathcal{C}^1} \leqslant c_1 \, |t-s|^a$$

for some positive constants c_1 *and* $a > 1$ *and all* $0 \leqslant s \leqslant u \leqslant t \leqslant T$.

For a partition $\pi_{ts} = \{s < t_1 < \cdots < t_n < t\}$ of $[s,t]$, set

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$$\mu_{\pi_{ts}} := \bigcirc_{i=0}^{n-1} \mu_{t_{i+1}t_i}.$$

▶ Theorem [B1, 12']. **A** $\mathcal{C}^1$**-approximate flow** μ **defines a unique flow** φ *st.* $\|\varphi_{ts} - \mu_{ts}\|_\infty \lesssim |t-s|^a$; *moreover*

$$\|\varphi_{ts} - \mu_{\pi_{ts}}\|_\infty \lesssim c_1^2 |\pi_{ts}|^{a-1},$$

and the φ_{ts} *are uniformly Lipschitz continuous.*

Rks. • *Elementary* and *short* proof.

• In practical situations, choice of μ_{ts} guided by local considerations on " *Taylor expansions*".

2.1 Flows generated by classical controlled ODEs

Given $h \in \mathcal{C}^\alpha, \alpha > \frac{1}{2}$ and $\mathrm{F} = \big(V_1, \ldots, V_\ell\big)$ vector fields on E, of class $\mathcal{C}_b^2$

$$dz_t = \mathrm{F}(z_t)\, dh_t. \tag{2.1}$$

▶ Definition. *A* **solution flow** *to equation* (2.1) *is a flow* φ *with a* "**uniform Taylor expansion**", *at any time s and any point x, of the form*

$$f\big(\varphi_{ts}(x)\big) = f\big(x\big) + h_{ts}^i (V_i f)(x) + O|t-s|^{>1}, \tag{2.2}$$

for all f regular enough.

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▶ **Method for constructing the solution flow to equation** (2.1)

1. Candidate for a map μ_{ts} with **good Taylor expansion**

   $$\mu_{ts}(x) = x + h^i_{ts} V_i(x).$$

   It satisfies (2.2) but is not a flow.
2. μ is a $\mathcal{C}^1$-**approximate flow**: $\big\|\mu_{tu}\circ\mu_{us} - \mu_{ts}\big\|_{\mathcal{C}^1} \leqslant c_1\,|t-s|^{2\alpha}$.
3. Its associated flow satisfies (2.2) since $\big\|\varphi_{ts} - \mu_{ts}\big\|_\infty \lesssim |t-s|^{2\alpha}$.

2.2 Flows generated by classical RDEs

$\mathrm{F} = \big(V_1, \dots, V_\ell\big)$: Lip_3 vector fields on E, **X** a weak geometric Hölder p-rough path over $\mathbb{R}^\ell$.

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$$f\big(\varphi_{ts}(x)\big) = f(x) + \sum_{|I| \leqslant [p]} X^I_{ts}(V_I f)(x) + O|t-s|^{>1},$$

with V_i *identified with a first order diff. operator and*

$$V_I f = V_{i_1} \cdots V_{i_k} f, \;\; \textit{if } I = \big(i_1, \dots, i_k\big).$$

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▶ **Coordinate-free formulation:** F a $\mathrm{Lip}_3(\mathrm{E},\mathrm{E})$-valued 1-form on $\mathbb{R}^\ell$. Set $\mathrm{F}^{\otimes}(1) = \mathrm{Id}$, and for $u, u_i \in \mathbb{R}^\ell$, $\mathbf{e}, \mathbf{e}' \in T^{[p]}_\ell$

$$\mathrm{F}^{\otimes}(u) = \mathrm{F}(u) : 1^{\mathrm{st}} \text{ order diff. op.}$$

$$\mathrm{F}^{\otimes}(u_1 \otimes \cdots \otimes u_k) = \mathrm{F}(u_1)\cdots\mathrm{F}(u_k) : k^{\mathrm{th}} \text{ order diff. op.}$$

Extend by linearity; we have

$$\mathrm{F}^{\otimes}(\mathbf{e}\mathbf{e}') = \mathrm{F}^{\otimes}(\mathbf{e})\mathrm{F}^{\otimes}(\mathbf{e}'), \text{ and } \mathrm{F}^{\otimes}\big([\mathbf{e},\mathbf{e}']\big) = \Big[\mathrm{F}^{\otimes}(\mathbf{e}), \mathrm{F}^{\otimes}(\mathbf{e}')\Big]$$

2.2 Flows generated by classical RDEs

In those terms, φ is a **solution flow** to the preceeding RDE **iff**

$$f\big(\varphi_{ts}(x)\big) = \big(\mathrm{F}^{\otimes}(\mathbf{X}_{ts})f\big)(x) + O|t-s|^{>1}.$$

Formalism adapted to *infinite dimensional* rough signals! Given an infinite dimensional weak geometric Hölder p-rough path $\mathbf{X}$, set

$$\mathbf{\Lambda}_{ts} = \log \mathbf{X}_{ts}.$$

Define μ_{ts} as the **time 1 map of the ODE**

$$\dot{y}_u = \mathrm{F}^{\otimes}(\mathbf{\Lambda}_{ts})(y_u), \quad 0 \leqslant u \leqslant 1.$$

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▶ Proposition [B1-B2, 12'-13']. *We have for all f nice enough*

$$\Big\| f \circ \mu_{ts} - \mathrm{F}^{\otimes}(\mathbf{X}_{ts}) f \Big\|_{\infty} \leqslant c(f, \mathbf{X})\, |t-s|^{a},$$

*for some $a > 1$, and μ is a $\mathcal{C}^1$-**approximate flow**.*

▶ Theorem [B1-B2, 12'-13']. The RDE $dz_t = \mathrm{F}(z_t)\,\mathbf{X}(dt)$ has a **unique solution flow** φ. It satisfies

$$\Big\| \varphi_{ts} - \mu_{\pi_{ts}} \Big\|_{\infty} \lesssim c_1^2\, \big|\pi_{ts}\big|^{a-1},$$

where c_1 is polynomial in the norm of $\mathbf{X}$.

2.2 Flows generated by classical RDEs

▶ Proof of proposition: idea. Based on the following **exact formula** obtained by using repeatedly the identity

$$f(y_r) = f(x) + \int_0^r \Big(\mathrm{F}^{\otimes}(\boldsymbol{\Lambda}_{ts}) f\Big)(y_u)\, du, \quad 0 \leqslant r \leqslant 1,$$

and the morphism property of $\mathrm{F}^{\otimes}$, and by separating terms according to their size.

$$\begin{aligned}
f(\mu_{ts}(x)) &= f(x) + \sum_{\ell=1}^{n} \frac{1}{\ell!} \sum_{k_1+\cdots+k_\ell \leqslant [p]} \Big(\mathrm{F}^{\otimes}(\pi_{k_\ell}\boldsymbol{\Lambda}_{ts} \cdots \pi_{k_1}\boldsymbol{\Lambda}_{ts}) f\Big)(x) \\
&+ \sum_{k_1+\cdots+k_n \leqslant [p]} \int_{\Delta_n} \Big\{\Big(\mathrm{F}^{\otimes}(\pi_{k_n}\boldsymbol{\Lambda}_{ts} \cdots \pi_{k_1}\boldsymbol{\Lambda}_{ts}) f\Big)(y_{s_n}) - \Big(\mathrm{F}^{\otimes}(\pi_{k_n}\boldsymbol{\Lambda}_{ts} \cdots \pi_{k_1}\boldsymbol{\Lambda}_{ts}) f\Big)(x)\Big\} ds \\
&+ \sum_{\ell=1}^{n} \frac{1}{\ell!} \sum_{k_1+\cdots+k_\ell \geqslant [p]+1} \Big(\mathrm{F}^{\otimes}(\pi_{k_\ell}\boldsymbol{\Lambda}_{ts} \cdots \pi_{k_1}\boldsymbol{\Lambda}_{ts}) f\Big)(x) \\
&+ \sum_{k_1+\cdots+k_n \geqslant [p]+1} \int_{\Delta_n} \Big\{\Big(\mathrm{F}^{\otimes}(\pi_{k_{[p]}}\boldsymbol{\Lambda}_{ts} \cdots \pi_{k_1}\boldsymbol{\Lambda}_{ts}) f\Big)(y_{s_n}) - \Big(\mathrm{F}^{\otimes}(\pi_{k_{[p]}}\boldsymbol{\Lambda}_{ts} \cdots \pi_{k_1}\boldsymbol{\Lambda}_{ts}) f\Big)(x)\Big\} ds
\end{aligned}$$

Same computation made independently by Boutaib, Gyurko, Lyons, Yang, 13', to prove the Taylor expansion of the proposition.

2.3 An illustration: stochastic mean field RDEs

Rk. The above results also hold for *time-dependent* vector fields, depending on time as a Lipschitz function.

Let $\mathbf{X}$ be a **random** (infinite dimensional) Holder weak geometric p-**rough path**. The equation

$$dz_t = V\big(z_t, \mathcal{L}(z_t)\big)\,dt + \mathrm{F}(z_t)\,\mathbf{X}(dt) \tag{2.4}$$

represents the dynamics in E of a *typical particle* in a large N limit of an *interacting particle system* with mean field interaction in the drift.

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represents the dynamics in E of a *typical particle* in a large N limit of an *interacting particle system* with mean field interaction in the drift.

$$\mathrm{d}(P,Q) := \sup\Big\{ (g,P) - (g,Q)\,;\ \|g\|_{\mathcal{C}^\gamma} \leqslant 1 \Big\}$$

▶ Theorem [B1, 12']. *Assume* F *is* $\mathcal{C}_b^{[p]+1}$ *and* $V(\cdot,P)$ *is* $\mathcal{C}_b^2$, *uniformly wrt* P, *and*

$$\big\|V(\cdot,P) - V(\cdot,Q)\big\|_\infty \leqslant \lambda\,\mathrm{d}(P,Q).$$

If $\mathbb{E}\,\|\mathbf{X}\|^{2([p]+1)} < \infty$, *then equation* (2.4) *has a* **unique solution**.

2.4 Variations and refinements

- **Refined** definition of $\mathcal{C}^1$-**approximate flow** and **theorem** on their **associated flow** give *existence* and *well-posedness* results for *classical RDEs* under *optimal regularity conditions* (with infinite dimensional rough signal and state space), [B1-B2, 12'-13'].

- A notion of **local** $\mathcal{C}^1$-**approximate flow** can be used to prove **existence** and **well-posedness** results for RDEs with **unbounded vector fields**, [B1, 12'].

3. From stochastic flows to rough flows

- **Ito** setting $\circ dx_t = V_i(x_t) \circ dB^i_t$ one can **separate space** (V_i) and **noise** (B)
- Le Jan-Watanabe-Kunita **stochastic flow** setting $dy_t = \mathrm{F}(y_t, \circ dB_t)$ one **cannot separate space** from **noise**

Stochastic flows: the fundamental object is a **vector field-valued semimartingale**.

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Stochastic flows: the fundamental object is a **vector field-valued semimartingale**.

- **Rough paths** setting **lift** B to a rough path $\mathbf{B}$ RDE $dx_t = \mathrm{F}(x_t)\,\mathbf{X}(dt)$
- **Rough flows** setting **lift** $\mathrm{F}(x, \circ dB_t)$ to **?** **?!**

3.1 Rough vector fields

Let $2 < p < 3$ be given, and $V(\cdot, t)$ a time-dependent **velocity field** on E. Set $V_{ts}(\cdot) = V(\cdot, t) - V(\cdot, s)$.

▶ Definition. *A* **(geometric Holder)** *p*-**rough vector field** *is a family* $(\mathbf{V}_{ts})_{0 \leqslant s \leqslant t \leqslant T}$*, where* $\mathbf{V}_{ts} = (V_{ts}, \mathbb{V}_{ts})$ *and* $\mathbb{V}_{ts}$ *is a* **second order differential operator** *s.t.*

(i) *the* **vector fields** V_{ts} *are* $\mathcal{C}_b^3$ *with*

$$\sup_{0 \leqslant s \leqslant t \leqslant T} \frac{\|V_{ts}\|_{\mathcal{C}^3}}{|t-s|^{\frac{1}{p}}} < \infty,$$

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(ii) *the* second order differential operators

$$W_{ts} := \mathbb{V}_{ts} - \frac{1}{2} V_{ts} V_{ts}$$

are actually **vector fields**, *and*

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(iii) *we have*

$$\mathbb{V}_{ts} = \mathbb{V}_{tu} + V_{us} V_{tu} + \mathbb{V}_{us}$$

for all $0 \leqslant s \leqslant u \leqslant t \leqslant T$.

3.2 Rough flows

- Definition. *We define a* **norm**

$$\|\mathbf{V}\| := \sup_{0\leqslant s\leqslant t\leqslant T} \left\{ \frac{\|V_{ts}\|_{\mathcal{C}^3}}{|t-s|^{\frac{1}{p}}} \vee \frac{\|W_{ts}\|_{\mathcal{C}^2}}{|t-s|^{\frac{2}{p}}} \right\}$$

and a **metric** *on the set of geometric p-rough vector fields setting*

$$d(\mathbf{V},\mathbf{V}') := \|\mathbf{V}-\mathbf{V}'\|.$$

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- Proposition [B3, 14']. *We have for all f nice enough*

$$\Big\| f\circ\mu_{ts} - \big(f + V_{ts}f + \mathbb{V}_{ts}f\big)\Big\|_\infty \leqslant c(f,\mathbf{V})\,|t-s|^{>1},$$

*and μ is a $\mathcal{C}^1$-**approximate flow** which depends continuously on* **V**.

The unique flow associated to μ is said to solve the **RDE on flows**

$$d\varphi = \mathbf{V}(\varphi, \circ dt),$$

and is called a **rough flow**; it is a continuous function of **V**.

3.3 Stochastic rough flows

Let V_t be a **vector valued semimartingale** on $\mathbb{R}^d$. Canonical decomposition "martingale+BV"

$$V_t(x) = M_t(x) + B_t(x).$$

Assume

1. $\left\langle M^i(x,\cdot), M^j(x,\cdot)\right\rangle_t = \int_0^t a^{ij}(x,y;s)\,ds$, for some $\mathcal{C}^{3+\epsilon}_{b;x,y}$-valued processes $\big(a(\cdot,\cdot,s)\big)_{0\leqslant s\leqslant T}$,
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▶ Theorem [B3, 14']. **One can lift** *V to a rough vector field* $\mathbf{V}$, *and the* **solution** *to the* **RDE on flows**

$$d\varphi = \mathbf{V}(\varphi, \circ dt)$$

is the **stochastic flow** *associated with the* **nonlinear SDE**

$$dx_t = V(x_t, \circ dt).$$

▶ One can also **lift Gaussian vector fields** to **rough vector fields**.

[B1, 12'] **Flows driven by rough paths**
(Submitted)

[B2, 13'] **Flows driven by Banach space-valued rough paths**
(To appear in Séminaire Prob., 14')

[B3, 14'] **Rough flows**
(To be submitted)

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Local $\mathcal{C}^1$-approximate flows with exponential growth

- Definition. *An E-valued map μ of class $\mathcal{C}^2$ defined on*

$$\bigcup_{x\in\mathrm{E}} \{x\} \times \Big\{(s,t)\,;\, 0 \leqslant t-s \leqslant c_0 e^{-c_1|x|}\Big\}.$$

  - *for all $R>0$ and all $0 \leqslant t-s \leqslant c_0 e^{-c_1 R}$, we have*

$$\Big\|\big(\mu_{ts}-\mathrm{Id}\big)_{|B_R}\Big\|_{\mathcal{C}^1} \leqslant c(R)\,|t-s|^{\frac{1}{2p}}$$

$$\Big\|\big(\mu_{ts}-\mathrm{Id}\big)_{|B_R}\Big\|_{\mathcal{C}^2} = o_{c'(R)\,|t-s|}(1).$$

Given $\epsilon>0, \exists\, R_\epsilon$ s.t. $\mu_{tu}\circ\mu_{us}$ **well-defined on** B_R, for $R>R_\epsilon$, if $0 \leqslant t-s \leqslant c_0 e^{-c_1 R}$ and $\epsilon \leqslant \frac{u-s}{t-s} \leqslant 1-\epsilon$.

  - *There are positive constants c_2 and $a>1$ s.t. for all $R>R_\epsilon$, all $0 \leqslant s \leqslant u \leqslant t$ as above, and $x\in B_R$, we have*

$$\big|\mu_{tu}\circ\mu_{us}(x)-\mu_{ts}(x)\big| \leqslant c\,e^{c_2|x|}|t-s|^a$$

$$\big|D_x\big(\mu_{tu}\circ\mu_{us}\big)-D_x\mu_{ts}\big| \leqslant c\,e^{c_2|x|}|t-s|^a.$$

Local $\mathcal{C}^1$-approximate flows with exponential growth

▶ Theorem [B1, 12']. **A local $\mathcal{C}^1$-approximate flows with exponential growth defines a unique flow on E** *to which one can associate a function* $\varepsilon : (0,\infty) \to \mathbb{R}_+$ *such that*

$$\left\| \left(\varphi_{ts} - \mu_{ts}\right)_{|B_R} \right\|_\infty \leqslant c(R)\,|t-s|^a$$

holds for all $R > 0$ *and* $t - s \leqslant \varepsilon(R)$.