Kinetic limits and imaging models for waves in random media

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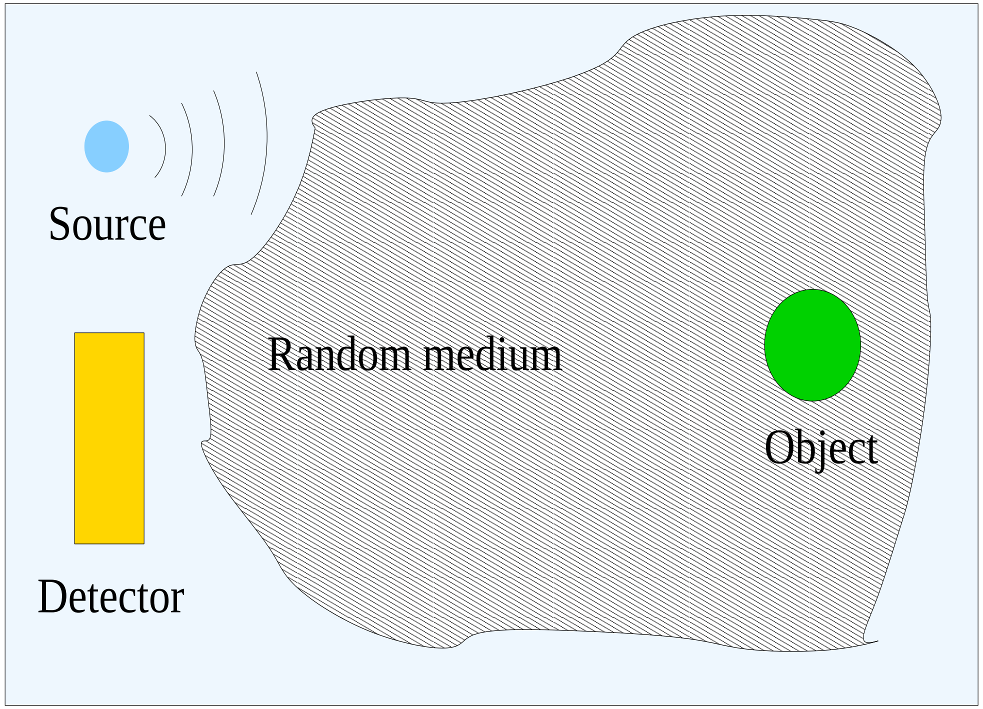
Introduction

Our generic problem : from detector measurements, image a buried inclusion in a random medium.

- The random medium models an unknown heterogeneous medium (atmosphere, forest, ocean, etc)
- We are interested in a regime in which the interaction between the wave and the medium is strong
- To make this precise, we need to identify the main parameters.



4 important parameters :

- the overall distance of propagation L
- the wavelength λ
- the correlation length l
- the strength of the fluctuations σ

Ex : ocean, $l = 1 - 25$m, σ a few %, sonar $\lambda = 1 - 60$m, L kilometers.

We use the scalar wave equation to model the propagation :

$$\frac{\partial^2 p}{\partial t^2} = c^2(\mathbf{x})\Delta p, \qquad p(0,\mathbf{x}) = p_0\left(\frac{\mathbf{x}}{\lambda}\right), \quad \partial_t p(0,\mathbf{x}) = p_1\left(\frac{\mathbf{x}}{\lambda}\right),$$

where $c(\mathbf{x}) = (\rho_0 \kappa(\mathbf{x}))^{-1/2}$ is the sound speed. We make the high frequency assumption that

$$\frac{\lambda}{L} = \varepsilon \ll 1.$$

We assume moreover that c takes the form

$$c^2(\mathbf{x}) = c_0^2 + \sigma V\left(\frac{\mathbf{x}}{l}\right),$$

where c_0 is the (slow varying) background sound speed (supposed to be known) and V is a mean-zero stationary process with correlation function R, that is

$$\mathbb{E}(V) = 0 \qquad ; \qquad \mathbb{E}(V(\mathbf{x}+\mathbf{y})V(\mathbf{y})) = R(\mathbf{x}).$$

- Several choices are possible for l : we assume here that it is of order of the wavelength so that the interaction between the wave and the medium is maximal. Thus $l \approx \lambda$. The alternatives $l \ll \lambda$ or $\lambda \ll l$ lead to different asymptotical regimes.
- the strength of the fluctuations is supposed to be $\sqrt{\varepsilon}$. This is the intensity that provides an effect of order one of the medium over large propagation distances.

Therefore c reads

$$c^2(\mathbf{x}) = c_0^2 + \sqrt{\varepsilon} V\left(\frac{\mathbf{x}}{\varepsilon}\right).$$

There is another important parameter : the mean free path $c_0\Sigma^{-1}$ (depends on λ and R), that can be interpreted as the distance between 2 interactions of the wave and the medium.

Let D be the distance between the inclusion and the source.

- If $2D \leq c_0\Sigma^{-1}$, coherent regime : the wave has weakly interacted with the medium, the wave front can me measured. In this regime, interferometry methods perform well (L. Borcea, G. Papanicolaou, K. Sølna, C. Tsogka)
- If $2D \gg c_0\Sigma^{-1}$, incoherent regime : the wave strongly interacts with the medium, the wave front is not available. We need a model to describe the multiple interactions : transport equations.

In our configuration : $2D \approx 320\lambda$, $c_0\Sigma^{-1} \approx 40\lambda$, so that $2D \approx 8c_0\Sigma^{-1}$. This is not the diffusive regime yet.

Introduction

These transport equations are obtained through a high frequency asymptotics of the random wave equation using Wigner transforms : one first recast the wave equation as a first-order system posed for $\mathbf{u}^\varepsilon = (\mathbf{v}^\varepsilon, p^\varepsilon)$

$$\rho_0 \frac{\partial \mathbf{v}^\varepsilon}{\partial t} + \nabla p^\varepsilon = 0, \qquad \kappa^\varepsilon \frac{\partial p^\varepsilon}{\partial t} + \nabla \cdot \mathbf{v}^\varepsilon = 0, \qquad \text{on } \mathbb{R}^d,$$

$$\kappa^\varepsilon(\mathbf{x}) = \kappa_0 + \sqrt{\varepsilon}\kappa_1\left(\frac{\mathbf{x}}{\varepsilon}\right).$$

The Wigner transform is defined as

$$W^\varepsilon(t,\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\,\mathbf{k}\cdot\mathbf{y}}\,\mathbf{u}^\varepsilon(t,\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2}) \otimes \mathbf{u}^\varepsilon(t,\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2})\,d\mathbf{y}.$$

and the wave energy is given by

$$\mathcal{E}^\varepsilon(t,\mathbf{x}) := \frac{1}{2}\left(\kappa^\varepsilon(\mathbf{x})|p^\varepsilon|^2(t,\mathbf{x}) + \rho_0|\mathbf{v}^\varepsilon|^2(t,\mathbf{x})\right).$$

Introduction

It is well-known, at least formally [Ryzhik-Papanicolaou-Keller 96], that, for irrotational initial conditions :

$$\lim_{\varepsilon\to 0}\mathbb{E}\{\mathcal{E}^\varepsilon\}(t,\mathbf{x}) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}(\mathsf{Tr} M_1\,\mathbb{E}\{W^\varepsilon\}\,M_2)(t,\mathbf{x},\mathbf{k})d\mathbf{k} = \int_{\mathbb{R}^d} a(t,\mathbf{x},\mathbf{k})d\mathbf{k},$$

where a is an amplitude solution to a radiative transfert equation

$$\frac{\partial a}{\partial t} + c_0\,\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} a + \Sigma\, a = Q(a), \qquad \hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|},$$

$$Q(a) = \int_{\mathbb{R}^d} a(t,\mathbf{x},\mathbf{p})\sigma(\mathbf{k},\mathbf{p})\delta(c_0|\mathbf{p}| - c_0|\mathbf{k}|)d\mathbf{p}, \quad \sigma(\mathbf{k},\mathbf{p}) = \frac{\pi c_0^2|\mathbf{k}|^2}{2(2\pi)^d}\hat{R}(\mathbf{k}-\mathbf{p}),$$

$$\Sigma = \int_{\mathbb{R}^d}\sigma(\mathbf{k},\mathbf{p})\delta(c_0|\mathbf{p}| - c_0|\mathbf{k}|)d\mathbf{p}.$$

Introduction

From measurements of $\mathcal{E}^\varepsilon$, our approach consists in using a and the related transport equations (that only depend on $\hat{R}$) to solve the inverse problem rather than using the wave description.

Main Problem : a describes the limit of $\mathbb{E}\{\mathcal{E}^\varepsilon\}$ and not that of $\mathcal{E}^\varepsilon$. In practical experiments, one cannot compute averages since one has access to one realization of the random medium only, the physical medium.

One therefore expects the following self-averaging property to hold when $\varepsilon \to 0$

$$\mathcal{E}^\varepsilon \sim \mathbb{E}\{\mathcal{E}^\varepsilon\}. \tag{1}$$

Quantifying precisely this relation is essential for our imaging problem since

$$\delta\mathcal{E}^\varepsilon = \mathcal{E}^\varepsilon - \mathbb{E}\{\mathcal{E}^\varepsilon\}$$

is the main source of noise.

Introduction

We present in this talk simplified situations where (1) can me made precise so as to

- obtain optimal error estimates
- quantify the dependance on some parameters of the problem as the regularity of the initial conditions or the size of the support of the test functions
- characterize the first-order corrector (at least its covariance)
- obtain convergence when the random medium has some long-range interactions.

This is done within the paraxial approximation in which the wave propagation is described by a Schrödinger equation.

There are many results in the literature adressing the convergence of $\mathbb{E}\{\mathcal{E}^\varepsilon\}$ (or of $\mathcal{E}^\varepsilon$ in probability) for the Schrödinger equation, see e.g. Erdös-Yau, Bal-Papanicolaou-Ryzhik, Fannjianng, Poupaud-Vasseur, without analyzing the convergence rate.

The first regime that allows a careful analysis is the Itô-Schrödinger regime.

Outline

- Itô-Schrödinger regime
- Time-independent potentials : simple and double scattering estimates
- Long-range correlations
- Reconstruction results

The Itô-Schrödinger equation reads, for $d \geq 2$:

$$d\psi_\eta(z,\mathbf{x}) = \frac{1}{2} i\eta \Delta_{\mathbf{x}} \psi_\eta(z,\mathbf{x}) dz + i\psi_\eta(z,\mathbf{x}) \circ dB\Big(\frac{\mathbf{x}}{\eta}, z\Big), \quad (z,\mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^d$$

where B is a Wiener process with correlation function

$$\mathbb{E}\{B(\mathbf{x},z)B(\mathbf{y},z')\} = R(\mathbf{x}-\mathbf{y}) z \wedge z', \qquad R \in L^1 \cap L^\infty.$$

We choose an initial condition of the form (a pure state)

$$\psi_\eta^0(\mathbf{x}) = \frac{1}{\eta^{\frac{d\alpha}{2}}} \chi\Big(\frac{\mathbf{x}-\mathbf{x}_0}{\eta^\alpha}\Big) e^{i\frac{(\mathbf{x}-\mathbf{x}_0)\cdot \mathbf{k}_0}{\eta}},$$

where $\chi \in \mathcal{S}(\mathbb{R}^d)$. ψ_η^0 is bounded in L^2 indep. of η.

The Itô-Schrödinger regime

The Wigner transform satisfies the random Wigner equation

$$(dW_\eta + \mathbf{k}\cdot\nabla_{\mathbf{x}} W_\eta\, dz)(z,\mathbf{x},\mathbf{k}) =$$
$$\frac{i}{(2\pi)^d}\int_{\mathbb{R}^d} d\mathbf{p}\, e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\eta}} \Big(W_\eta(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}) - W_\eta(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})\Big)\circ d\hat{B}(\mathbf{p},z),$$

with $\hat{B} = \mathcal{F}_{\mathbf{x}\to\mathbf{p}} B$ and the related initial condition reads

$$W_{\eta 0}(\mathbf{x},\mathbf{k}) = \frac{1}{\eta^d} W_0\left(\frac{\mathbf{x}-\mathbf{x}_0}{\eta^\alpha}, \frac{\mathbf{k}-\mathbf{k}_0}{\eta^{1-\alpha}}\right).$$

The main tool in the analysis is the scintillation function J_η , defined as

$$J_\eta(z,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) = \mathbb{E}\{W_\eta(z,\mathbf{x},\mathbf{k})W_\eta(z,\mathbf{y},\mathbf{p})\} - \mathbb{E}\{W_\eta(z,\mathbf{x},\mathbf{k})\}\mathbb{E}\{W_\eta(z,\mathbf{y},\mathbf{p})\},$$

whose weak convergence to zero implies the convergence in probability :

$$\mathbb{P}\Big(|\langle W_\eta(z),\varphi\rangle - \langle \mathbb{E}\{W_\eta\}(z),\varphi\rangle| \geq \delta\Big) \leq \frac{1}{\delta^2}\langle J_\eta(z),\varphi\otimes\varphi\rangle.$$

The key point the is that J_η satisfies a closed-form equation (it is a consequence of the fast decorrelation in time)

$$\Big(\frac{\partial}{\partial t} + \mathcal{T}_2 + 2R(0) - \mathcal{Q}_2 - \mathcal{K}_\eta\Big) J_\eta = \mathcal{K}_\eta a_\eta \otimes a_\eta \quad \text{on } \mathbb{R}^{2d} \times \mathbb{R}^{2d},$$

$$J^\eta(t=0) = 0,$$

$$\mathcal{T}_2 = \mathbf{k}\cdot\nabla_{\mathbf{x}} + \mathbf{p}\cdot\nabla_{\mathbf{y}}, \qquad a_\eta = \mathbb{E}\{W_\eta\},$$

$$\mathcal{Q}_2 h = \int_{\mathbb{R}^{2d}} \Big(\hat{R}(\mathbf{k}-\mathbf{k}')\delta(\mathbf{p}-\mathbf{p}') + \hat{R}(\mathbf{p}-\mathbf{p}')\delta(\mathbf{k}-\mathbf{k}')\Big) h(\mathbf{x},\mathbf{k}',\mathbf{y},\mathbf{p}')d\mathbf{k}'d\mathbf{p}',$$

$$\mathcal{K}_\eta h = \sum_{\epsilon_i,\epsilon_j=\pm 1} \epsilon_i\epsilon_j \int_{\mathbb{R}^d} \hat{R}(\mathbf{u}) e^{i\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{u}}{\eta}} h\left(\mathbf{x}, \mathbf{k}+\epsilon_i\frac{\mathbf{u}}{2}, \mathbf{y}, \mathbf{p}+\epsilon_j\frac{\mathbf{u}}{2}\right) d\mathbf{u}.$$

Performing the limit $\eta \to 0$ in the system above yields the following result :

The Itô-Schrödinger regime

Theorem [Bal - P., CPDE 2010] For all $\alpha \in [0,1]$, $J_\eta \to 0$ weakly in $\mathcal{S}'$ and more precisely

$$J_\eta = g_1(\eta,\alpha,d)J_1 + g_2(\eta,\alpha,d)J_2 + r_\eta,$$

where r_η is negligible in $\mathcal{S}'$ compared to the first 2 terms. J_1 and J_2 satisfy 4-transport equations of the form

$$\Big(\frac{\partial}{\partial t} + \mathcal{T}_2 + 2R(0) - \mathcal{Q}_2\Big)J_i = S_i(\alpha,d) \quad \text{on } \mathbb{R}^{2d}\times\mathbb{R}^{2d},$$

$$J_i(t=0) = J_i^0(\alpha,d),$$

The data for J_1 are linear with respect to $\hat{R}$ (simple scattering) while that of J_2 are quadratic (double scattering). Higher-order scattering events therefore produce negligible instabilities compared to that of the simple and double scattering.

The simple scattering contribution dominates when $\alpha > \frac{2}{3}$.

The Itô-Schrödinger regime

When $\alpha > \frac{1}{2}$, $S_i = 0 \Rightarrow$ instabilities generated by an initial condition

(This is consistant with a recent result of Komorowski-Ryzhik)

When $\alpha \leq \frac{1}{2}$, $J_i^0 = 0 \Rightarrow$ instabilities generated by a source term.

Most stable configuration : $\alpha = 0 \Rightarrow g_2 = \eta^d$.

Least stable configuration : $\alpha = 1 \Rightarrow g_1 = \eta$.

We also have the following result : consider a test function of the form

$$\varphi_{s_1}(\mathbf{x}, \mathbf{k}) = \frac{1}{\eta^{s_1 d}} \varphi\Big(\frac{\mathbf{x}}{\eta^{s_1}}, \mathbf{k}\Big).$$

Then, for $\alpha = 0$

$$\langle J_\eta, \varphi_{s_1} \otimes \varphi_{s_1} \rangle = \mathcal{O}\big(\eta^{d(1-s_1)}\big).$$

Hence, when the initial condition has a support of order one compared to η, statistical stability holds when the detector is of size $\varepsilon^{-s_1 d}$, $s_1 < 1$.

Ideas of the proof

One needs a functional setting adapted to

- Wigner transforms,
- the analysis of the operator $\mathcal{K}_\eta$,
- the scintillation equation.

Prototype spaces :

$$\sup_{\mathbf{u},\boldsymbol{\xi}\in\mathbb{R}^d} |\mathcal{F}a_\eta(\mathbf{u},\boldsymbol{\xi})| < \infty \qquad \int_{\mathbb{R}^{4d}} \omega(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})|\mathcal{F}h(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})|d\boldsymbol{\xi}d\mathbf{u}d\mathbf{v}d\boldsymbol{\zeta} < \infty$$

Other ingredients :

- Integral formulation of the scintillation equation
- decomposition of a_η in terms of multiple scatterings :

$$a_\eta = \text{ballistic part } + \text{multiple scatterings}$$

- regularizing effect of the collisions
- dispersive effect of type $\|a_\eta(t)\| \leq \frac{C_\eta}{t^d}.$

The Schrödinger equation with time-independent potential reads in the weak coupling regime :

$$\left(i\eta \frac{\partial}{\partial t} + \frac{\eta^2}{2}\Delta_{\mathbf{x}} + \sqrt{\eta} V\left(\frac{\mathbf{x}}{\eta}\right)\right) \psi_\eta(t,\mathbf{x}) = 0, \quad t > 0, \quad \mathbf{x} \in \mathbb{R}^d,$$

$\psi_\eta(0,\mathbf{x}) = \psi_\eta^0(\mathbf{x})$ bounded in L^2.

We know from the Erdös-Yau result that when $d \geq 2$, $\mathbb{E}\{W_\eta\}$ converges to a solution to a radiative transfer equation. Up to our knowledge, there is no result about the convergence of the whole process W_η.

We analyze here the scintillation of the simple and double scattering contributions (that are expected to be dominant in some regimes according the Itô-Schrödinger case) and also obtain optimal estimates.

Another regime : time independent potential

The random Wigner equation reads :

$$\left(\frac{\partial}{\partial t} + \mathbf{k}\cdot\nabla_{\mathbf{x}}\right) W_\eta(z,\mathbf{x},\mathbf{k}) =$$
$$\frac{i}{\sqrt{\eta}(2\pi)^d}\int_{\mathbb{R}^d} d\mathbf{p}\, d\hat{V}(\mathbf{p})\, e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\eta}} \left(W_\eta(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}) - W_\eta(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})\right)$$

We then formally expand W_η in terms of multiple scattering and only retain the terms at most quadratic in $\hat{R}$. The corresponding scintillation is

$$J_\eta = J_\eta^S + J_\eta^D.$$

We consider again an initial condition of the form

$$W_{\eta 0}(\mathbf{x},\mathbf{k}) = \frac{1}{\eta^d} W_0\left(\frac{\mathbf{x}-\mathbf{x}_0}{\eta^\alpha}, \frac{\mathbf{k}-\mathbf{k}_0}{\eta^{1-\alpha}}\right).$$

Let

$$w_\eta^S(t) = \langle J_\eta^S(t), \varphi \otimes \varphi \rangle.$$

Then, pointwise in time, we have, $\forall d \geq 1$:

$$\eta^{-d(1-\alpha)-1+\alpha\wedge(1-\alpha)} w_\eta^S(t) \to w^S(t), \qquad a \wedge b = \min(a,b).$$

This means that the scintillation of the simple scattering is of order

$$\eta^{d(1-\alpha)+1-\alpha\wedge(1-\alpha)}$$

- Most stable configuration : $\alpha = 0 \Rightarrow \eta^{d+1}$ (to be compared with η^{d+2} for Itô-S.).
- Least stable configuration : $\alpha = 1$ ($\alpha \geq \frac{1}{2}$ in 1D) $\Rightarrow \eta$ (same as Itô-S.)
  - The simple scattering is always stable, even in 1D
  - Localization is generated in 1D by higher order scattering events

Let

$$w_\eta^D(t) = \langle J_\eta^D(t), \varphi \otimes \varphi \rangle.$$

Then, pointwise in time, we have, for $d = 1$:

$$\eta^{-\alpha} w_\eta^D(t) \to w^D(t),$$

and for $d \geq 2$

$$\eta^{-d(1-\alpha)-(2\alpha-1)-0\wedge(2\alpha-1)} w_\eta^D(t) \to w^D(t).$$

This means that the scintillation of the double scattering is of order

$$\eta^\alpha \quad \text{for} \quad d = 1 \quad ; \quad \eta^{d(1-\alpha)+(2\alpha-1)+0\wedge(2\alpha-1)} \quad \text{for} \quad d \geq 2.$$

- For $d \geq 2$: simple scattering dominates when $\alpha > \frac{2}{3}$ (same as Itô-S.)
- For $d = 1$: double scattering dominates when $\alpha < 1$

Double scattering

- Least stable configuration for $d \geq 2 : \eta^{\frac{d}{2}}$.
- Least stable configuration for $d = 1$: scintillation of order 1 when $\alpha = 0$
  - Localization when $\alpha = 0$
  - Stability when $\alpha > 0$
  - It is known that localization arises in 1D for plane waves (see eg the book of Fouque-Garnier-Papanicolaou-Sølna). What happens for $\alpha > 0$, ie localized initial conditions ?
  - one needs to go beyond the double scattering to figure...

Long-range interactions

We assume the correlation function is slowly decreasing and satisfies

$$R(\mathbf{x}) \approx_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{\delta-d}, \qquad 0<\delta<d$$

so that R is not integrable. This yields for the Fourier transform a singularity near the origin like

$$\hat{R}(\mathbf{k}) \approx_{|\mathbf{k}|\to 0} |\mathbf{k}|^{-\delta}.$$

Short-range correlations correspond to L^1 functions R and therefore bounded $\hat{R}$.

In this context, for the Itô-Schrödinger regime, the simple and double scattering, we have the following result, pointwise in time : $\forall d \geq 1, \forall \alpha \in [0,1]$ ($\alpha \neq 0$ in 1D), $\forall \delta \in (0,d)$

$$\langle J_\eta(t), \varphi \otimes \varphi \rangle \to 0.$$

We also have some bounds from below that show (for $d \geq 2$) that the scintillation grows as $\delta \to d$.

Owing the results of statistical stability for the paraxial regime, we assume they also hold for the wave equation.

Imaging procedure (in 2D) :

- We measure the energy $\mathcal{E}^{\varepsilon}$ for one realization of the random medium.
- We assume we can form differential measurements (ie both in absence and presence of the inclusion)
  - this removes a substantial amount of noise
  - imaging with direct measurements only is possible if the inclusion is large enough or if the mfp is important
- We find the deterministic transport prediction that best fits the (weakly) random measurements.
- The procedure is carried out over 20 realizations to quantify the variance of the reconstructed parameters.

- First step : estimation of the transport parameters

For simplicity, we use a random medium associated with an isotropic cross-section, and an initial condition with only one frequency content.

- one parameter to reconstruct, the mean free time Σ^{-1}.
- we minimize over Σ^{-1}

$$\int_0^T |\mathcal{E}^\varepsilon(t) - \mathcal{A}(t)|^2 dt$$

$$\mathcal{E}^\varepsilon(t) = \int_{\mathcal{D}} \mathcal{E}^\varepsilon(t,\mathbf{x})d\mathbf{x} \qquad \mathcal{A}(t) = \int_{\mathcal{D}}\int_{S^1} a(t,\mathbf{x},|\mathbf{k}_0|\hat{\mathbf{k}})d\mathbf{x}d\hat{\mathbf{k}}$$

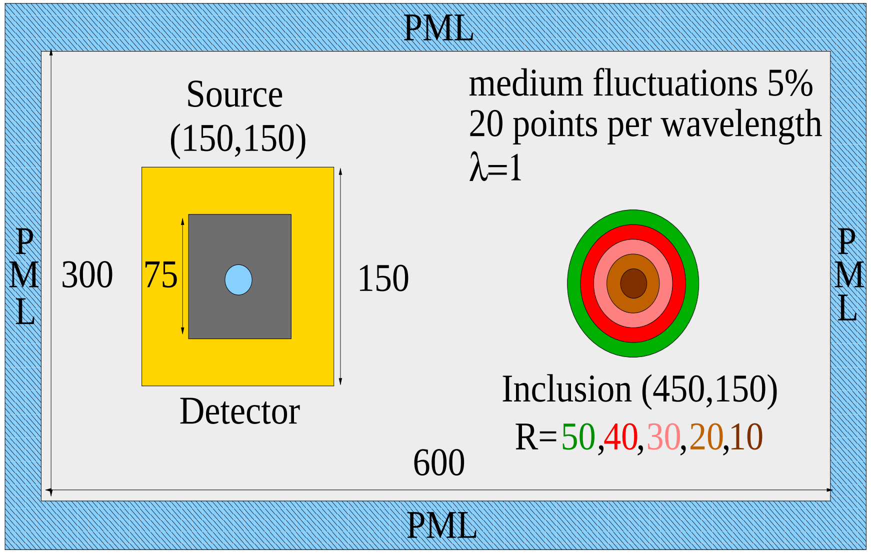
- Second step : reconstruction of the inclusion

We suppose the inclusion is spherical with an infinite sound speed.

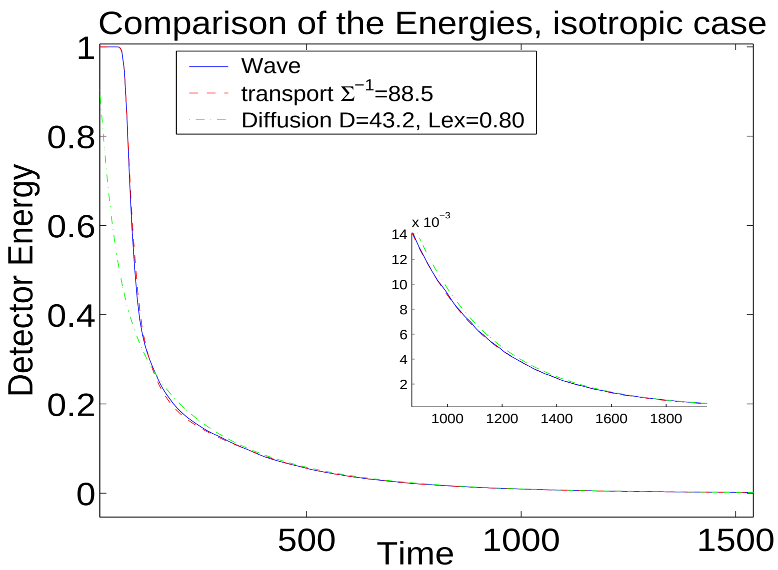
- we minimize over (X, Y, R)

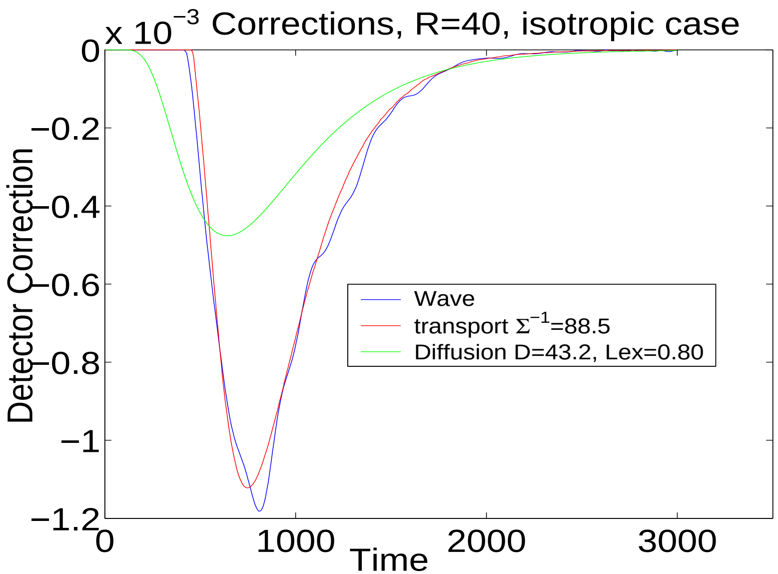
$$\int_0^T |\delta\mathcal{E}^\varepsilon(t) - \delta\mathcal{A}[X,Y,R](t)|^2 dt$$

Validity of transport



- $\lambda = 1, \varepsilon = \frac{1}{300}$
- Isotropic cross section, mean free path ≈ 90
- Isotropic initial condition with only one frequency content

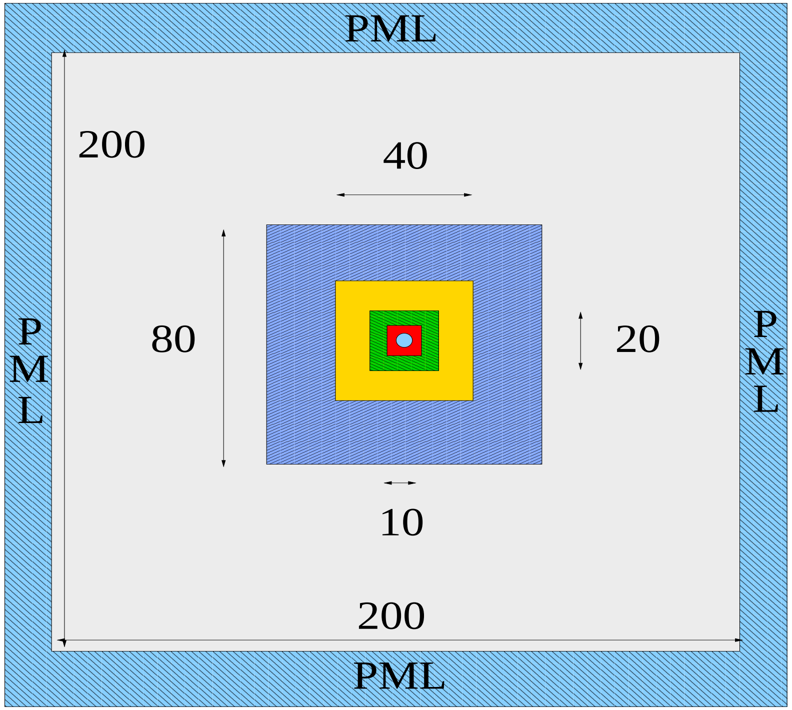


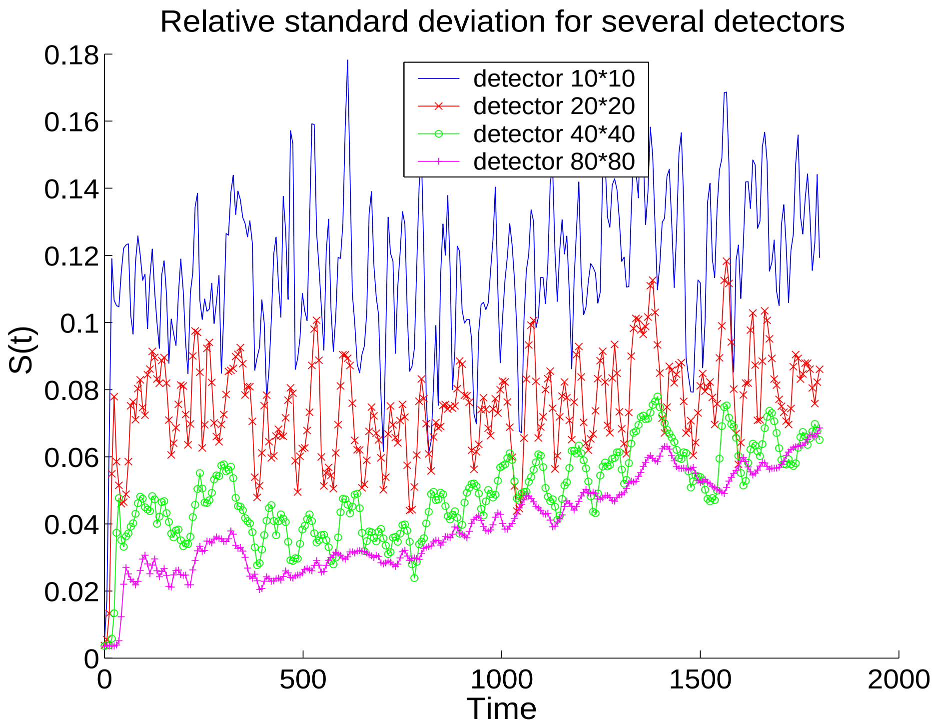


Statistical Stability

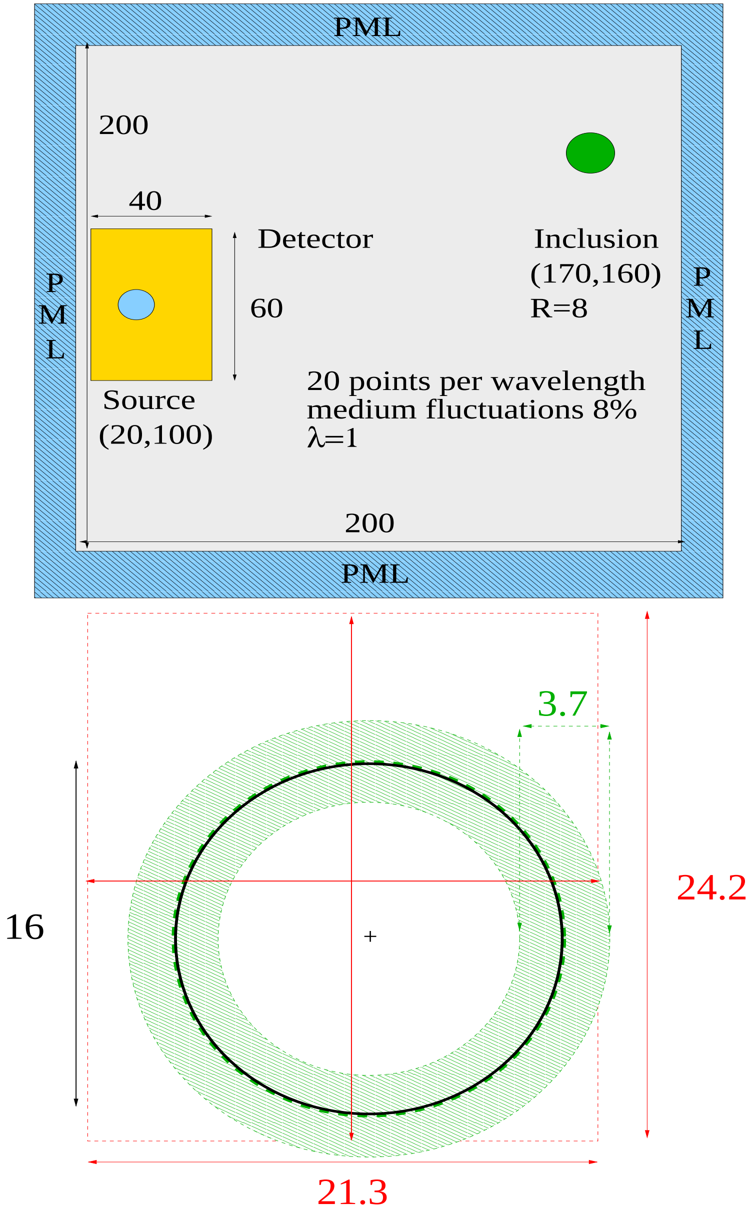
Wavelength $\lambda = 1$ ($\varepsilon = \frac{1}{200}$), mean free path ≈ 40, isotropic cross section.

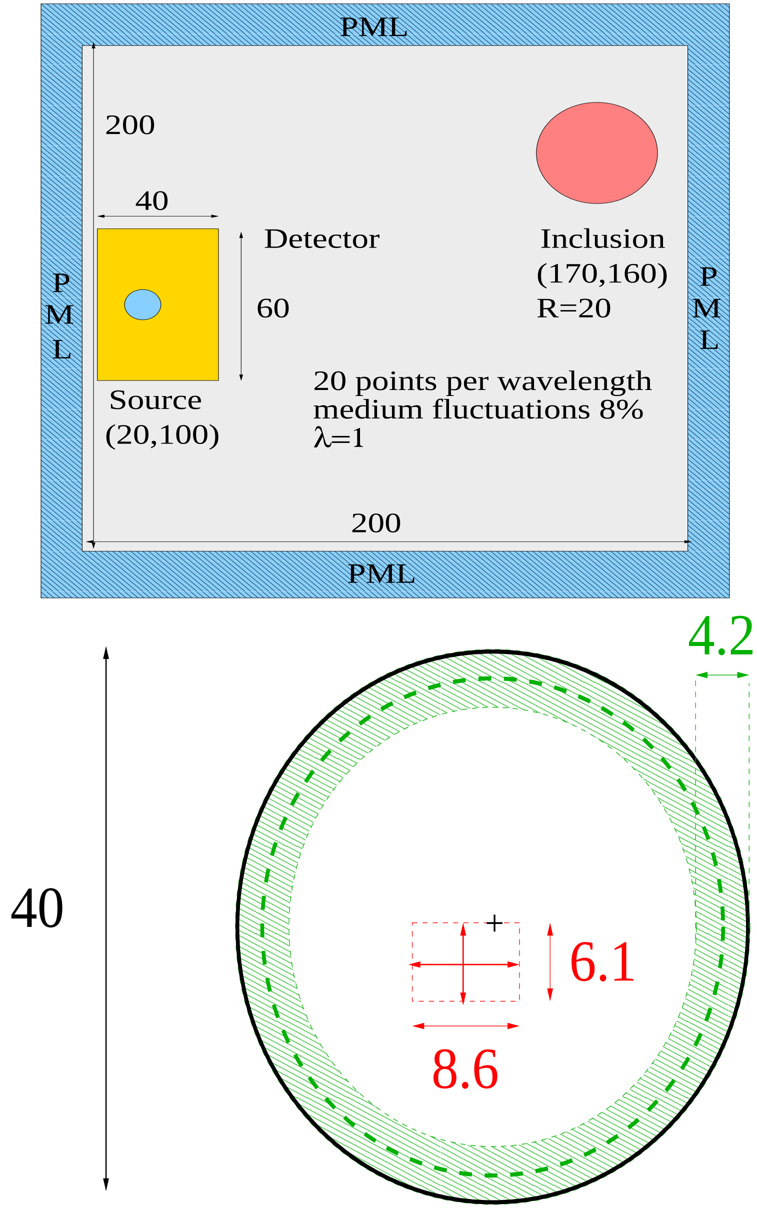
Display of $S(t) = \dfrac{\sigma\{\mathcal{E}^\varepsilon\}(t)}{\mathbb{E}\{\mathcal{E}^\varepsilon\}(t)}$ for 20 realizations.





Reconstructions (mean free path ≈ 40)





Improving the stability

The stability can be improved if one is able to form the correlation of the wavefield in presence and in absence of the inclusion :

$$\mathcal{C}^{\varepsilon}(t,\mathbf{x}) = \frac{1}{2}\left(\kappa_1^{\frac{1}{2}}\kappa_2^{\frac{1}{2}}(\mathbf{x})p_{\varepsilon}^{1}(t,\mathbf{x})\,p_{\varepsilon}^{2}(t,\mathbf{x}) + \rho_0\mathbf{v}_{\varepsilon}^{1}(t,\mathbf{x})\cdot\mathbf{v}_{\varepsilon}^{2}(t,\mathbf{x})\right).$$

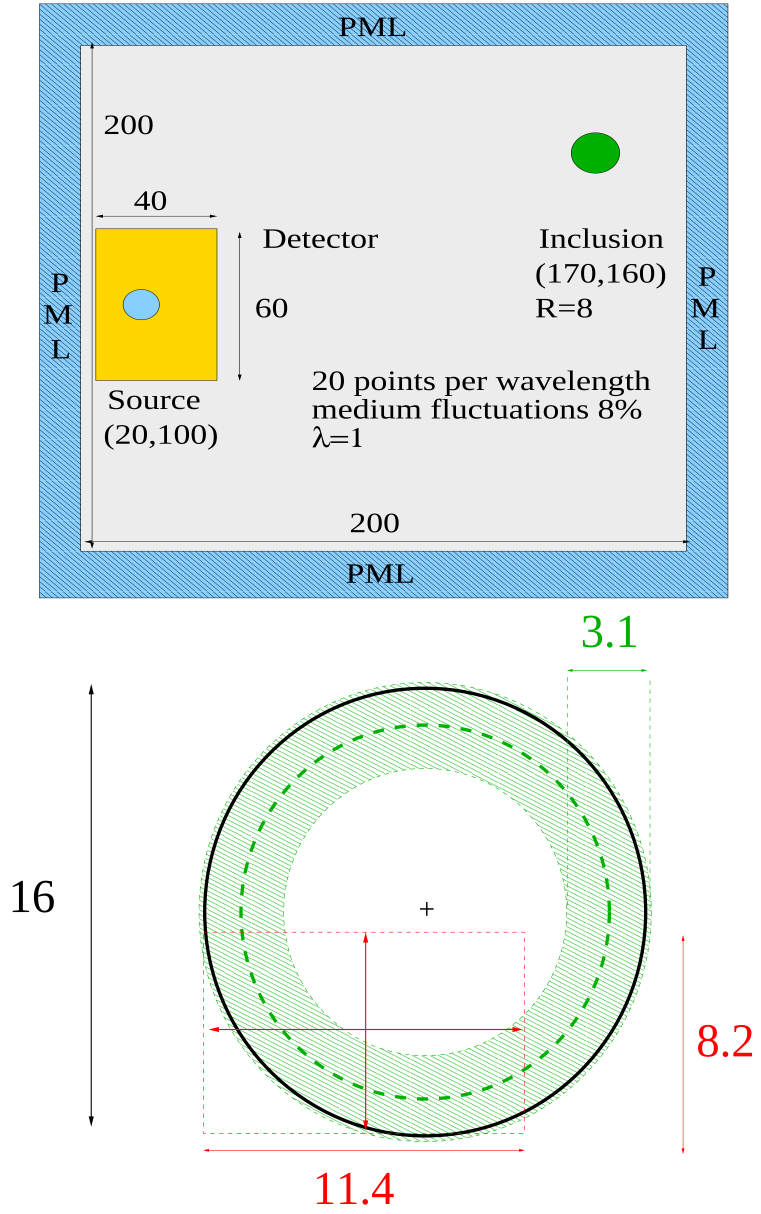
In this case, the corresponding transport solution satisfies Dirichlet boundary conditions at the inclusion boundary rather than specular conditions for the energies.

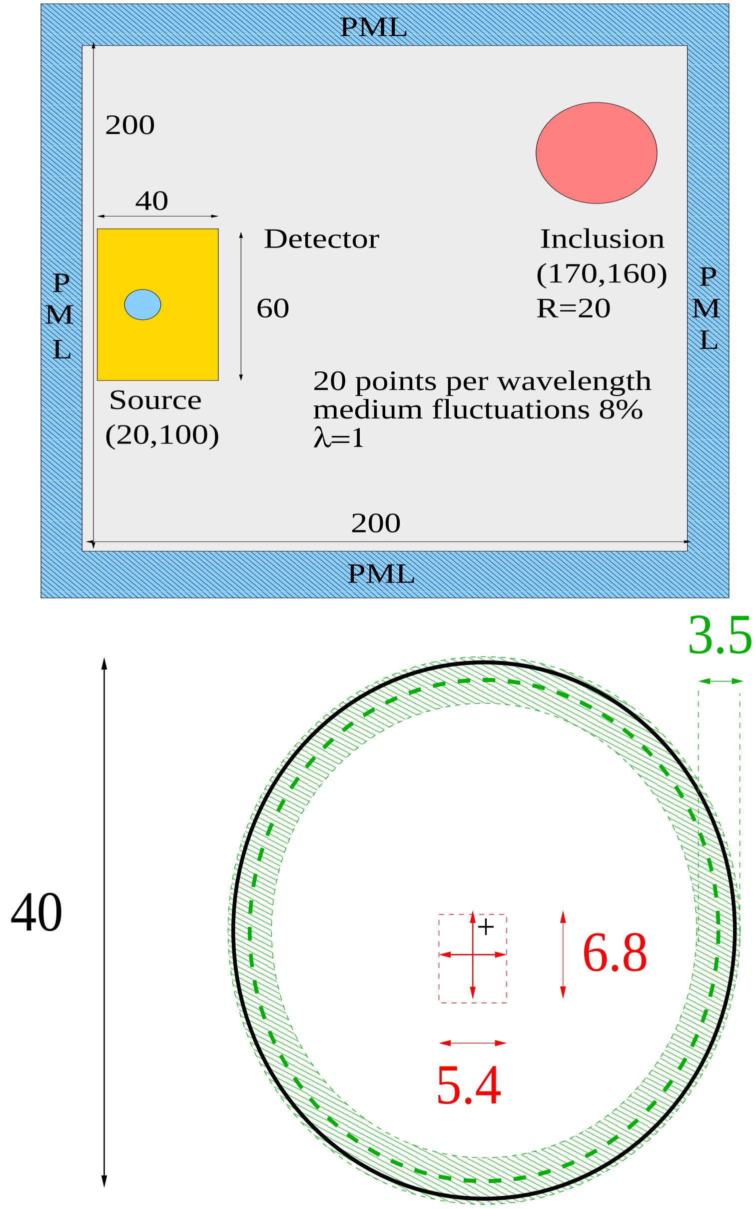
Formal calculations then show that, in the transport regime, the correlation should be more stable than the energy.

In the diffusion regime, we have, for $d \geq 3$:

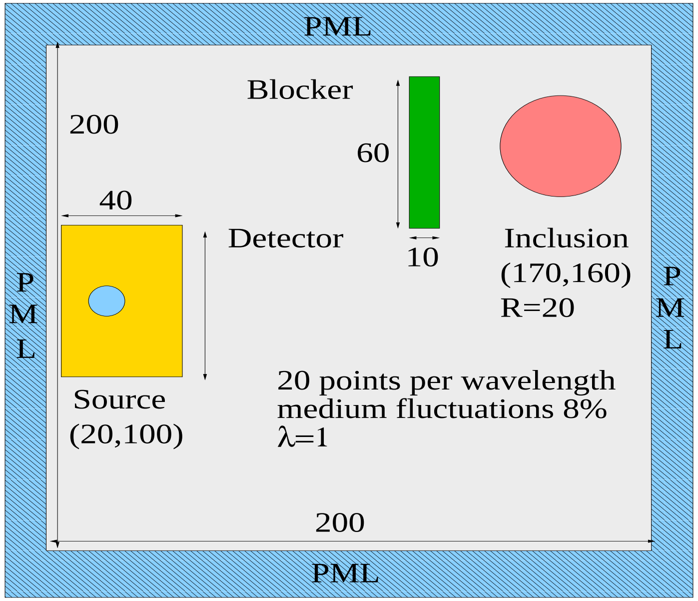
$$\delta E^{\varepsilon} = \mathcal{O}(R^{d}), \qquad \delta C^{\varepsilon} = \mathcal{O}(R^{d-2})$$

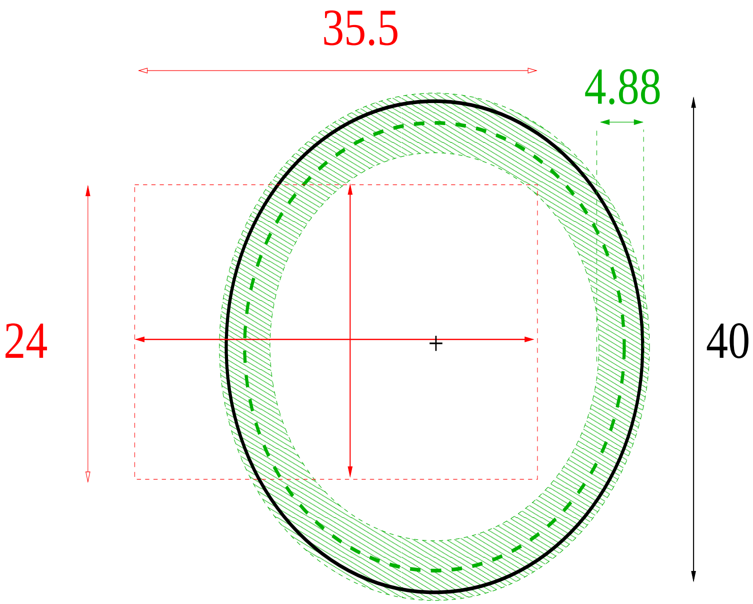
Reconstructions (mean free path ≈ 40)

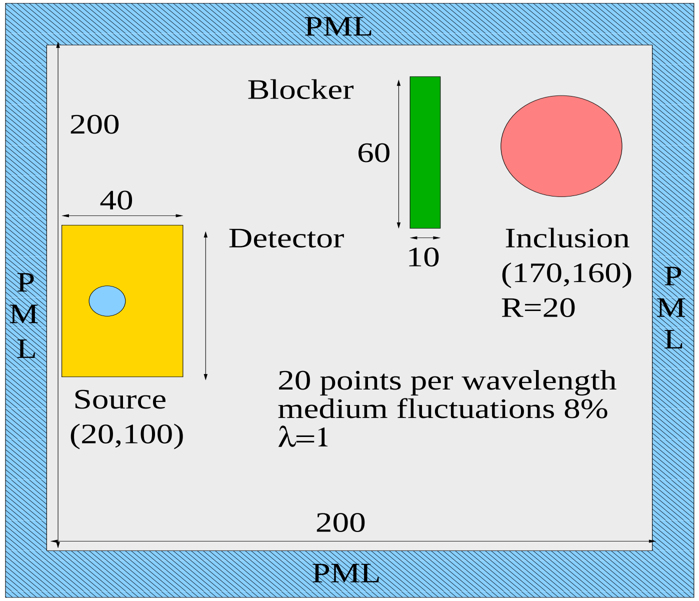


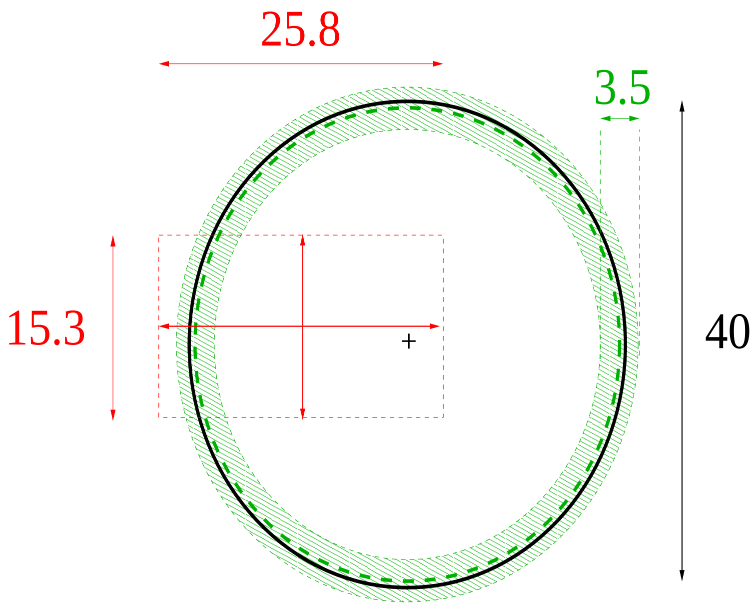


Reconstructions (mean free path ≈ 40)









Conclusion

- Models for imaging in strongly heterogeneous environments were proposed and validated
- They consist in solving an inverse transport problem rather than imaging using a wave description
- Quantifying their precision requires a careful analysis of the limit random wave equation $\rightarrow$ transport equation
- In simplified settings some optimal rates of convergence were obtained and the corrector was characterized

Some open questions

- Kinetic limit for the wave equation
- Comparison transport-based and interferometry methods

Elements of proof

Typically, w_η^S and w_η^D involves terms of the form

$$f_\eta^S(t) = g^S(\eta) \int_{\mathbb{R}^d} d\xi \hat{R}(\xi) |F^\eta(\tau, \xi)|^2,$$

$$f_\eta^D(t) = g^D(\eta) \int_{\mathbb{R}^{2d}} d\xi d\mu \hat{R}(\xi - \mu) \hat{R}(\mu) |G^\eta(\tau, \xi, \mu)|^2,$$

where F^η and G^η involve one or several oscillatory integrals.

These integrals tend to localize F^η and G^η on $\xi = 0$ at the limit, and G^η on a closed surface of $\mathbb{R}^{d-1}$ in η to which belongs the origin.

This is problematic when $\hat{R}$ is singular near zero (long-range correlations) and one has to decompose $\mathbb{R}^{2d}$ into various subdomains to obtain the right limit.