Stochastic Burgers-Type Equations

M. Hairer

University of Warwick

IPAM, January 25, 2011

Motivation (Path Sampling)

Take μ_0 Brownian bridge measure and μ the law of

$$du = b(u)\,dt + dB(t)\ .$$

By Girsanov's formula, one has

$$\mu(du) = \exp\Big(\int b(u)\,du(t) - \frac{1}{2}\int |b(u)|^2\,dt\Big)\,\mu_0(du)\ ,$$

with $\Phi(u) = |b(u)|^2 - \operatorname{div} b(u)$. Thus, formally, μ is invariant for

$$du = \partial_t^2 u\,d\tau - \frac{1}{2}\nabla\Phi(u)\,d\tau - \big(Db(u) - Db(u)^T\big)\partial_t u\,d\tau + \sqrt{2}\,dW\ .$$

(Work with A. Stuart and J. Voss in the case $b = \nabla V$. See also Zabczyk, Vanden Eijnden, Reznikoff & Maths Physics literature)
Problem: What does this actually mean if b is not a gradient?

Motivation (Path Sampling)

Take μ_0 Brownian bridge measure and μ the law of

$$du = b(u)\,dt + dB(t)\;.$$

By Girsanov's formula, one has

$$\mu(du) = \exp\Big(\int b(u)\circ du(t) - \frac{1}{2}\int \Phi(u)\,dt\Big)\,\mu_0(du)\;,$$

with $\Phi(u) = |b(u)|^2 - \operatorname{div} b(u)$. Thus, formally, μ is invariant for

$$du = \partial_t^2 u\,d\tau - \frac{1}{2}\nabla\Phi(u)\,d\tau - \big(Db(u) - Db(u)^T\big)\partial_t u\,d\tau + \sqrt{2}\,dW\;.$$

(Work with A. Stuart and J. Voss in the case $b = \nabla V$. See also Zabczyk, Vanden Eijnden, Reznikoff & Maths Physics literature)
Problem: What does this actually mean if b is not a gradient?

Motivation (Path Sampling)

Take μ_0 Brownian bridge measure and μ the law of

$$du = b(u)\,dt + dB(t)\ .$$

By Girsanov's formula, one has

$$\mu(du) = \exp\Big(\int b(u) \circ du(t) - \frac{1}{2}\int \Phi(u)\,dt\Big)\,\mu_0(du)\ ,$$

with $\Phi(u) = |b(u)|^2 - \operatorname{div} b(u)$. Thus, formally, μ is invariant for

$$du = \partial_t^2 u\,d\tau - \frac{1}{2}\nabla\Phi(u)\,d\tau - \big(Db(u) - Db(u)^T\big)\partial_t u\,d\tau + \sqrt{2}\,dW\ .$$

(Work with A. Stuart and J. Voss in the case $b = \nabla V$. See also Zabczyk, Vanden Eijnden, Reznikoff & Maths Physics literature)

Problem: What does this actually mean if b is not a gradient?

Motivation (Path Sampling)

Take μ_0 Brownian bridge measure and μ the law of

$$du = b(u)\,dt + dB(t)\;.$$

By Girsanov's formula, one has

$$\mu(du) = \exp\Big(\int b(u)\circ du(t) - \frac{1}{2}\int \Phi(u)\,dt\Big)\,\mu_0(du)\;,$$

with $\Phi(u) = |b(u)|^2 - \operatorname{div} b(u)$. Thus, formally, μ is invariant for

$$du = \partial_t^2 u\,d\tau - \frac{1}{2}\nabla\Phi(u)\,d\tau - \big(Db(u) - Db(u)^T\big)\partial_t u\,d\tau + \sqrt{2}\,dW\;.$$

(Work with A. Stuart and J. Voss in the case $b = \nabla V$. See also Zabczyk, Vanden Eijnden, Reznikoff & Maths Physics literature)

Problem: What does this actually mean if b is not a gradient?

Object of interest

Stochastic "Burgers-type" equations of the form

$$\partial_t u = \partial_x^2 u + f(u) + g(u)\,\partial_x u + \xi\ , \qquad x \in S^1\ ,$$

with ξ space-time white noise, $\mathbf{E}\xi(x,t)\xi(y,s) = \delta(t-s)\delta(x-y)$.

Important: x is one-dimensional, but $u(x,t) \in \mathbf{R}^n$. We do not assume that there exists G such that $g(u)\partial_x u = \partial_x G(u)$!

Questions: Are there solutions, what do they look like, and what does it even mean to be a solution?

Object of interest

Stochastic "Burgers-type" equations of the form

$$\partial_t u = \partial_x^2 u + f(u) + g(u)\,\partial_x u + \xi\;, \qquad x \in S^1\;,$$

with ξ space-time white noise, $\mathbf{E}\xi(x,t)\xi(y,s) = \delta(t-s)\delta(x-y)$.

Important: x is one-dimensional, but $u(x,t) \in \mathbf{R}^n$. We do not assume that there exists G such that $g(u)\partial_x u = \partial_x G(u)$!

Questions: Are there solutions, what do they look like, and what does it even mean to be a solution?

Object of interest

Stochastic "Burgers-type" equations of the form

$$\partial_t u = \partial_x^2 u + f(u) + g(u)\,\partial_x u + \xi\;, \qquad x \in S^1\;,$$

with ξ space-time white noise, $\mathbf{E}\xi(x,t)\xi(y,s) = \delta(t-s)\delta(x-y)$.

Important: x is one-dimensional, but $u(x,t) \in \mathbf{R}^n$. We do not assume that there exists G such that $g(u)\partial_x u = \partial_x G(u)$!

Questions: Are there solutions, what do they look like, and what does it even mean to be a solution?

What is the problem?

Consider the linearised equation

$$\partial_t \psi = \partial_x^2 \psi + \xi \; .$$

Law at any $t > 0$ equivalent to Wiener measure.

Consequence: Cannot give classical meaning to

$$\int \varphi(x) g(u(x)) \, \partial_x u \, dx \; ,$$

even for smooth φ. No good notion of weak solution!

Expected behaviour: Solution might depend on space-discretisation, even in the simple cases where $g(u)\partial_x u = \partial_x G(u)$! Just like Itô integral $\neq$ Stratonovich integral.

What is the problem?

Consider the linearised equation

$$\partial_t \psi = \partial_x^2 \psi + \xi \;.$$

Law at any $t > 0$ equivalent to Wiener measure.

Consequence: Cannot give classical meaning to

$$\int \varphi(x) g(u(x))\, \partial_x u\, dx \;,$$

even for smooth φ. No good notion of weak solution!

Expected behaviour: Solution might depend on space-discretisation, even in the simple cases where $g(u)\partial_x u = \partial_x G(u)$! Just like Itô integral $\neq$ Stratonovich integral.

What is the problem?

Consider the linearised equation

$$\partial_t \psi = \partial_x^2 \psi + \xi \; .$$

Law at any $t > 0$ equivalent to Wiener measure.

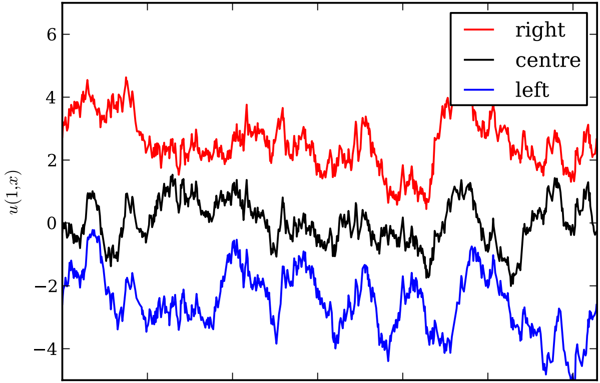
Consequence: Cannot give classical meaning to

$$\int \varphi(x) g(u(x)) \, \partial_x u \, dx \; ,$$

even for smooth φ. No good notion of weak solution!

Expected behaviour: Solution might depend on space-discretisation, even in the simple cases where $g(u)\partial_x u = \partial_x G(u)$! Just like Itô integral $\neq$ Stratonovich integral.

Numerical Evidence (With J. Voß)



Numerical solution of stochastic Burgers at time 1 with three discretisations for the nonlinear term $u\,\partial_x u$. The centred discretisation converges to the "correct" solution.

Theoretical result (With J. Maas)

Take $u \in \mathbf{R}$ (say) and consider approximations of the form

$$\partial_t u_\varepsilon = f(-\varepsilon^2 \partial_x^2)\partial_x^2 u_\varepsilon + g(u_\varepsilon) D_\varepsilon u_\varepsilon + \sigma(-\varepsilon^2\partial_x^2)\xi\;, \qquad (\star)$$

where $f(0) = 1$, $\sigma(0) = 1$, and $D_\varepsilon u(x) = \frac{1}{\varepsilon}\int u(x+\varepsilon y)\,\mu(dy)$ with $\int d\mu = 0$ and $\int x\,d\mu = 1$. (μ signed measure of finite variance.)

Theorem: Let K be the constant given by

$$K = \frac{1}{2\pi}\int_0^\infty \frac{\sigma^2(t^2)}{t^2 f(t^2)} \int_{\mathbf{R}} \big(1-\cos(yt)\big)\,d\mu(y)\,dt\;.$$

Then solutions to $(\star)$ converge as $\varepsilon \to 0$ to solutions to

$$\partial_t u = \partial_x^2 u + g(u)\partial_x u - K g'(u) + \xi\;,$$

Theoretical result (With J. Maas)

Take $u \in \mathbf{R}$ (say) and consider approximations of the form

$$\partial_t u_\varepsilon = f(-\varepsilon^2 \partial_x^2)\partial_x^2 u_\varepsilon + g(u_\varepsilon) D_\varepsilon u_\varepsilon + \sigma(-\varepsilon^2 \partial_x^2)\xi \;, \qquad (\star)$$

where $f(0) = 1$, $\sigma(0) = 1$, and $D_\varepsilon u(x) = \frac{1}{\varepsilon}\int u(x+\varepsilon y)\,\mu(dy)$ with $\int d\mu = 0$ and $\int x\,d\mu = 1$. (μ signed measure of finite variance.)

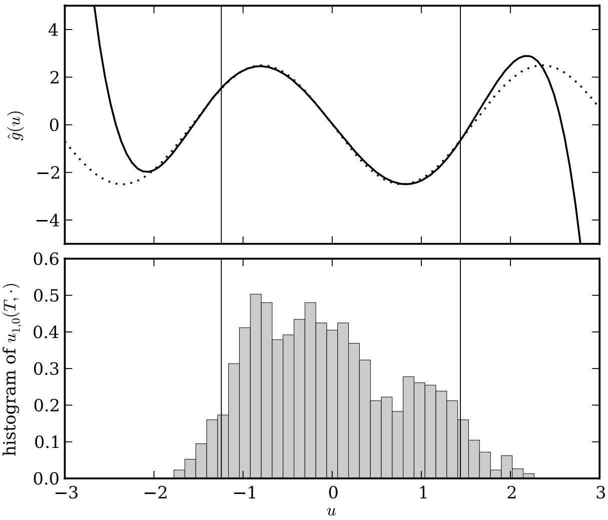
Theorem: Let K be the constant given by

$$K = \frac{1}{2\pi}\int_0^\infty \frac{\sigma^2(t^2)}{t^2 f(t^2)} \int_{\mathbf{R}} \bigl(1 - \cos(yt)\bigr)\, d\mu(y)\, dt \;.$$

Then solutions to ($\star$) converge as $\varepsilon \to 0$ to solutions to

$$\partial_t u = \partial_x^2 u + g(u)\partial_x u - K g'(u) + \xi \;,$$

Numerical Evidence 4



Fifth order fit for correction term in the case $g'(u) = \sin u$.

A standard trick

Write $u = \psi + v$ with ψ solution to linearised equation. New form of equation:

$$\partial_t v = \partial_x^2 v + g(\psi + v)\partial_x v + g(\psi + v)\partial_x \psi \ .$$

Problem: Last term still just as badly behaved...

Advantage: The 'worst' part is $\partial_x \psi$, which is considered given. Solution v is expected to be more regular.

A standard trick

Write $u = \psi + v$ with ψ solution to linearised equation. New form of equation:

$$\partial_t v = \partial_x^2 v + g(\psi + v)\partial_x v + g(\psi + v)\partial_x \psi \ .$$

Problem: Last term still just as badly behaved...

Advantage: The 'worst' part is $\partial_x \psi$, which is considered given. Solution v is expected to be more regular.

Possible approach

Could try to define concept of weak solution be interpreting the nonlinearity as

$$\int \varphi(x) g(u_t(x)) \circ d\psi_t(x) \,. \qquad (\star)$$

Problems:

1. Integrand not adapted to filtration generated by ψ_t.
2. Complicated dependency of u_t on ψ_t and additional randomness $\Rightarrow$ not easy to control Malliavin derivatives.
3. How to stitch together bounds for different values of t?

Solution: Use Lyon's rough path theory to interpret $(\star)$.

Possible approach

Could try to define concept of weak solution be interpreting the nonlinearity as

$$\int \varphi(x) g(u_t(x)) \circ d\psi_t(x) \; . \qquad (\star)$$

Problems:

1. Integrand not adapted to filtration generated by ψ_t.
2. Complicated dependency of u_t on ψ_t and additional randomness $\Rightarrow$ not easy to control Malliavin derivatives.
3. How to stitch together bounds for different values of t?

Solution: Use Lyon's rough path theory to interpret $(\star)$.

Possible approach

Could try to define concept of weak solution be interpreting the nonlinearity as

$$\int \varphi(x) g(u_t(x)) \circ d\psi_t(x) \, . \tag{$\star$}$$

Problems:

1. Integrand not adapted to filtration generated by ψ_t.
2. Complicated dependency of u_t on ψ_t and additional randomness $\Rightarrow$ not easy to control Malliavin derivatives.
3. How to stitch together bounds for different values of t?

Solution: Use Lyon's rough path theory to interpret $(\star)$.

Crash course in rough paths I

Aim: Give meaning to $\int Y\,dX$ for X, Y rougher than just $\mathcal{C}^\alpha$ with $\alpha > \frac{1}{2}$. **Additional ingredient:** Assume that we know *a priori* how to define $\int X\,dX$:

$$\int_x^y \delta X_{x,z} \otimes dX_z := \mathbf{X}_{x,y}\;, \quad \mathbf{X}_{x,y} + \mathbf{X}_{y,z} = \mathbf{X}_{x,z} - \delta X_{x,y} \otimes \delta X_{y,z}\;.$$

A rough path $(X, \mathbf{X})$ consists of both. If $\alpha < \frac{1}{2}$, $\mathbf{X}$ is never determined by X! We say $(X, \mathbf{X}) \in \mathcal{D}^\alpha$ if

$$\|X\|_\alpha := \sup_{x \neq y} \frac{|\delta X_{x,y}|}{|x-y|^\alpha} < \infty\;, \quad \|\mathbf{X}\|_{2\alpha} := \sup_{x \neq y} \frac{|\mathbf{X}_{x,y}|}{|x-y|^{2\alpha}} < \infty\;.$$

Crash course in rough paths I

Aim: Give meaning to $\int Y\,dX$ for X, Y rougher than just $\mathcal{C}^\alpha$ with $\alpha > \frac{1}{2}$. **Additional ingredient:** Assume that we know *a priori* how to define $\int X\,dX$:

$$\int_x^y \delta X_{x,z} \otimes dX_z := \mathbf{X}_{x,y}\;, \quad \mathbf{X}_{x,y} + \mathbf{X}_{y,z} = \mathbf{X}_{x,z} - \delta X_{x,y} \otimes \delta X_{y,z}\;.$$

A rough path $(X, \mathbf{X})$ consists of both. If $\alpha < \frac{1}{2}$, $\mathbf{X}$ is never determined by X! We say $(X, \mathbf{X}) \in \mathcal{D}^\alpha$ if

$$\|X\|_\alpha := \sup_{x \neq y} \frac{|\delta X_{x,y}|}{|x-y|^\alpha} < \infty\;, \quad \|\mathbf{X}\|_{2\alpha} := \sup_{x \neq y} \frac{|\mathbf{X}_{x,y}|}{|x-y|^{2\alpha}} < \infty\;.$$

Crash course in rough paths II

Idea (Lyons, Gubinelli): We can give meaning to $\int Y\,dX$ if $Y \approx X$ at small scales. Say $Y \in \mathcal{C}^\alpha_X$ if there is Y' such that

$$\delta Y_{x,y} = Y'_x\,\delta X_{x,y} + R_{x,y}\;,$$

with

$$\|Y'\|_\alpha < \infty\;,\qquad \|R\|_{2\alpha} < \infty\;.$$

Given $(X, \mathbf{X})$ and (Y, Y'), define $\int Y\,dX$ by

$$\int_0^1 Y\,dX = \lim_{|\mathcal{P}|\to 0} \sum_{[x,y]\in\mathcal{P}} \Big(Y_x\,\delta X_{x,y} + Y'_x\,\mathbf{X}_{x,y}\Big)$$

Fact: Operation continuous for $\alpha > \frac{1}{3}$!

Crash course in rough paths II

Idea (Lyons, Gubinelli): We can give meaning to $\int Y\,dX$ if $Y \approx X$ at small scales. Say $Y \in \mathcal{C}^\alpha_X$ if there is Y' such that

$$\delta Y_{x,y} = Y'_x\,\delta X_{x,y} + R_{x,y}\;,$$

with

$$\|Y'\|_\alpha < \infty\;, \qquad \|R\|_{2\alpha} < \infty\;.$$

Given $(X,\mathbf{X})$ and (Y,Y'), define $\int Y\,dX$ by

$$\int_0^1 Y\,dX = \lim_{|\mathcal{P}|\to 0} \sum_{[x,y]\in\mathcal{P}} \Big(Y_x\,\delta X_{x,y} + Y'_x\,\mathbf{X}_{x,y}\Big)$$

Fact: Operation continuous for $\alpha > \frac{1}{3}$!

Crash course in rough paths II

Idea (Lyons, Gubinelli): We can give meaning to $\int Y\,dX$ if $Y \approx X$ at small scales. Say $Y \in \mathcal{C}^\alpha_X$ if there is Y' such that

$$\delta Y_{x,y} = Y'_x\,\delta X_{x,y} + R_{x,y}\;,$$

with

$$\|Y'\|_\alpha < \infty\;, \qquad \|R\|_{2\alpha} < \infty\;.$$

Given $(X, \mathbf{X})$ and (Y, Y'), define $\int Y\,dX$ by

$$\int_0^1 Y\,dX = \lim_{|\mathcal{P}|\to 0} \sum_{[x,y]\in\mathcal{P}} \Big(Y_x\,\delta X_{x,y} + Y'_x\,\mathbf{X}_{x,y}\Big)$$

Fact: Operation continuous for $\alpha > \frac{1}{3}$!

A concept of solution

Using probabilistic techniques, one can lift ψ canonically to a continuous $\mathcal{D}^\alpha$-valued process $(\Psi_t, \boldsymbol{\Psi}_t)$ for every $\alpha < \frac{1}{2}$ (see Coutin & Qian, Friz & Victoir).

Easy to check that if $g \in \mathcal{C}^2$ and $v \in \mathcal{C}^{2\alpha}$, then

$$x \mapsto \varphi(x) g\big(v(x) + \Psi(x)\big)$$

belongs to $\mathcal{C}^\alpha_\Psi$ for every $(\Psi, \boldsymbol{\Psi}) \in \mathcal{D}^\alpha \Rightarrow$ concept of weak solutions for $u = \psi + v$ with $v \in \mathcal{C}^{2\alpha}$.

A concept of solution

Using probabilistic techniques, one can lift ψ canonically to a continuous $\mathcal{D}^\alpha$-valued process $(\Psi_t, \boldsymbol{\Psi}_t)$ for every $\alpha < \frac{1}{2}$ (see Coutin & Qian, Friz & Victoir).

Easy to check that if $g \in \mathcal{C}^2$ and $v \in \mathcal{C}^{2\alpha}$, then

$$x \mapsto \varphi(x) g\big(v(x) + \Psi(x)\big)$$

belongs to $\mathcal{C}^\alpha_\Psi$ for every $(\Psi, \boldsymbol{\Psi}) \in \mathcal{D}^\alpha \Rightarrow$ concept of weak solutions for $u = \psi + v$ with $v \in \mathcal{C}^{2\alpha}$.

Uniqueness of solutions?

Picard iteration $\Rightarrow$ local well-posedness. Global well-posedness if g and all its derivatives and bounded.

However, solution depends on the choice of Ψ_t!

Perform Picard iteration with $(\Psi, \tilde{\Psi})$ for

$$(\tilde{\Psi}_t)_{x,y} = (\Psi_t)_{x,y} + \int_x^y h(t,z)\, dz \;.$$

Proposition: Solution is same as with (Ψ, Ψ), but with the nonlinearity replaced by $g_{ij}(u)\, \partial_x u_j + \partial_k g_{ij}(u)\, h_{jk}$. Explains form of correction term

Uniqueness of solutions?

Picard iteration $\Rightarrow$ local well-posedness. Global well-posedness if g and all its derivatives and bounded.

However, solution depends on the choice of $\mathbf{\Psi}_t$!

Perform Picard iteration with $(\Psi, \tilde{\mathbf{\Psi}})$ for

$$(\tilde{\mathbf{\Psi}}_t)_{x,y} = (\mathbf{\Psi}_t)_{x,y} + \int_x^y h(t,z)\,dz \;.$$

Proposition: Solution is same as with $(\Psi, \mathbf{\Psi})$, but with the nonlinearity replaced by $g_{ij}(u)\,\partial_x u_j + \partial_k g_{ij}(u)\,h_{jk}$. Explains form of correction term

Uniqueness of solutions?

Picard iteration $\Rightarrow$ local well-posedness. Global well-posedness if g and all its derivatives and bounded.

However, solution depends on the choice of $\boldsymbol{\Psi}_t$!

Perform Picard iteration with $(\Psi, \tilde{\boldsymbol{\Psi}})$ for

$$(\tilde{\boldsymbol{\Psi}}_t)_{x,y} = (\boldsymbol{\Psi}_t)_{x,y} + \int_x^y h(t,z)\,dz\;.$$

Proposition: Solution is same as with $(\Psi, \boldsymbol{\Psi})$, but with the nonlinearity replaced by $g_{ij}(u)\,\partial_x u_j + \partial_k g_{ij}(u)\,h_{jk}$. Explains form of correction term

Approximation results

The following equations are classically well-posed:

$$\partial_t u = \partial_x^2 u + g(u)\,\partial_x u + \xi_\varepsilon\;,$$
$$\partial_t u = \partial_x^2 u - \varepsilon^2 \partial_x^4 u + g(u)\,\partial_x u + \xi\;,$$

where ξ_ε is the spatial convolution of ξ with a mollifier.

Theorem: Both sequences converge as $\varepsilon \to 0$ to the same limit, which is given by the solution to

$$\partial_t u = \partial_x^2 u + g(u)\,\partial_x u + \xi\;,$$

interpreted as above.

Work in progress (with J. Maas and H.Weber): Appearance of correction term for full discretisation.

Approximation results

The following equations are classically well-posed:

$$\partial_t u = \partial_x^2 u + g(u)\,\partial_x u + \xi_\varepsilon\;,$$
$$\partial_t u = \partial_x^2 u - \varepsilon^2 \partial_x^4 u + g(u)\,\partial_x u + \xi\;,$$

where ξ_ε is the spatial convolution of ξ with a mollifier.

Theorem: Both sequences converge as $\varepsilon \to 0$ to the same limit, which is given by the solution to

$$\partial_t u = \partial_x^2 u + g(u)\,\partial_x u + \xi\;,$$

interpreted as above.

Work in progress (with J. Maas and H.Weber): Appearance of correction term for full discretisation.

Approximation results

The following equations are classically well-posed:

$$\partial_t u = \partial_x^2 u + g(u)\,\partial_x u + \xi_\varepsilon \;,$$
$$\partial_t u = \partial_x^2 u - \varepsilon^2 \partial_x^4 u + g(u)\,\partial_x u + \xi \;,$$

where ξ_ε is the spatial convolution of ξ with a mollifier.

Theorem: Both sequences converge as $\varepsilon \to 0$ to the same limit, which is given by the solution to

$$\partial_t u = \partial_x^2 u + g(u)\,\partial_x u + \xi \;,$$

interpreted as above.

Work in progress (with J. Maas and H.Weber): Appearance of correction term for full discretisation.

Back to original problem (kind of)

Let μ_0 be the Gaussian measure on $\mathcal{C}(S^1, \mathbf{R}^n)$ with covariance operator $(1 + \partial_x^2)^{-1}$. For $\mathcal{C}_b^\infty$ functions F and G, set

$$\frac{d\mu}{d\mu_0}(u) = Z^{-1} \exp\Big(\int_0^{2\pi} G(u_t) \circ du_t + \int_0^{2\pi} F(u_t)\, dt\Big)\ .$$

(Well-defined Stratonovich integral since μ_0 locally equivalent to Wiener measure.)

Theorem: The measure μ is invariant for the equation

$$\partial_t u = \partial_x^2 u + g(u)\, \partial_x u + f(u) + \sqrt{2}\xi\ ,$$

where $f_i(u) = \partial_i F(u) - u_i$ and $g_{ij}(u) = \partial_i G_j(u) - \partial_j G_i(u)$.

Proof: Show it for μ_0 replaced by μ_ε and pass to limit.

Back to original problem (kind of)

Let μ_0 be the Gaussian measure on $\mathcal{C}(S^1, \mathbf{R}^n)$ with covariance operator $(1 + \partial_x^2)^{-1}$. For $\mathcal{C}_b^\infty$ functions F and G, set

$$\frac{d\mu}{d\mu_0}(u) = Z^{-1} \exp\Big(\int_0^{2\pi} G(u_t) \circ du_t + \int_0^{2\pi} F(u_t)\, dt\Big) .$$

(Well-defined Stratonovich integral since μ_0 locally equivalent to Wiener measure.)

Theorem: The measure μ is invariant for the equation

$$\partial_t u = \partial_x^2 u + g(u)\, \partial_x u + f(u) + \sqrt{2}\xi ,$$

where $f_i(u) = \partial_i F(u) - u_i$ and $g_{ij}(u) = \partial_i G_j(u) - \partial_j G_i(u)$.

Proof: Show it for μ_0 replaced by μ_ε and pass to limit.

Back to original problem (kind of)

Let μ_0 be the Gaussian measure on $\mathcal{C}(S^1, \mathbf{R}^n)$ with covariance operator $(1 + \partial_x^2)^{-1}$. For $\mathcal{C}_b^\infty$ functions F and G, set

$$\frac{d\mu}{d\mu_0}(u) = Z^{-1} \exp\Big(\int_0^{2\pi} G(u_t) \circ du_t + \int_0^{2\pi} F(u_t)\, dt\Big) \;.$$

(Well-defined Stratonovich integral since μ_0 locally equivalent to Wiener measure.)

Theorem: The measure μ is invariant for the equation

$$\partial_t u = \partial_x^2 u + g(u)\, \partial_x u + f(u) + \sqrt{2}\xi \;,$$

where $f_i(u) = \partial_i F(u) - u_i$ and $g_{ij}(u) = \partial_i G_j(u) - \partial_j G_i(u)$.

Proof: Show it for μ_0 replaced by μ_ε and pass to limit.

Multiplicative noise (With H. Weber)

What about equations of the form

$$\partial_t u = \partial_x^2 u + f(u) + g(u)\,\partial_x u + h(u)\xi \qquad ?$$

Key observation: For θ an adapted process, set

$$d\psi_t = \partial_x^2 \psi_t\,dt + dW(t)\;, \qquad d\psi_t^\theta = \partial_x^2 \psi_t^\theta\,dt + \theta_t\,dW(t)\;,$$

in the Itô sense.

Proposition: Let θ be α-Hölder in space and $\frac{\alpha}{2}$-Hölder in time for some $\alpha \in (\frac{1}{3}, \frac{1}{2})$. Then, for any $t > 0$, ψ_t^θ is a rough path (in x) controlled by $(\Psi_t, \mathbf{\Psi}_t)$ with "derivative process" θ_t.

Leads to concept of mild solution as before.

Multiplicative noise (With H. Weber)

What about equations of the form

$$\partial_t u = \partial_x^2 u + f(u) + g(u)\,\partial_x u + h(u)\xi \qquad ?$$

Key observation: For θ an adapted process, set

$$d\psi_t = \partial_x^2 \psi_t\,dt + dW(t)\;, \qquad d\psi_t^\theta = \partial_x^2 \psi_t^\theta\,dt + \theta_t\,dW(t)\;,$$

in the Itô sense.

Proposition: Let θ be α-Hölder in space and $\frac{\alpha}{2}$-Hölder in time for some $\alpha \in (\frac{1}{3}, \frac{1}{2})$. Then, for any $t > 0$, ψ_t^θ is a rough path (in x) controlled by $(\Psi_t, \boldsymbol{\Psi}_t)$ with “derivative process” θ_t.

Leads to concept of mild solution as before.

Multiplicative noise (With H. Weber)

What about equations of the form

$$\partial_t u = \partial_x^2 u + f(u) + g(u)\,\partial_x u + h(u)\xi \qquad ?$$

Key observation: For θ an adapted process, set

$$d\psi_t = \partial_x^2 \psi_t\,dt + dW(t)\;, \qquad d\psi_t^\theta = \partial_x^2 \psi_t^\theta\,dt + \theta_t\,dW(t)\;,$$

in the Itô sense.

Proposition: Let θ be α-Hölder in space and $\frac{\alpha}{2}$-Hölder in time for some $\alpha \in (\frac{1}{3}, \frac{1}{2})$. Then, for any $t > 0$, ψ_t^θ is a rough path (in x) controlled by $(\Psi_t, \boldsymbol{\Psi}_t)$ with "derivative process" θ_t.

Leads to concept of mild solution as before.

Well-posedness

Idea: Combine Itô calculus (in time) with rough path analysis (in space).

"Inner loop": For a fixed rough-path valued process $(\Phi, \mathbf{\Phi})$, solve

$$\partial_t v^\Phi = \partial_x^2 v^\Phi + g(\Phi + v^\Phi)\partial_x v^\Phi + g(\varphi + v^\Phi)\partial_x \Phi \;,$$

in the same sense as before.

"Outer loop": Solve the fixed point equation

$$u = v^{\Psi^{h(u)}} + \Psi^{h(u)} \;.$$

Make judicious use of cut-offs to get contraction for short times, then concatenate solutions.

Well-posedness

Idea: Combine Itô calculus (in time) with rough path analysis (in space).

"Inner loop": For a fixed rough-path valued process $(\Phi, \boldsymbol{\Phi})$, solve

$$\partial_t v^\Phi = \partial_x^2 v^\Phi + g(\Phi + v^\Phi)\partial_x v^\Phi + g(\varphi + v^\Phi)\partial_x \Phi \;,$$

in the same sense as before.

"Outer loop": Solve the fixed point equation

$$u = v^{\Psi^{h(u)}} + \Psi^{h(u)} \;.$$

Make judicious use of cut-offs to get contraction for short times, then concatenate solutions.

Well-posedness

Idea: Combine Itô calculus (in time) with rough path analysis (in space).

"Inner loop": For a fixed rough-path valued process $(\Phi, \mathbf{\Phi})$, solve

$$\partial_t v^\Phi = \partial_x^2 v^\Phi + g(\Phi + v^\Phi)\partial_x v^\Phi + g(\varphi + v^\Phi)\partial_x \Phi \;,$$

in the same sense as before.

"Outer loop": Solve the fixed point equation

$$u = v^{\Psi^{h(u)}} + \Psi^{h(u)} \;.$$

Make judicious use of cut-offs to get contraction for short times, then concatenate solutions.

Outlook

- What about numerical schemes for state-dependent noise? One can guess answer and we “almost” have a proof.
- Higher space dimensions? (What is the “right” class of equations in this case?)
- What about starting from SDEs with multiplicative noise?
- What about KPZ equation in 1D? Seems quite far off (one derivative), but one can exploit bilinear structure to gain something.
- Other KPZ-related dynamics. For example: motion of a random string constrained on a manifold.

Outlook

- What about numerical schemes for state-dependent noise? One can guess answer and we "almost" have a proof.
- Higher space dimensions? (What is the "right" class of equations in this case?)
- What about starting from SDEs with multiplicative noise?
- What about KPZ equation in 1D? Seems quite far off (one derivative), but one can exploit bilinear structure to gain something.
- Other KPZ-related dynamics. For example: motion of a random string constrained on a manifold.

Outlook

- What about numerical schemes for state-dependent noise? One can guess answer and we "almost" have a proof.
- Higher space dimensions? (What is the "right" class of equations in this case?)
- What about starting from SDEs with multiplicative noise?
- What about KPZ equation in 1D? Seems quite far off (one derivative), but one can exploit bilinear structure to gain something.
- Other KPZ-related dynamics. For example: motion of a random string constrained on a manifold.

Outlook

- What about numerical schemes for state-dependent noise? One can guess answer and we "almost" have a proof.
- Higher space dimensions? (What is the "right" class of equations in this case?)
- What about starting from SDEs with multiplicative noise?
- What about KPZ equation in 1D? Seems quite far off (one derivative), but one can exploit bilinear structure to gain something.
- Other KPZ-related dynamics. For example: motion of a random string constrained on a manifold.

Outlook

- What about numerical schemes for state-dependent noise? One can guess answer and we "almost" have a proof.
- Higher space dimensions? (What is the "right" class of equations in this case?)
- What about starting from SDEs with multiplicative noise?
- What about KPZ equation in 1D? Seems quite far off (one derivative), but one can exploit bilinear structure to gain something.
- Other KPZ-related dynamics. For example: motion of a random string constrained on a manifold.