Pulse propagation in random media with short- and long-range correlations

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- Wave propagation in a one-dimensional random medium with long- or short-range correlations.

- Pulse propagation is characterized by
1) a *random time shift*,
2) a *deterministic spreading* described by a pseudo-differential operator. In some regimes, this operator has the form of a fractional derivative.

- Relation between the statistics of the random fluctuations of the medium and the effective attenuation/dispersion of the wave.

Acoustic wave equation in one-dimensional random medium

Acoustic wave equation:

$$\rho(z)\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial p^\varepsilon}{\partial z} = 0$$
$$\frac{1}{K(z)}\frac{\partial p^\varepsilon}{\partial t} + \frac{\partial u^\varepsilon}{\partial z} = 0$$

A section of random medium is sandwiched in between two homogeneous half-spaces:

$$\frac{1}{K(z)} = \begin{cases} \frac{1}{K_0}\left(1+\varepsilon\nu\left(\frac{z}{\varepsilon^2}\right)\right) & \text{for } z \in [0,L] \\ \frac{1}{K_0} & \text{for } z \in (-\infty,0)\cup(L,\infty) \end{cases}$$
$$\rho(z) = \rho_0 \text{ for all } z$$

The background velocity is $c_0 = \sqrt{K_0/\rho_0}$.

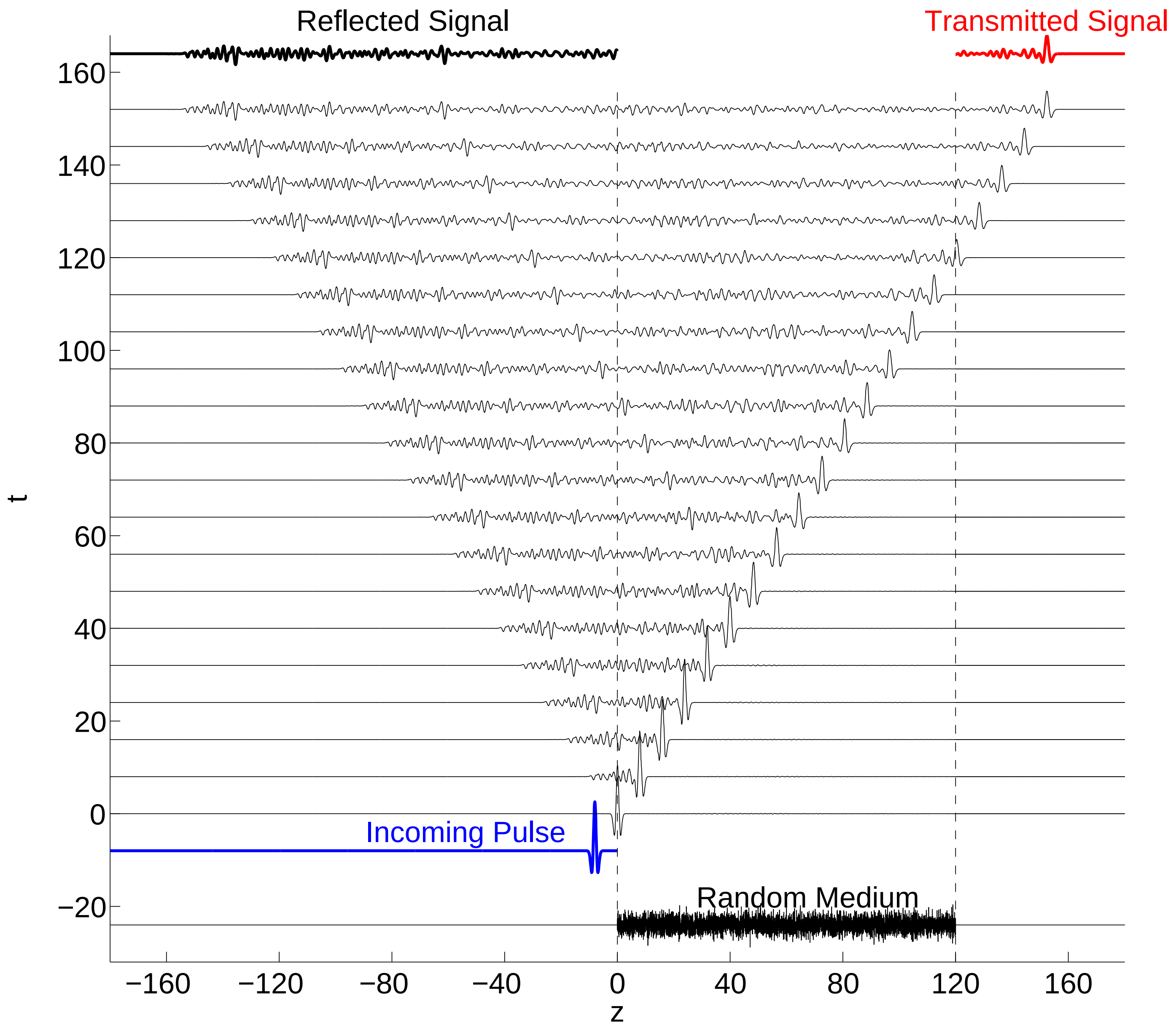
$\nu(z)$ is a zero-mean, stationary, random process with covariance function:

$$\phi(z) := \mathbb{E}[\nu(y)\nu(y+z)]$$

Pulse wave incoming from the left with pulse profile

$$p^\varepsilon(t,z=0) = p_0\left(\frac{t}{\varepsilon^2}\right)$$

Numerical experiment in one-dimensional random medium



The mixing case (1/5)

$\phi(z) = \mathbb{E}[\nu(y)\nu(y+z)]$ decays fast enough so that it is integrable (+technical conditions).

The correlation length l_c can be defined by:

$$l_c = \frac{1}{\phi(0)} \int_{-\infty}^{\infty} \phi(z)dz$$

Introduce the random travel time

$$\tau^\varepsilon(L) = \frac{L}{c_0} + \varepsilon^2 \tau_c^\varepsilon(L), \qquad \tau_c^\varepsilon(L) = \frac{1}{2c_0\varepsilon} \int_0^L \nu\left(\frac{z}{\varepsilon^2}\right) dz$$

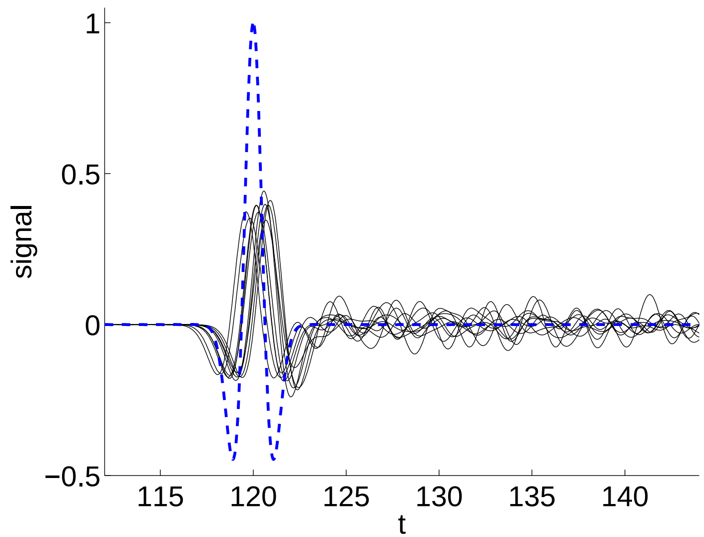
In the limit $\varepsilon \to 0$, at $z = L$,

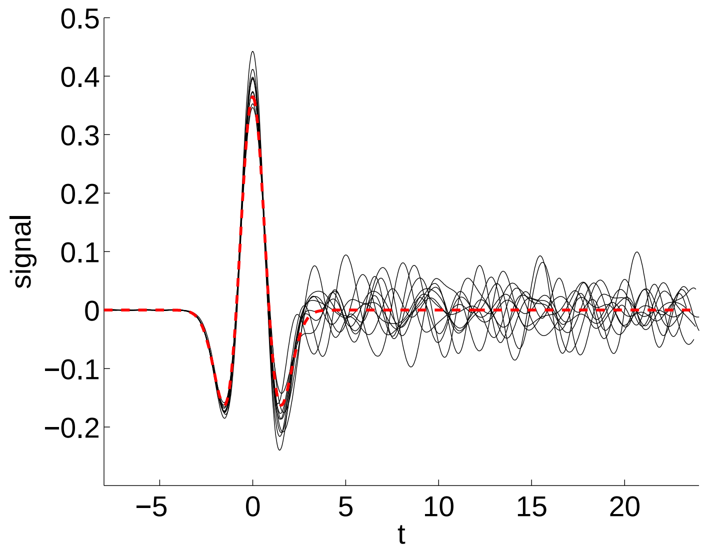
1) the *random time shift* $\tau_c^\varepsilon(L)$ converges in distribution to a zero-mean random variable with variance of order one,

2) the pulse profile converges in probability to a *deterministic profile*:

$$p^\varepsilon\left(z = L, t = \tau^\varepsilon(L) + \varepsilon^2 \tau\right) \stackrel{\varepsilon \to 0}{\longrightarrow} p_L(\tau)$$

The mixing case (2/5)





Time profiles of the transmitted wave at $z = L$

Without time shift

Black: ten numerical results obtained with ten realizations of the random medium

Blue: original pulse $p_0(t)$

With time shift

Red: theoretical (deterministic) transmitted pulse shape $p_L(t)$

The mixing case (3/5)

More precisely:

1) the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a Gaussian random variable T_L (proportional to W_L) with mean zero and variance:

$$\mathbb{E}\big[T_L^2\big] = \frac{\phi(0) l_c}{8c_0^2} L$$

2) the pulse profile experiences a deterministic deformation:

$$p_L(\tau) = \frac{1}{2\pi} \int \hat{p}_0(\omega) \exp\Big(- i\omega\tau - \frac{\omega^2 [\hat{\phi}_c(\omega) + i\hat{\phi}_s(\omega)]}{8c_0^2} L\Big) d\omega$$

$$\hat{\phi}_c(\omega) := 2\int_0^\infty \phi(z) \cos\Big(\frac{2\omega z}{c_0}\Big) dz, \qquad \hat{\phi}_s(\omega) := 2\int_0^\infty \phi(z) \sin\Big(\frac{2\omega z}{c_0}\Big) dz$$

- Remark 1: Known as O'Doherty-Anstey theory.

Cf: J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, Wave propagation and time reversal in randomly layered media, Springer, 2007.

The mixing case (4/5)

If $l_c \ll$ wavelength (the wave probes the tail of $\phi(z)$), then:

$$p_L(\tau) = \frac{1}{2\pi} \int \hat{p}_0(\omega) \exp\Big(- i\omega\tau - \frac{\phi(0)}{8} \frac{\omega^2 l_c^2}{c_0^2} \frac{L}{l_c} \Big) d\omega$$

$\hookrightarrow$ Effective second-order diffusion (attenuation), no effective dispersion.

- Remark 2: The mean field approach is quantitatively wrong:

$$p_L^{\mathrm{MF}}(\tau) = \lim_{\varepsilon \to 0} \mathbb{E}\Big[p^\varepsilon \big(z = L, t = \frac{L}{c_0} + \varepsilon^2 \tau \big) \Big] = \frac{1}{2\pi} \int \hat{p}_0(\omega) \exp\Big(- i\omega\tau - \frac{\phi(0)}{4} \frac{\omega^2 l_c^2}{c_0^2} \frac{L}{l_c} \Big) d\omega$$

(additional frequency-dependent attenuation that comes from the averaging with respect to the random time shift).

- Remark 3: Only $\int_{-\infty}^{\infty} \phi(z) dz = \phi(0) l_c$ can be estimated from experiments. This estimation is robust (statistically stable).

- Remark 4: What happens if $\int_{-\infty}^{\infty} \phi(z) dz = \infty$?

The mixing case (5/5)

Assume (typical Markovian model):

$$\phi(z) = \phi(0)\Big(1 - d_1 \frac{|z|}{l_c} + o\Big(\frac{|z|}{l_c}\Big)\Big), \qquad |z| \ll l_c$$

If wavelength $\ll l_c$ (the wave probes the small-z behavior of $\phi(z)$), then:

$$p_L(t) = \frac{1}{2\pi} \int \hat{p}_0(\omega) \exp\Big(-i\omega t - \frac{\phi(0)d_1}{16}\frac{L}{l_c} - i\frac{\phi(0)}{8}\frac{\omega l_c}{c_0}\frac{L}{l_c}\Big) d\omega$$

$\hookrightarrow$ Effective constant attenuation, no effective dispersion.

- Remark 5: Only $\phi(0)d_1$ can be estimated from experiments.
- Remark 6: What happens if $\phi(z)$ is not smooth at zero ?

Long-range and short-range correlations

- Long-range correlations: ϕ is not integrable and has a power decay at infinity:

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_H \Big|\frac{z}{l_c}\Big|^{2H-2}$$

where $r_H > 0$ and $H \in (1/2, 1)$ (i.e., $2H - 2 \in (-1, 0)$).
l_c is the critical length scale beyond which the power law behavior is valid.

- Short-range correlations: ϕ is not smooth at zero:

$$\phi(z) \overset{|z|\to 0}{\simeq} \phi(0)\Big(1 - d_H \Big|\frac{z}{l_c}\Big|^{2H} + O\Big(\Big|\frac{z}{l_c}\Big|\Big)\Big)$$

where $d_H > 0$ and $H \in (0, 1/2)$ (i.e., $2H \in (0, 1)$).
l_c is the critical length scale below which the expansion is valid.

Model 0: Fractional Brownian motion

Fractional Brownian motion with Hurst index $H \in (0,1)$:

$$W_H(z)$$

Gaussian process, with zero mean and covariance:

$$\phi(z) = \mathbb{E}[W_H(y)W_H(y+z)] = \frac{1}{2}\left(|y+z|^{2H} + |y|^{2H} - |z|^{2H}\right)$$

Self-similar process:

$$W_H(az) \overset{dist.}{\sim} a^H W_H(z) \quad \text{for all } a > 0$$

Stationary increments:

$$\mathbb{E}\left[(W_H(y+z) - W_H(y))^2\right] = |z|^{2H}$$

$H = 1/2$: standard Brownian motion (independent increments).
$H < 1/2$: short-range correlations (negatively-correlated increments). The realizations are continuous but irregular.
$H > 1/2$: long-range correlations (positively-correlated increments). The realizations are continuous and more regular (but not differentiable).

However, the process itself is <u>not stationary</u> !

Model 1: Fractional Ornstein Uhlenbeck (OU) process

$$\nu(z) := \frac{\sigma}{\sqrt{H\Gamma(2H) l_c^H}} \left[W_H(z) - \frac{1}{l_c} \int_{-\infty}^{z} e^{\frac{y-z}{l_c}} W_H(y) dy \right]$$

where W_H is a fractional Brownian motion with Hurst index $H \in (0, 1)$.

Looks like a fBm, but with a restoring force $\to$ stationary.

The fractional OU process is a zero-mean, variance σ^2, stationary, Gaussian process and its covariance function is

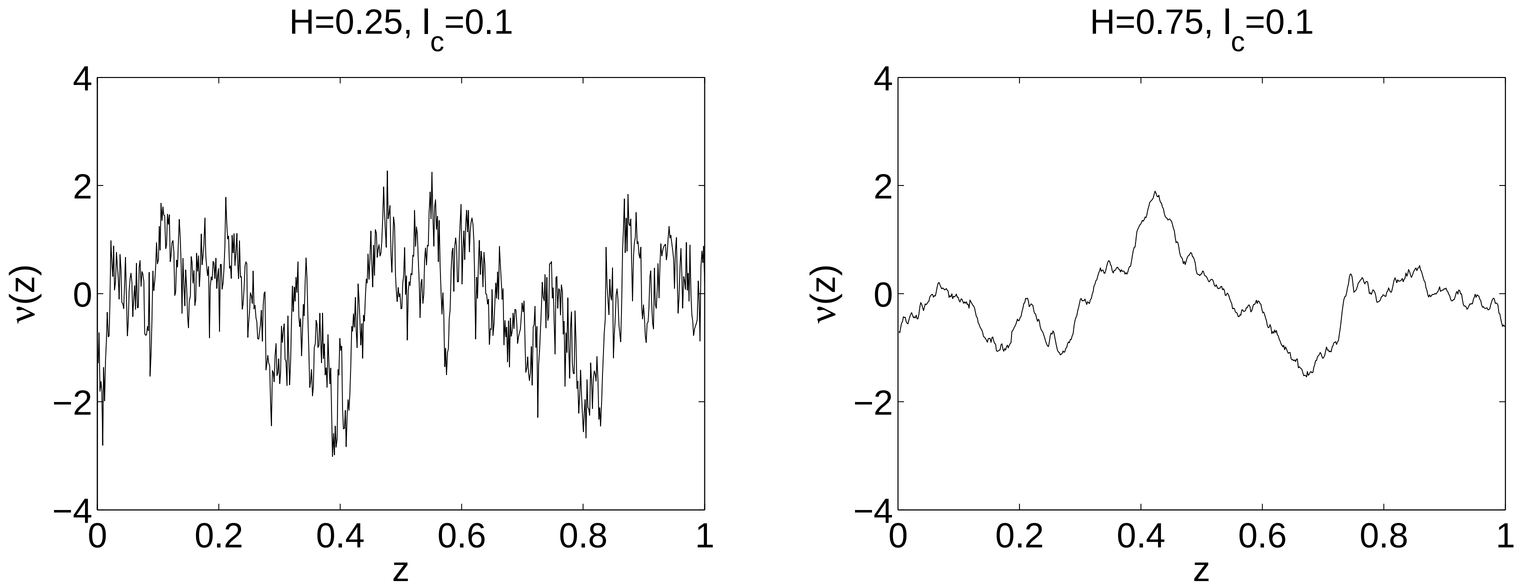
$$\phi(z) = \frac{\sigma^2}{H\Gamma(2H) l_c^{2H}} \left[\frac{1}{4 l_c} \int_{-\infty}^{\infty} e^{-\frac{|y|}{l_c}} |z+y|^{2H} dy - \frac{1}{2} |z|^{2H} \right]$$

If $H \in (1/2, 1)$, then the large-z behavior of the covariance function is

$$\phi(z) \overset{|z| \to \infty}{\simeq} r_H \left| \frac{z}{l_c} \right|^{2H-2}, \text{ with } r_H = \frac{\sigma^2 (2H-1)}{\Gamma(2H)}$$

If $H \in (0, 1/2)$, then the small-z behavior of the covariance function is

$$\phi(z) \overset{|z| \to 0}{\simeq} \sigma^2 \left(1 - d_H \left| \frac{z}{l_c} \right|^{2H} + o\left(\left| \frac{z}{l_c} \right| \right) \right), \text{ with } d_H = 1$$



Realizations of the fractional OU process with Hurst index H and correlation length l_c. The trajectories are more regular when H is larger.

Model 2: Fractional white noise medium

$$\nu(z) := \frac{\sigma}{l_c^H}\Big[W_H(z) - W_H(z+l_c)\Big]$$

where W_H is a fractional Brownian motion with Hurst index $H \in (0,1)$.

Stationary, zero-mean, variance σ^2, Gaussian process with covariance function

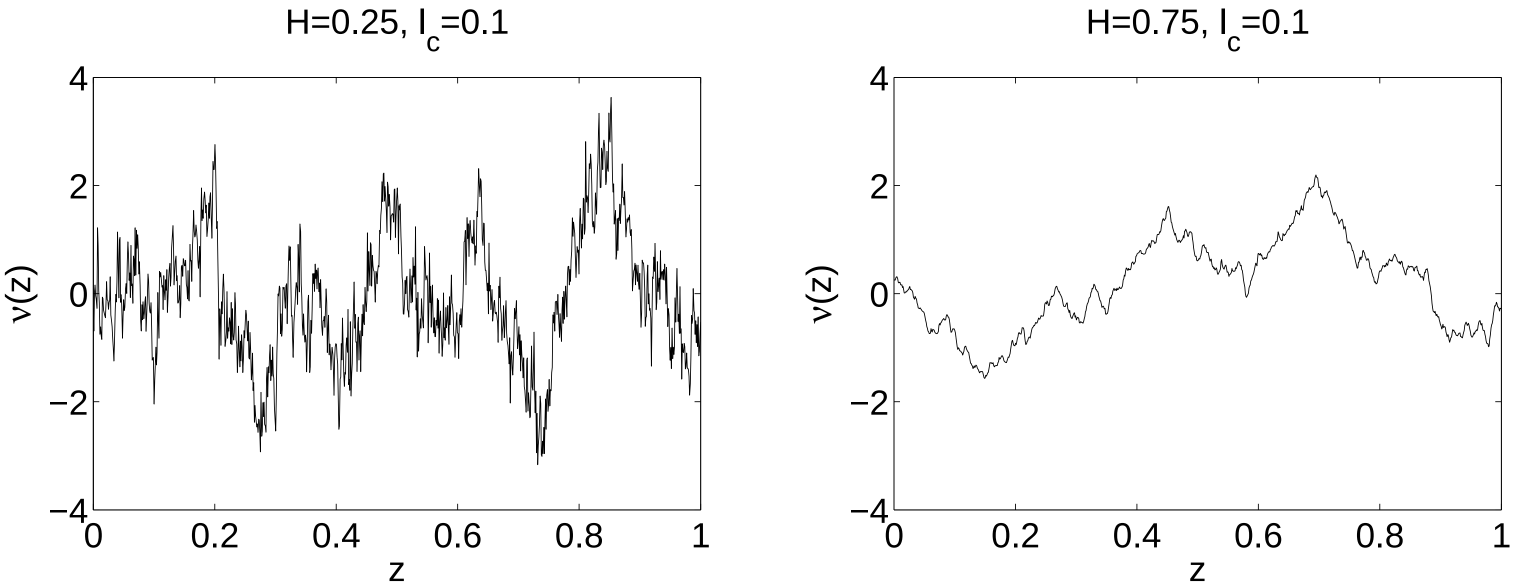
$$\phi(z) = \frac{\sigma^2}{2l_c^{2H}}\big(|z+l_c|^{2H} + |z-l_c|^{2H} - 2|z|^{2H}\big)$$

If $H \in (1/2,1)$ then

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_H\Big|\frac{z}{l_c}\Big|^{2H-2}, \text{ with } r_H = \sigma^2 H(2H-1)$$

If $H \in (0,1/2)$, then

$$\phi(z) \overset{|z|\to 0}{\simeq} \sigma^2\Big(1 - d_H\Big|\frac{z}{l_c}\Big|^{2H} + o\big(\Big|\frac{z}{l_c}\Big|\big)\Big), \text{ with } d_H = 1$$



Realizations of the fractional white noise with Hurst index H and correlation length l_c. The trajectories are more regular when H is larger.

Model 3: Binary medium - long-range correlations

The process ν is stepwise constant.
$(l_j)_{j\geq 0}$: lengths of the elementary intervals.
$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.
The random variables n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The random variables l_j are i.i.d. with the pdf ($H \in (1/2, 1)$):

$$p_{l_1}(z) = \frac{3-2H}{l_c} \frac{l_c^{4-2H}}{z^{4-2H}} \mathbf{1}_{[l_c,\infty)}(z)$$

Note: The average length of the intervals is $\frac{3-2H}{2-2H} l_c$ while the variance is infinite.

The process ν is (almost) stationary. The covariance function is

$$\phi(z) = \sigma^2 \left[\frac{1}{3-2H} \frac{l_c^{2-2H}}{|z|^{2-2H}} \mathbf{1}_{[l_c,\infty)}(|z|) + \left(1 - \frac{2-2H}{3-2H} \frac{|z|}{l_c}\right) \mathbf{1}_{[0,l_c)}(|z|) \right]$$

We have

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_H \left|\frac{z}{l_c}\right|^{2H-2}, \text{ with } r_H = \frac{\sigma^2}{3-2H}$$

Model 3: Binary medium - short-range correlations

The process ν is stepwise constant.

$(l_j)_{j\geq 0}$: lengths of the elementary intervals.

$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The random variables n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The random variables l_j are i.i.d. with the pdf ($H \in (0, 1/2)$):

$$p_{l_1}(z) = \frac{1-2H}{l_c[(l_i/l_c)^{2H-1} - 1]} \frac{l_c^{2-2H}}{z^{2-2H}} \mathbf{1}_{[l_i, l_c]}(z)$$

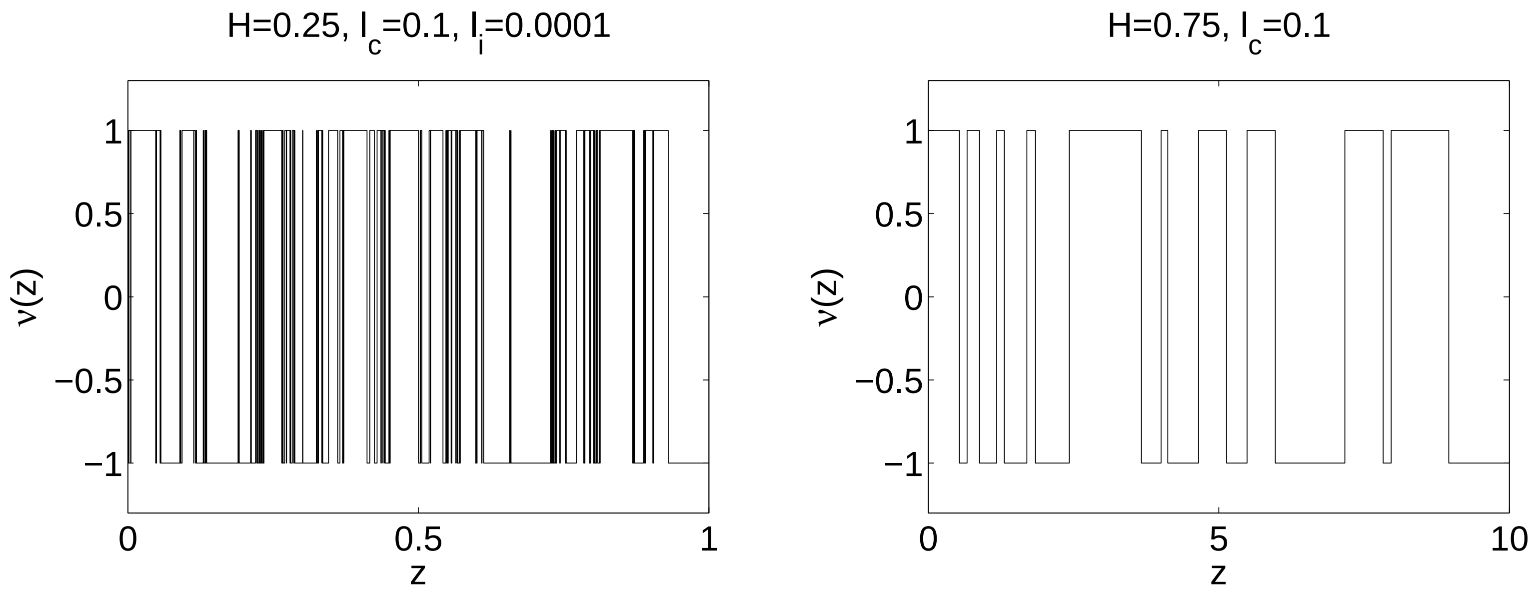
with $l_i \ll l_c$.

The process ν is (almost) stationary. The covariance function is

$$\phi(z) \simeq \sigma^2 \Big(1 - \frac{1}{1-2H} \frac{|z|^{2H}}{l_c^{2H}} + \frac{2H}{1-2H} \frac{|z|}{l_c} \Big) \mathbf{1}_{(0, l_c]}(|z|) \,,$$

We have

$$\phi(z) \overset{|z|\to 0}{\simeq} \sigma^2 \Big(1 - d_H \big| \frac{z}{l_c} \big|^{2H} + o\big(\big| \frac{z}{l_c} \big| \big) \Big) \,, \text{ with } d_H = \frac{1}{1-2H}$$



Realizations of a binary medium with the index H.

When $H < 1/2$: accumulation of very small intervals responsible for the short-range correlation property (left).

When $H > 1/2$: generation of intervals longer than the average responsible for the long-range correlation property (right).

Analysis of pulse propagation (1/2)

Introduce the right- and left-going wave amplitudes:

$$\begin{bmatrix} A^\varepsilon(t,z) \\ B^\varepsilon(t,z) \end{bmatrix} = \begin{bmatrix} \zeta^\varepsilon(z)^{-1/2} p^\varepsilon(t,z) + \zeta^\varepsilon(z)^{1/2} u^\varepsilon(t,z) \\ -\zeta^\varepsilon(z)^{-1/2} p^\varepsilon(t,z) + \zeta^\varepsilon(z)^{1/2} u^\varepsilon(t,z) \end{bmatrix}$$

where the local impedance and sound speed are given by

$$\zeta^\varepsilon(z) = \sqrt{K(z)\rho(z)} = \frac{\zeta_0}{\sqrt{1+\varepsilon\nu(z/\varepsilon^2)}}, \qquad c^\varepsilon(z) = \sqrt{K(z)/\rho(z)} = \frac{c_0}{\sqrt{1+\varepsilon\nu(z/\varepsilon^2)}}$$

The mode amplitudes satisfy

$$\frac{\partial}{\partial z}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} = -\frac{1}{c^\varepsilon(z)}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} + \frac{\partial_z \zeta^\varepsilon(z)}{2\zeta^\varepsilon(z)}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix}$$

Incoming wave:

$$A^\varepsilon(t,z) = f\Big(\frac{t - z/c_0}{\varepsilon^2}\Big), \qquad B^\varepsilon(t,z) = 0, \qquad t < 0$$

Main phenomena:

- transport along the random characteristics with the local sound speed $c^\varepsilon(z)$.
- coupling between the right- and left-going modes, proportional to the derivative of the logarithmic impedance $\zeta^\varepsilon(z)$.

Analysis of pulse propagation (2/2)

$$\frac{\partial}{\partial z}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} = -\frac{1}{c^\varepsilon(z)}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} + \frac{\partial_z \zeta^\varepsilon(z)}{2\zeta^\varepsilon(z)}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix}$$

Main steps of the analysis:

- rewrite the evolution equations of the modes by centering along the characteristic of the right-going mode: we obtain an upper-triangular system
- apply a limit theorem to this system in order to establish an effective equation for the wave front.

Cf: J. Garnier, K. Sølna, SIAM Multiscale Model. Simul. 7, 1302 (2009); J. Acoust. Soc. Am. 127, 62 (2010).

Effective pulse propagation - long-range correlations (1/8)

Covariance function:

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_H \left| \frac{z}{l_c} \right|^{2H-2}, \quad H \in (1/2, 1)$$

Introduce the random travel time

$$\tau^\varepsilon(L) = \frac{L}{c_0} + \varepsilon^{3-2H} \tau_c^\varepsilon(L), \qquad \tau_c^\varepsilon(L) = \frac{1}{2c_0\varepsilon^{2-2H}} \int_0^L \nu\left(\frac{z}{\varepsilon^2}\right) dz$$

In the limit $\varepsilon \to 0$, at $z = L$,

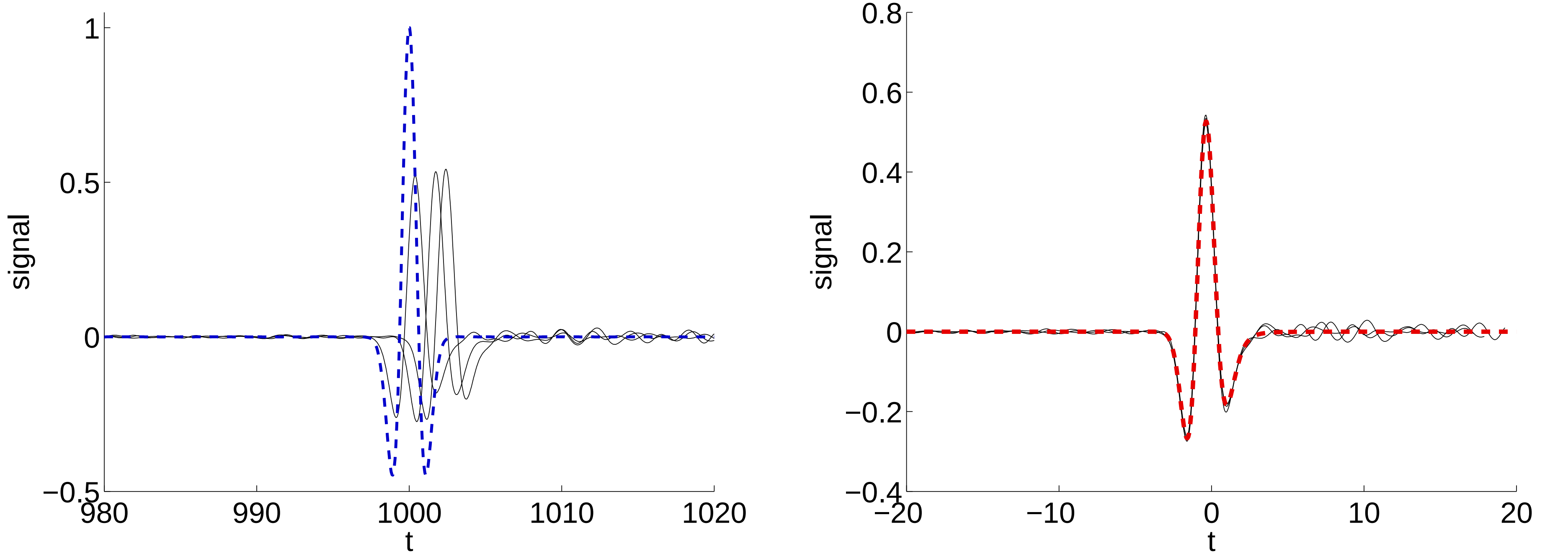
1) the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a zero-mean random variable with variance of order one,

2) the pulse profile converges in probability to a deterministic profile:

$$p\big(z = L, t = \tau^\varepsilon(L) + \varepsilon^2 \tau\big) \xrightarrow{\varepsilon\to 0} p_L(\tau)$$

Note that the time shift is of order $\varepsilon^{3-2H} \gg \varepsilon^2$ (here $H > 1/2$, i.e. $3 - 2H < 2$): the random time shift is larger than the deterministic deformation.

Effective pulse propagation - long-range correlations (2/8)



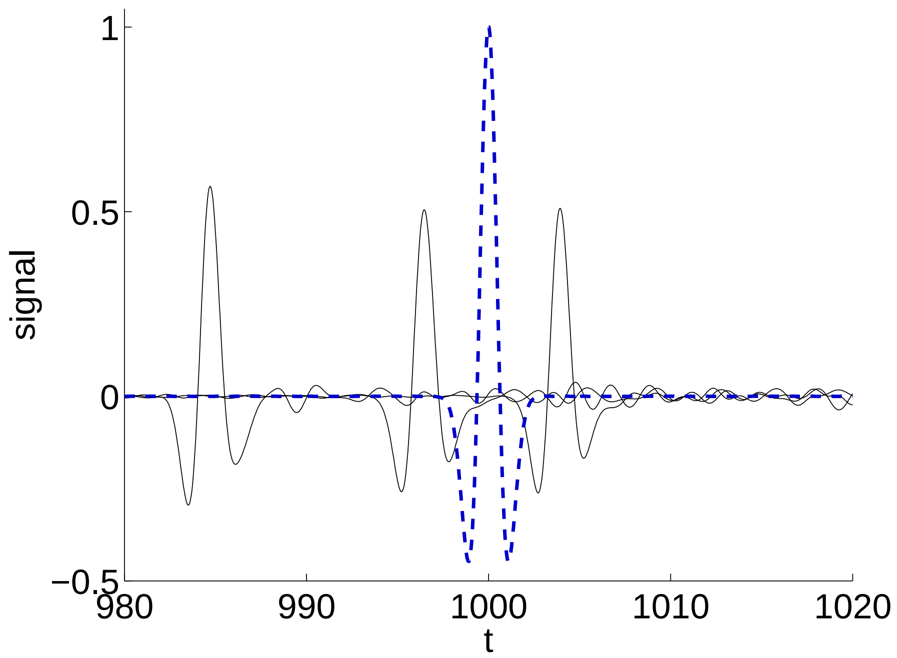
Blue: original pulse profile $p_0(t)$

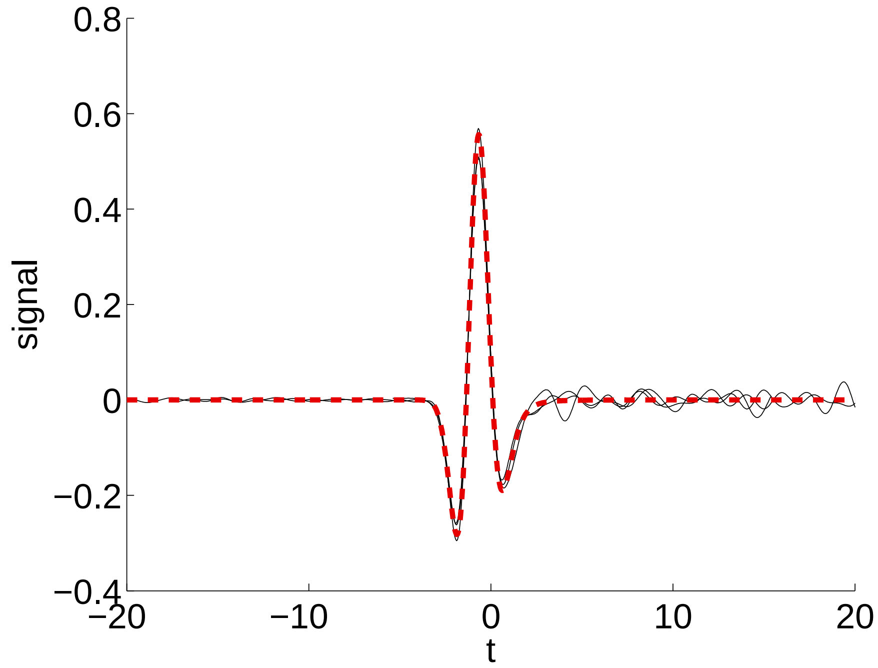
Red: theoretical transmitted profile $p_L(t)$

Black: results of three numerical simulations (with centering in the right picture).

Here $H = 0.75$.

Effective pulse propagation - long-range correlations (3/8)





Blue: original pulse profile $p_0(t)$

Red: theoretical transmitted profile $p_L(t)$

Black: results of three numerical simulations (with centering in the right picture).

Here $H = 0.875$.

Effective pulse propagation - long-range correlations (4/8)

More precisely:

1) the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a Gaussian random variable T_L with mean zero and variance ($\sim W_H(L)$):

$$\mathbb{E}\big[T_L^2\big] = \frac{l_c^{2-2H}}{c_0^2} \frac{r_H}{4H(2H-1)} L^{2H}$$

2) If $l_c \ll$ wavelength, then

$$p_L(\tau) = \frac{1}{2\pi} \int d\omega \hat{p}_0(\omega) \exp\Big(- i\omega\tau - \frac{r_H}{2} \frac{\Gamma(2H-1)}{2^{2H}} \cos\Big((H-\frac{1}{2})\pi\Big) \Big(\frac{|\omega| l_c}{c_0}\Big)^{3-2H} \frac{L}{l_c} - i \frac{r_H}{2} \frac{\Gamma(2H-1)}{2^{2H}} \sin\Big((H-\frac{1}{2})\pi\Big) \Big(\frac{|\omega| l_c}{c_0}\Big)^{2-2H} \frac{\omega l_c}{c_0} \frac{L}{l_c} \Big)$$

$\hookrightarrow$ Effective fractional diffusion (attenuation) $\sim |\omega|^{3-2H}$, $3-2H \in (1,2)$.

$\hookrightarrow$ Effective fractional dispersion $\sim |\omega|^{2-2H}\omega$.

Effective pulse propagation - long-range correlations (5/8)

Effective fractional wave equation in the "original" frame (up to random time correction):

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = \frac{r_H l_c^{2-2H}}{2^{2H} c_0^{4-2H}} \int_0^\infty \frac{1}{s^{2-2H}} \frac{\partial^3 p}{\partial t^3}(t-s)ds$$

→ same form as wave equations used in sound propagation in lossy media;
→ slightly different from the standard models [M. Caputo, Geophys. J. R. Astron. Soc. 1, 529 (1967), T. L. Szabo, J. Acoust. Soc. Am. 96, 491 (1994)];
→ equivalent to the model proposed in [W. Chen and S. Holm, J. Acoust. Soc. Am. 114, 2570 (2003)];
→ respects causality (dispersion relation satisfies Kramers-Kronig relation).

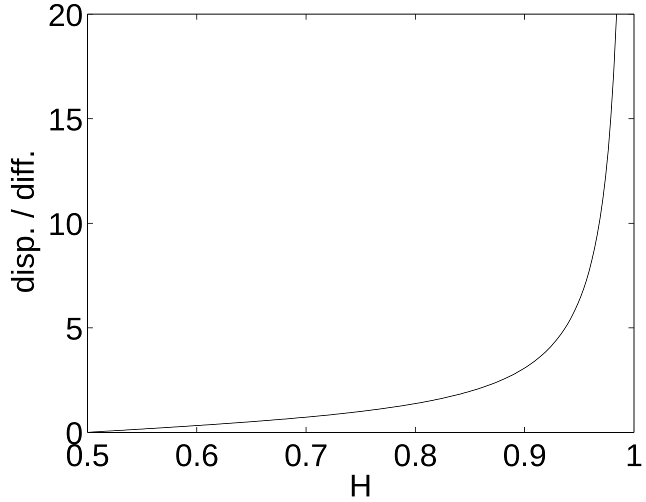
Effective pulse propagation - long-range correlations (6/8)

$$\phi(z) \overset{|z|\to\infty}{\sim} r_H \left|\frac{z}{l_c}\right|^{2H-2}, \quad H \in (1/2,1)\,, \qquad \text{and } l_c \ll \text{ wavelength}$$

$$\Downarrow$$

$$p_L(\tau) = \frac{1}{2\pi}\int d\omega \hat{p}_0(\omega)\exp\Big(-i\omega\tau - \frac{r_H}{2}\frac{\Gamma(2H-1)}{2^{2H}}\cos\Big((H-\frac{1}{2})\pi\Big)\Big(\frac{|\omega| l_c}{c_0}\Big)^{3-2H}\frac{L}{l_c}$$
$$-i\frac{r_H}{2}\frac{\Gamma(2H-1)}{2^{2H}}\sin\Big((H-\frac{1}{2})\pi\Big)\Big(\frac{|\omega| l_c}{c_0}\Big)^{2-2H}\frac{\omega l_c}{c_0}\frac{L}{l_c}\Big)$$

- Remark 1: Only r_H and H can be estimated from experiments (statistically stable).
- Remark 2: Dispersion (resp. diffusion) is dominant for H close to 1 (resp. 1/2).



Effective pulse propagation - long-range correlations (7/8)

- Remark 3: The mean field approach is qualitatively wrong.

$$p^{\mathrm{MF},\varepsilon}(L,\tau) := \mathbb{E}\left[p^{\varepsilon}\left(L, \frac{L}{c_0} + \varepsilon^2 \tau\right)\right]$$

The analysis of the mean field exhibits an additional frequency-dependent decay that comes from the averaging with respect to the random time shift. This term is strong, it becomes of order one for a small propagation distance, of order $\varepsilon^{2-\frac{1}{H}}$:

$$p^{\mathrm{MF},\varepsilon}\left(\varepsilon^{2-\frac{1}{H}} z, \tau\right) \stackrel{\varepsilon\to 0}{\longrightarrow} p^{\mathrm{MF}}(z,\tau)$$

where the asymptotic mean field is

$$p^{\mathrm{MF}}(z,\tau) = \frac{1}{2\pi}\int d\omega \hat{p}_0(\omega) \exp\Big(-i\omega\tau - \frac{r_H}{8H(2H-1)} \frac{\omega^2 l_c^2}{c_0^2} \frac{z^{2H}}{l_c^{2H}}\Big)$$

Thus, the mean field is described by an anomalous diffusion equation.

Effective pulse propagation - long-range correlations (8/8)

If the initial pulse has the Gaussian shape:

$$p_0(\tau) = a_0 \exp\Big(-\frac{\tau^2}{2T_0^2}\Big)$$

then the output mean pulse is given by

$$p^{\mathrm{MF}}(z,\tau) = a(z) \exp\Big(-\frac{\tau^2}{2T(z)^2}\Big)$$

where the width of the mean pulse increases as

$$T(z)^2 = T_0^2 + \frac{r_H l_c^2}{4H(2H-1)c_0^2} \frac{z^{2H}}{l_c^{2H}}$$

and its amplitude $a(z)$ decays as

$$a(z) = a_0 \frac{T_0}{T(z)}$$

This anomalous diffusion is strong, much larger than the diffusion for the randomly centered wave front, but it is not physical as it is determined by the averaging with respect to the random time shift.

Effective pulse propagation - short-range correlations (1/3)

$$\phi(z) \overset{|z|\to 0}{\simeq} \phi(0)\Big(1 - d_H \Big|\frac{z}{l_c}\Big|^{2H} + O\Big(\Big|\frac{z}{l_c}\Big|\Big)\Big), \quad H \in (0, 1/2)$$

In the limit $\varepsilon \to 0$, at $z = L$,
the pulse profile converges in probability to a deterministic profile:

$$p\big(z = L, t = \frac{L}{c_0} + \varepsilon^2 \tau\big) \xrightarrow{\varepsilon \to 0} p_L(\tau)$$

Note that the time shift is vanishing.

If wavelength $\ll l_c$, then

$$p_L(\tau) = \frac{1}{2\pi} \int d\omega \hat{p}_0(\omega) \exp\Big(- i\omega\tau - \frac{\phi(0) d_H}{8} \frac{\Gamma(1+2H)}{2^{2H}} \sin(H\pi) \Big(\frac{|\omega| l_c}{c_0}\Big)^{1-2H} \frac{L}{l_c} - i \frac{\phi(0)}{8} \frac{\omega l_c}{c_0} \frac{L}{l_c} + i \frac{\phi(0) d_H}{8} \frac{\Gamma(1+2H)}{2^{2H}} \cos(H\pi) \Big(\frac{|\omega| l_c}{c_0}\Big)^{-2H} \frac{\omega l_c}{c_0} \frac{L}{l_c}\Big)$$

$\hookrightarrow$ Effective fractional diffusion (attenuation) $\sim |\omega|^{1-2H}$, $1 - 2H \in (0, 1)$.
$\hookrightarrow$ Effective fractional dispersion $\sim |\omega|^{-2H} \omega$.

Effective pulse propagation - short-range correlations (2/3)

Effective fractional wave equation in the "original" frame (up to random time correction):

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = \frac{H d_H}{2^{1+2H} c_0^{2-2H} l_c^{2H}} \int_0^\infty \frac{1}{s^{1-2H}} \frac{\partial^2 p}{\partial t^2}(t-s) ds$$

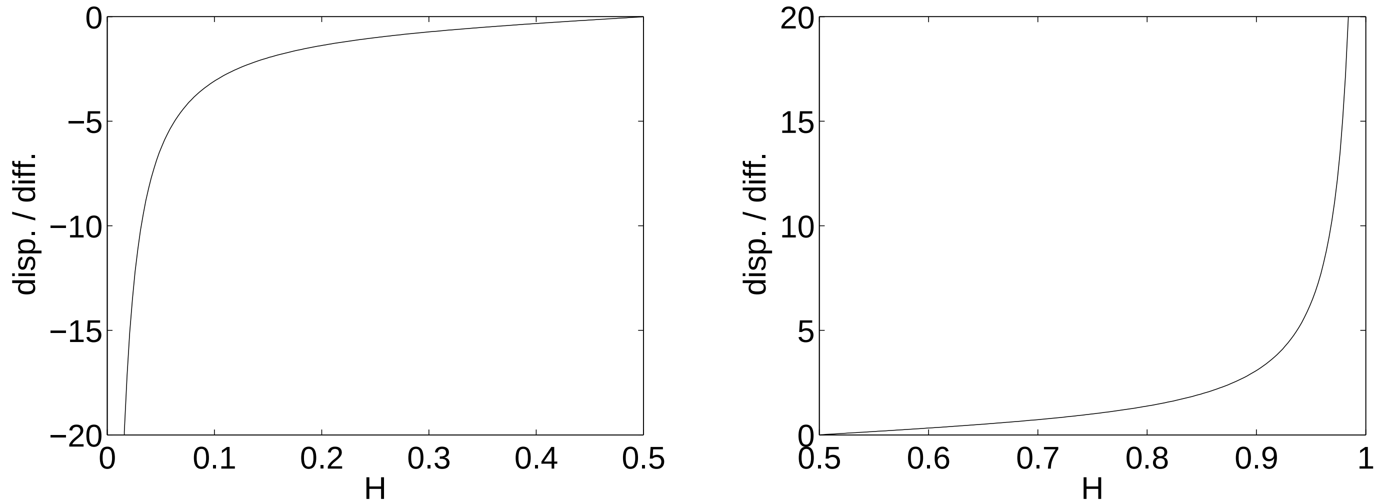
Effective pulse propagation - short-range correlations (3/3)

$$\phi(z) \overset{|z|\to 0}{\simeq} \phi(0)\Big(1 - d_H\Big|\frac{z}{l_c}\Big|^{2H} + O\Big(\Big|\frac{z}{l_c}\Big|\Big)\Big), \quad H \in (0,1/2), \quad \text{and wavelength} \ll l_c$$

$$\Downarrow$$

$$p_L(\tau) = \frac{1}{2\pi}\int d\omega \hat{p}_0(\omega)\exp\Big(- i\omega\tau - \frac{\phi(0)d_H}{8}\frac{\Gamma(1+2H)}{2^{2H}}\sin(H\pi)\Big(\frac{|\omega|l_c}{c_0}\Big)^{1-2H}\frac{L}{l_c} - i\frac{\phi(0)}{8}\frac{\omega l_c}{c_0}\frac{L}{l_c} + i\frac{\phi(0)d_H}{8}\frac{\Gamma(1+2H)}{2^{2H}}\cos(H\pi)\Big(\frac{|\omega|l_c}{c_0}\Big)^{-2H}\frac{\omega l_c}{c_0}\frac{L}{l_c}\Big)$$

- Remark 1: Only $\phi(0)d_H$ and H can be estimated from experiments.
- Remark 2: Dispersion (resp. diffusion) is dominant for H close to 0 (resp. 1/2).



- Remark 3: The mean field approach is correct.

Conclusion

• When a long-wavelength pulse propagates in a random medium with an autocorrelation function that decays at infinity as $|z|^{2H-2}$, $H \in (1/2, 1)$, then the attenuation has a power law frequency-dependence $\alpha(\omega) = \alpha_0 |\omega|^y$ with $y = 3 - 2H \in (1, 2)$. The random time shift is large (described in terms of a fBM).

• When a short-wavelength pulse propagates in a random medium with an autocorrelation function that behaves at zero like $1 - d_H |z|^{2H}$, $H \in (0, 1/2)$, then the attenuation has a power law frequency-dependence $\alpha(\omega) = \alpha_0 |\omega|^y$ with $y = 1 - 2H \in (0, 1)$. The random time shift is negligible.

• A special frequency-dependent phase is associated to the frequency-dependent attenuation and it ensures that causality and Kramers-Kronig relations are respected.

• Effective fractional wave equations can be written that have the form of equations studied in the literature in the context of wave propagation in lossy media.

• Existence of fractional precursors.

Cf: J. Garnier and K. Sølna, Waves in Random and Complex Media 20, 122 (2010).

Fractional precursors

- When a broadband pulse of the form

$$p_0(\tau) = \sin(\omega_0 \tau) \mathbf{1}_{[0,\infty)}(\tau)$$

penetrates into a dissipative and dispersive medium, phase dispersion and frequency-dependent attenuation alter the pulse in a way that results in the appearance of a precursor field with an algebraic decay.

- Precursors exist in non-dispersive, non-dissipative, but randomly heterogeneous and multiscale media.

- The shape of the precursor and its fractional power law decay with propagation distance L depend on the random medium class.

(i) In mixing random media, the precursor has a Gaussian profile in time τ and a peak amplitude that decays as $L^{-1/2}$.

(ii) In short-range correlation media ($H < 1/2$), the precursor has a skewed shape with a right tail that decays in time as $\tau^{-(2-2H)}$ and a peak amplitude that decays as $L^{-1/(1-2H)}$.

(iii) In long-range correlation media ($H > 1/2$), the precursor has a skewed shape with a right tail that decays in time as $\tau^{-(4-2H)}$ and a peak amplitude that decays as $L^{-1/(3-2H)}$.