Wave Equations in Random Media

Numerical Analysis of Sparse Tensor gpc and MLMC Methods

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Numerical models in engineering can be solved with high accuracy

if input data are known exactly.

Often, however,

input data are not known exactly

and

accurate numerical solutions are of limited use.

- Mathematical description of uncertainty in input data and solution?
- How to *propagate* data uncertainty through an engineering simulation?
- How to process statistical information in engineering simulations?

Goal:

given statistics of input data, compute (deterministic) solution statistics.

Tool:

Formulation and solution of *Stochastic Partial Differential Equation (SPDE)*

1. Deterministic operator w. stochastic data:

   $u : \Omega \ni \omega \to \mathcal{X}$ such that

   $$Au = f(\cdot, \omega), \qquad f : \Omega \ni \omega \to \mathcal{Y}'$$

2. Stochastic operator $A(\omega) \in L(\mathcal{X}, \mathcal{Y}')$ and deterministic data $f \in \mathcal{Y}'$:
   $u : \Omega \ni \omega \to \mathcal{X}$ such that

   $$A(\omega)u = f \ .$$

Choices of $\mathcal{X}$ and $\mathcal{Y}$: elliptic PDEs $\mathcal{X} = \mathcal{Y}$, parabolic/hyperbolic PDEs: $\mathcal{X} \neq \mathcal{Y}$.

Examples

$(\Omega, \mathcal{A}, \mathbb{P})$ P.-space, $\omega \in \Omega$, $x \in D \subset \mathbb{R}^d$ bounded Lipschitz domain

1. Diffusion (random medium, random source term) (Dettinger & Wilson (1985), ...)

$$-\nabla \cdot (a(x;\omega)\nabla u(x;\omega)) = f(x;\omega) \quad \text{in} \quad D, \qquad u(\cdot;\omega)|_{\partial D} = 0 \; .$$

2. Random Eigenvalue Problem (R. Andreev & CS (2010))

$$-\nabla \cdot (a(x;\omega)\nabla w(x;\omega)) = \lambda(\omega) w(x;\omega) \quad \text{in} \quad D, \qquad w(\cdot;\omega)|_{\partial D} = 0 \; .$$

3. Diffusion in Random Medium

$$\rho(x,t;\omega)\partial_t u - \nabla \cdot (a(x,t;\omega)\nabla u(x,t;\omega)) = f(x,t;\omega) \quad \text{in} \quad D, \qquad u(\cdot,\cdot;\omega)|_{\partial D} = 0 \; ,$$

$$u(x,0;\omega) = u_0(x;\omega) \quad \text{in} \quad D \; .$$

4. Wave Propagation in Random Medium

$$\rho(x,t;\omega)\partial^2_{tt} u - \nabla \cdot (a(x,t;\omega)\nabla u(x,t;\omega)) = f(x,t;\omega) \quad \text{in} \quad D, \qquad u(\cdot,\cdot;\omega)|_{\partial D} = 0 \; ,$$

$$u(x,0;\omega) = u_0(x;\omega) \; , \quad \partial_t u(x,0;\omega) = u_1(x;\omega) \quad \text{in} \quad D \; .$$

Computational Approaches

Monte Carlo FEM
Sampling Methods w. *Sparse Tensor Estimation* of Correlations (CS and von Petersdorff (2006)) ,
Multilevel Monte Carlo: Heinrichs (2000), Giles (2006)
Diffusion (Barth, CS & Zollinger (2010) MLMC-FEM) , SCL (Mishra & CS (2010) MLMC-FVM) .

Perturbation Methods "First Order Second Moment" (FOSM)
Asymptotics: J. B. Keller (1964), L. Borcea, G. Papanicolau et al., M. Kleiber, T.D. Hien (1992)
Sparse Tensor FEM: CS and R.A. Todor (2003), CS and T. von Petersdorff (2006) , CS and H. Harbrecht, CS and A. Chernov (2009)

Stochastic Collocation and Galerkin WPC, KL, gPC
R. G. Ghanem, P. D. Spanos (1991)
I. Babuška , R. Tempone etal. SINUM (2003- 2005)
G. E. Karniadakis, D. Xiu et al. SIAM J. Sci. Comp. (2002)
H. Matthies etal. CMAME (2005)
F. Nobile, R. Tempone and F. Webster, SINUM (2006 -)
A. Cohen, R. DeVore & CS (2010,2011), V.H. Hoang & CS (2011)

Outline

Wave Equation with Random Coefficient

$I = (0, T)$, $T < \infty$, $D \subset \mathbb{R}^d$ bounded Lipschitz. In $Q_T = I \times D$ consider

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (a(x, \omega) \nabla u) = g(t, x), \ u|_{\partial D \times I} = 0, \ u|_{t=0} = g_1, \ u_t|_{t=0} = g_2.$$

Assume

- $a(x, \omega)$ is random field (Ω, Σ, P) over $L^\infty(D)$
- forcing g and initial data g_1 and g_2 are deterministic
- There are constants $0 < a_{\min} \leq a_{\max} < \infty$ so that

$$\forall \, \omega \in \Omega : \quad 0 < a_{\min} \leq \operatorname{essinf}\{a(x, \omega) : x \in D\} \leq \|a(\cdot, \omega)\|_{L^\infty(D)} \leq a_{\max} \ .$$

Wave Equation with Random Coefficient

Weak form: $V := H_0^1(D)$, $H := L^2(D)$ and assume

$$g \in L^2(I; H), \quad g_1 \in V, \quad g_2 \in H$$

Bochner spaces:

$$\mathcal{X} = L^2(I; V) \cap H^1(I; H) \cap H^2(I; V'), \qquad \mathcal{Y} = L^2(I; V) \times V \times H \ .$$

A *weak random solution of (1)* is any random field $u : \Omega \mapsto \mathcal{X}$ such that $\mathbb{P}$-a.s.

$$\int_I \langle \frac{d^2 u}{dt^2}(t, \cdot), v_0(t, \cdot)\rangle_H dt + \int_I \int_D a(x; \omega) \nabla u(t, x; \omega) \cdot \nabla v_0(t, x) dx dt + \langle u(0), v_1\rangle_V + \langle u_t(0), v_2\rangle_H$$
$$= \int_I \int_D g(t, x) v_0(t, x) dx dt + \langle g_1, v_1\rangle_V + \langle g_2, v_2\rangle_H, \quad \forall v = (v_0, v_1, v_2) \in \mathcal{Y}.$$

Wave Equation with Random Coefficient

- For every $\omega \in \Omega$, problem (3) admits a unique weak solution $u \in \mathcal{X}$.
- There holds

$$\sup_{\omega \in \Omega} \|u(\cdot,\omega)\|_{\mathcal{X}} \leq C(\|g\|_{L^2(I;H)} + \|g_1\|_V + \|g_2\|_H),$$

- u is a random field (measuable as a mapping $(\Omega, \mathcal{A}, \mathbb{P}) \to \mathcal{X}$),
- for every $0 < p \leq \infty$,

$$\|u\|_{L^p(\Omega,d\mathbb{P};\mathcal{X})} \leq C\big(\|g\|_{L^p(\Omega,\mathcal{A},\mathbb{P};L^2(I;H))} + \|g_1\|_{L^p(\Omega,\mathcal{A},\mathbb{P};V)} + \|g_2\|_{L^p(\Omega,\mathcal{A},\mathbb{P},H)}\big) \, .$$

Karhúnen-Loève expansion of random input

Structural assumptions: a characterized by a sequence of infinitely many, scalar random variables Y_j:

$$a(x,\omega) = \bar{a}(x) + \sum_{j\geq 1} Y_j(\omega)\psi_j(x), \quad \psi_j \in L^\infty(D) .$$

Affine parameter dependence highly ambiguous (Y_j and ψ_j could be rescaled).

Assumption: $\Gamma_j = \mathrm{supp}(\mathrm{law}(Y_j))) = (-1,1)$, and $\bar{a}(x)$ and ψ_j satisfy

$$\sum_{j\geq 1} \|\psi_j\|_{L^\infty(D)} \leq \frac{\kappa}{1+\kappa}\bar{a}_{\min}$$

with $\bar{a}_{\min} = \mathrm{essinf}_{x\in D}\bar{a}(x) > 0$ and some $\kappa > 0$. Then

$$a(x,\omega) \geq a_{\min} := \bar{a}_{\min} - \frac{\kappa}{1+\kappa}\bar{a}_{\min} = \frac{1}{1+\kappa}\bar{a}_{\min}.$$

Karhúnen-Loève expansion of random input

Example (Karhunen-Loève Expansion)
If $a \in L^2(\Omega, d\mathbb{P}; L^\infty(D))$ then in $L^2(\Omega, d\mathbb{P}; L^2(D))$,

$$a(x,\omega) = \bar{a}(x) + \sum_{j\geq 1} \psi_j(x) Y_j(\omega) = \mathbb{E}[a](x) + \sum_{j\geq 1} \sqrt{\lambda_j} \varphi_j(x) Y_j(\omega),$$

- $(\lambda_j, \varphi_j)_{j\geq 1}$ eigensequence of (cpt., s.a.) **covariance operator**

$$\mathcal{C}[a] : L^2(D) \to L^2(D) \quad (\mathcal{C}[a]v)(x) := \int_D R_a(x,x') v(x') dx' \quad \forall v \in L^2(D),$$

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$$R_a = \mathbb{E}\left[(a - \mathbb{E}[a]) \otimes (a - \mathbb{E}[a])\right] \in L^\infty(D \times D)$$

-

$$Y_j(\omega) := \frac{1}{\sqrt{\lambda_j}} \int_D (a(x,\omega) - \mathbb{E}[a](x)) \varphi_j(x) dx : \Omega \to \Gamma_j \subseteq \mathbb{R} \quad j = 1, 2, ...$$

$$a(x,\omega) = \mathbb{E}[a](x) + \sum_{j\geq 1} \sqrt{\lambda_j} \varphi_j(x) Y_j(\omega)$$

Karhúnen-Loève expansion of random input

KL expansion converges in $L^2(\Omega; L^2(D))$.
To ensure $L^\infty(\Omega; L^\infty(D))$ convergence, must

- - estimate decay rate of KL eigenvalues λ_m: Schwab & Todor JCP (2006),
- - bound $||\varphi_j||_{L^\infty(D)}$: Todor Diss ETH (2005), SINUM (2006)
- - **assume**: bounds for $||Y_j||_{L^\infty(\Omega)}$

Rescaling:

$$\forall j : \quad \Gamma_j = \text{supp}(Y_j) = [-1,1], \quad y = (y_j)_{j\geq 1} \in U = [-1,1]^{\mathbb{N}} \ .$$

Probability: σ-algebra $\Theta = \bigotimes_{j\geq 1} \mathcal{B}^1([-1,1])$ where $\mathcal{B}^1([-1,1])$ Borel σ-algebra on $[-1,1]$.
Probability measure on (U, Θ):

$$\rho(dy) := \bigotimes_{j\geq 1} dy_j/2 \ .$$

Since $\frac{1}{2}dy_j$ is a probability measure on $(-1,1)$, so is $\rho(dy)$ on (U,Θ) and hence (U,Θ,ρ) is a probability space.
As $Y_j \sim \mathcal{U}(-1,1)$, for any $S = \prod_{j=1}^\infty S_j$ with $S_j \in \mathcal{B}^1([-1,1])$,

$$\rho(S) = \prod_{j=1}^\infty \mathbb{P}\{\omega : y_j(\omega) \in S_j\}.$$

Parametric deterministic system

For each $y \in U$ consider the parametric, deterministic problem

$$\frac{\partial^2 u(t,x;y)}{\partial t^2} - \nabla_x \cdot (a(x;y)\nabla_x u(t,x;y)) = g(t,x) \text{ in } Q_T,\ u(t,x,y)|_{\partial D\times I} = 0,\ u|_{t=0} = g_1,\ u_t|_{t=0} = g_2,$$

where $U \ni y \mapsto a(\cdot;y) \in L^\infty(D)$ is defined as

$$a(x;y) = \bar{a}(x) + \sum_{j=1}^{\infty} y_j \psi_j(x).$$

For each $y \in U$, define the parametric bilinear form $U \ni y \mapsto b : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ by

$$b(y;w,(v_0,v_1,v_2)) = \int_I \langle \frac{d^2 w}{dt^2}(t,\cdot), v_0(t,\cdot)\rangle_H dt + \int_I \int_D a(x;y)\nabla w(t,x)\cdot\nabla v_0(t,x)dxdt + \langle u(0), v_1\rangle_V + \langle u_t(0), v_2$$

Linear form on $\mathcal{Y}$

$$f(v) = \int_I \int_D g(t,x)v_0(t,x)dxdt + \langle g_1, v_1\rangle_V + \langle g_2, v_2\rangle_H.$$

For every $y \in U$, find $u(y) \in \mathcal{X}$ such that

$$b(y;u,v) = f(v) \quad \forall v = (v_0,v_1,v_2) \in \mathcal{Y}\ .$$

Parametric deterministic system

Basic Properties of the parametric problem:

1. For every $y \in U$, the parametric problem admits a unique weak solution $u(y) \in \mathcal{X}$.
2. The weak solutions $\{u(y) : y \in U\} \subset \mathcal{X}$ satisfy the apriori estimate

$$\forall y \in U : \quad \|u(\cdot,\cdot,y)\|_{\mathcal{X}} \leq C(\|g\|_{L^2(I;H)} + \|g_1\|_V + \|g_2\|_H),$$

3. The map $u : U \to \mathcal{X}$ is strongly measurable.

p-summability? Bochner Spaces:
For $0 < p \leq \infty$, and a separable Hilbert space $\mathcal{V}$,
define $L^p(U, \mathcal{V}, \rho)$ space of ρ-measurable mappings $v : U \to \mathcal{V}$ s.t. $\|v\|_{\mathcal{V}}^p$ is ρ-integrable.
$p = 2$:

$$\underline{\mathcal{X}} := L^2(U, \mathcal{X}, \rho) \simeq L^2(U, \rho) \otimes \mathcal{X}, \quad \underline{\mathcal{Y}} := L^2(U, \mathcal{Y}, \rho) \simeq L^2(U, \rho) \otimes \mathcal{Y} .$$

Define

$$B(u, v) := \int_U b(y; u, v)\rho(dy) : \underline{\mathcal{X}} \times \underline{\mathcal{Y}} \to \mathbb{R}, \quad F(v) := \int_U f(v)\rho(dy) : \underline{\mathcal{Y}} \to \mathbb{R} .$$

Consider the ("ensemble-averaged") variational problem: find

$$u \in \underline{\mathcal{X}} \quad \text{such that} \quad B(u, v) = F(v) \qquad \forall v \in \underline{\mathcal{Y}}.$$

4. Problem (6) admits a unique solution $u \in \underline{\mathcal{X}}$.

gpc Expansion of u

Polynomial spaces in U:

$$(L_n)_{n\geq 0} \text{ univariate Legendre polynomials,} \quad \int_{-1}^{1} |L_n(t)|^2 \frac{dt}{2} = 1.$$

$$\mathcal{F} = \{\nu \in \mathbb{N}_0^{\mathbb{N}} : |\nu| < \infty\} \ .$$

Tensorized Legendre polynomials:

$$L_\nu(y) = \bigotimes_{j\geq 1} L_{\nu_j}(y_j), \qquad \nu \in \mathcal{F},\ y \in U \ .$$

$$\{L_\nu : \nu \in \mathcal{F}\} \text{ is an orthonormal basis of } L^2(U,\rho).$$

gpc representation of $u \in L^2(U, \mathcal{X}, \rho)$:

$$\forall u \in \underline{\mathcal{X}} : \quad u = \sum_{\nu\in\mathcal{F}} u_\nu L_\nu, \qquad u_\nu = \int_U u(\cdot,\cdot;y) L_\nu(y)\rho(dy) \in \mathcal{X} \ .$$

gpc Expansion of u

Stochastic Galerkin: for any $\Lambda \subset \mathcal{F}$, $N = \#\Lambda < \infty$ define $\underline{\mathcal{X}}_\Lambda \subset \underline{\mathcal{X}}$, $\underline{\mathcal{Y}}_\Lambda \subset \underline{\mathcal{Y}}$ by

$$\underline{\mathcal{X}}_\Lambda := \{u_\Lambda(t,x,y) = \sum_{\nu \in \Lambda} u_\nu(t,x) L_\nu(y) : \ u_\nu \in \mathcal{X}\}, \quad \underline{\mathcal{Y}}_\Lambda = \{v_\Lambda(t,x,y) = \sum_{\nu \in \Lambda} v_\nu(t,x) L_\nu(y) : \ v_\nu \in \mathcal{Y}\} .$$

Find

$$u_\Lambda \in \underline{\mathcal{X}}_\Lambda \quad \text{such that} \quad B(u_\Lambda, v_\Lambda) = F(v_\Lambda) \quad \forall v_\Lambda \in \underline{\mathcal{Y}}_\Lambda.$$

$\exists! u_\Lambda \in \underline{\mathcal{X}}_\Lambda$. Error? Note: nonelliptic problem, can not use Lax-Milgram. Still, have "quasioptimality" of error.

$$\mathcal{Z} := H^1(I;V) \cap H^2(I;H) \subset C^0(\overline{I};V) \cap C^1(\overline{I};H) \subset \mathcal{X} .$$

Proposition:

Assume $u \in L^2(U, \mathcal{Z}, \rho)$. Then $\forall \nu \in \mathcal{F} : \ u_\nu \in \mathcal{Z}$.

Assume further $u_\Lambda \in L^2(U, \mathcal{Z}, \rho(dy))$ for a finite subset $\Lambda \subset \mathcal{F}$. Then

$$\|u - u_\Lambda\|_{L^2(U,\mathcal{X},\rho)} \leq c \| \sum_{\nu \in \mathcal{F}\setminus\Lambda} u_\nu L_\nu \|_{L^2(U,\mathcal{Z},\rho)} = c \left(\sum_{\nu \in \mathcal{F}\setminus\Lambda} \|u_\nu\|_{\mathcal{Z}}^2 \right)^{1/2} .$$

Here, the constant $c > 0$ depends only on the coefficient bounds a_{min} and a_{max}.

Choice of $\Lambda \subset \mathcal{F}$? Rate of convergence as $N = \#\Lambda \to \infty$?

Best N-term gpc approximation of u

Lemma (Stechkin) Let $\alpha = (\alpha_\nu)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$ and let $q \geq p > 0$.
If $\Lambda_N \in \mathcal{F}$ is a set of indices corresponding to a set of N largest $|\alpha_\nu|$, then

$$\|(\alpha_\nu)\|_{\ell^q(\mathcal{F}\backslash\Lambda_N)} = \left(\sum_{\nu\in\mathcal{F}\backslash\Lambda_N} |\alpha_\nu|^q\right)^{1/q} \leq N^{-\sigma}\|(\alpha_\nu)\|_{\ell^p(\mathcal{F})} \qquad \sigma := \frac{1}{p} - \frac{1}{q} > 0 \; .$$

Here: $\alpha_\nu = \|u_\nu\|_{\mathcal{Z}}$, $q = 2$ and $0 < p < 1$. Then $\sigma = \frac{1}{p} - \frac{1}{q} > 1/2$ and there is $c > 0$ ind. of N s.t.

$$\|u - u_{\Lambda_N}\|_{L^2(U,\mathcal{X},\rho)} \leq c \left(\sum_{\nu\in\mathcal{F}\backslash\Lambda_N} \|u_\nu\|_{\mathcal{Z}}^2\right)^{1/2} \leq N^{-(\frac{1}{p}-\frac{1}{2})}\,\|(\|u_\nu\|_{\mathcal{Z}})\|_{\ell^p(\mathcal{F})} \; .$$

When is $(\|u_\nu\|_{\mathcal{Z}})_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$?
Theorem (Hoang and CS (2010)): for $0 < p \leq 1$,

$$\left\|(\|\psi_j\|_{W^{1,\infty}(D)})\right\|_{\ell^p(\mathbb{N})} < \infty \quad \Longrightarrow \quad \|(\|u_\nu\|_{\mathcal{Z}})\|_{\ell^p(\mathcal{F})} < \infty \; .$$

- $\forall \nu \in \mathcal{F}$: u_ν solution of one scalar wave-equation, u_{Λ_N} soln. of N wave equations.
- $p = 1$ implies rate $\sigma = 1/2$ (same as MC), $0 < p < 1$ implies $\sigma > 1/2$.

Proof by sev. complex variables and sequence factorization argument Cohen, DeVore & CS (J. FoCM 2010)

Best N-term gpc approximation of u: Computational Aspects

Q1 given R_a resp. $\mathrm{Cov}[a]$, how to find ψ_j?

Q2 given N and $(\psi_j)_{j\in\mathbb{N}}$, how to choose $\Lambda_N \subset \mathcal{F}$?

Q3 given $\Lambda_N \subset \mathcal{F}$, how to compute the $\{u_\nu : \nu \in \Lambda_N\}$?

A1 Karhúnen-Loève Eigensolver: gFMM + Krylov subspace solver (Todor & CS JCP (2006))

A2a **a-priori** choice of Λ_N based on $\|\psi_j\|_{W^{1,\infty}(D)}$ (Andreev, Bieri & CS SISC (2009)) or

A2b **adaptive** choice of Λ_N (C.J. Gittelson & CS Acta Numerica (2011)) (only elliptic pblms.).

A3 Solution of $N = \#\Lambda_N$ many deterministic wave eqns. for the u_ν:
Numerical solution with
N_x space DOF, N_t time steps: $O(N_x N_t)$ operations per gpc coefficient u_ν
Complexity: $O(N N_x N_t)$

Sparse Tensor approximation of $u(\cdot, \cdot; y)$

MC

- given N, draw N many coefficient samples $a_j(\cdot) = a(\cdot, \omega_j)$, $j = 1, ..., N$,
- Solve linear wave equation with coefficient $a_j \Rightarrow u(x, t; \omega_j)$
- Estimate $\mathbb{E}[u]$ by sample average: $E_N[u](x,t) = N^{-1} \sum_{j=1}^{N} u(x, t; \omega_j)$

E.g. first order scheme for N *linear wave equations*:
space step h and time step $k \sim O(h)$ (CFL), i.e.
$N_x = O(h^{-d})$, $N_t = O(k^{-1})$ DOFs $\Rightarrow$

$$\|u(\cdot, \cdot; \omega_j) - u_j^{hk}\|_{W^0} \leq C(h+k) \sup_{\omega \in \Omega} \|u(\cdot, \cdot; \omega)\|_{W^1} \ .$$

Here $\{W^r\}_{r \geq 0}$ is a family of smoothness spaces for the wave-equn.
MC - Accuracy: if $N = O(h^{-2}) = O(k^{-2})$ then for any $u \in L^2(\Omega; W^1)$
$\|\mathbb{E}[u] - E_N[u]\|_{L^2(\Omega;W)} \leq C(N^{-1/2} + h + k) = O(N^{-1/2}) = O(N_{dof}^{-1/(d+3)})$

where
MC - Complexity: $N_{dof} = O(N N_x N_t) = O(N h^{-d} k^{-1}) = O(N h^{-d-1}) = O(h^{-(d+3)})$.

Sparse Tensor approximation of $u(\cdot, \cdot; y)$

gpc:

- Assume given N, $\Lambda_N \subset \mathcal{F}$ with $\#\Lambda_N = N$ has been determined.
- $\forall \nu \in \Lambda_N$, select $N_x(\nu)$, $N_t(\nu)$ such that $\|u_\nu - u_\nu^{h,k}\|_{W^0} \leq C(h_\nu + k_\nu)\|u_\nu\|_{W^1}$.

Theorem: Assume $(\|\psi_j\|_{W^{1,\infty}(D)}) \in \ell^p(\mathbb{N})$ for $0 < p \leq 1$. Then $(\|u_\nu\|_{W^1}) \in \ell^p(\mathcal{F})$.
For $N = 1, 2, \ldots$, ex. $\Lambda_N \subset \mathcal{F}$ s.t. $\#\Lambda_N \leq N$ and choice $h(\nu) = O(k(\nu))$ such that

gpc - Accuracy:

$$\|\mathbb{E}[u] - \sum_{\nu \in \Lambda_N} u_\nu^{h(\nu),k(\nu)} L_\nu\|_{L^2(\Omega;W^0)} \leq C N_{dof}^{-\min\{\frac{1}{p}-\frac{1}{2}, \frac{1}{d+1}\}} \|(\|u_\nu\|_{W^1})\|_{\ell^p(\mathcal{F})} .$$

Note: Minimal assumption $p = 1 \Longrightarrow$

$$\|\mathbb{E}[u] - \sum_{\nu \in \Lambda_N} u_\nu^{h(\nu),k(\nu)} L_\nu\|_{L^2(\Omega;W^0)} \leq C N_{dof}^{-\min\{\frac{1}{2}, \frac{1}{d+1}\}} \|(\|u_\nu\|_{W^1})\|_{\ell^p(\mathcal{F})} .$$

Superior to MC - rate.
Analogous for *methods of higher order* $r \geq 1$:

$$\|\mathbb{E}[u] - \sum_{\nu \in \Lambda_N} u_\nu^{h(\nu),k(\nu)} L_\nu\|_{L^2(\Omega;W^0)} \leq C N_{dof}^{-\min\{\frac{1}{p}-\frac{1}{2}, \frac{r}{d+1}\}} \|(\|u_\nu\|_{W^r})\|_{\ell^p(\mathcal{F})} .$$

Sparse Tensor approximation of $u(\cdot,\cdot;y)$

MC dead? No!
MLMC and and first order (x,t) discretization $\Longrightarrow$

$$\|\mathbb{E}[u] - E_L^{MLMC}[u]\|_{L^2(\Omega;W)} \leq CN_{dof}^{-\min\{\frac{1}{2},\frac{1}{d+1}\}}$$

where (convention: $u^{h_{-1},k_{-1}} := 0$)

$$E_L^{MLMC}[u] = \sum_{\ell=0}^{L} E_{N_\ell}\left[u^{h_\ell,k_\ell} - u^{h_{\ell-1},k_{\ell-1}}\right]$$

and
MLMC - Complexity: total work proportional to

$$N_{dof} = \sum_{\ell=0}^{L} O(N_\ell N_{\ell,x} N_{\ell,t}) \; .$$

MLMC and higher order in (x,t)?
Order $r \geq 1$ discretization in (x,t) $\Longrightarrow$

$$\|\mathbb{E}[u] - E_N[u]\|_{L^2(\Omega;W^0)} \leq CN_{dof}^{-\min\{\frac{1}{2},\frac{r}{d+1}\}}\|u\|_{L^2(\Omega;W^r)}$$

Nonlinear Hyperbolic PDEs: SCL in $\mathbb{R}^d$ (Mishra & CS & J. Šukys (2010-))

Consider the *deterministic SCL in* $\mathbb{R}^d$ with *random initial data*:

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (f_j(u)) = 0, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,\ t > 0 \quad u(x,t;\omega)|_{t=0} = u_0(x;\omega) \ .$$

For every $0 < T < \infty$, ex. unique *random entropy solution* $u : \Omega \ni \omega \mapsto C_b((0,T); L^1(\mathbb{R}^d))$ given by

$$u(\cdot, t; \omega) = S(t) u_0(\cdot, \omega) \ , \quad t > 0,\ \omega \in \Omega \ .$$

$S(t) \in \mathrm{Lip}(L^1(\mathbb{R}^d))$ implies that for every $k \geq m \geq 1$ and for every $0 \leq t \leq T < \infty$ holds $\mathbb{P}$-a.s.

$$\|u\|_{L^k\left(\Omega; C(0,T;L^1(\mathbb{R}^d))\right)} \leq \ \|u_0\|_{L^k\left(\Omega; L^1(\mathbb{R}^d)\right)} \ ,$$

$$\|S(t)\, u_0(\cdot,\omega)\|_{(L^1 \cap L^\infty)(\mathbb{R}^d)} \leq \ \|u_0(\cdot,\omega)\|_{(L^1 \cap L^\infty)(\mathbb{R}^d)}$$

Nonlinear Hyperbolic PDEs: SCL in $\mathbb{R}^d$ (Mishra & CS (2010))

Assume further that for some $k \in \mathbb{N}$

$$u_0 \in L^k(\Omega; L^1(\mathbb{R}^d)) .$$

Then, for every $0 < T < \infty$ and every

$$0 < t_1, t_2, \ldots, t_k \leq T < \infty$$

the spatial k-point correlation function

$$u(x_1, t_1; \omega) \otimes \cdots \otimes u(x_k, t_k; \omega)$$

is well-defined as an element of $L^1(\Omega; L^1(\mathbb{R}^{kd}))$. In particular, the k-th moment

$$(\mathcal{M}^k u)(t_1, \ldots, t_k) := \mathbb{E}[u(\cdot, t_1; \omega) \otimes \cdots \otimes u(\cdot, t_k; \omega)]$$

is well-defined for any choice of t_j as in (12) as an element of $L^1(\mathbb{R}^{kd})$, and it satisfies

$$\left\| \left(\mathcal{M}^k u\right)(t_1, ..., t_k) \right\|_{(L^1(\mathbb{R}^d))^{(k)}} \leq \left\| \bigotimes_{j=1}^{k} u(\cdot, t_j; \cdot) \right\|_{L^1(\Omega; (L^1(\mathbb{R}^d))^{(k)})} \leq \|u_0\|^k_{L^k(\Omega; L^1(\mathbb{R}^d))} .$$

S. Mishra and CS (2010):
Complete analysis of convergence rates of MLMC-FVM for this problem.

Systems of Nonlinear Hyperbolic CLs in $\mathbb{R}^d$ (Mishra & CS & J. Šukys (2011)

We apply the MLMC-FVM to nonlinear, hyperbolic systems of CL's in Multi-d.

www.sam.math.ethz.ch/ reports/2011

Notation. Notation for different combinations of ML(MC) and FVM methods:

MC	Monte Carlo with 1st order FVM scheme	$M = \mathcal{O}(\Delta x^{-1})$
MC2	Monte Carlo with 2nd order FVM scheme	$M = \mathcal{O}(\Delta x^{-2})$
MLMC	multilevel MC with 1st order FVM scheme	$M_\ell = M_L 2^{(L-\ell)}$
MLMC2	multilevel MC with 2nd order FVM scheme	$M_\ell = M_L 4^{(L-\ell)}$

- Compressible Euler
- Compressible Ideal HMD
- Wave Equation in Random Medium

Code: ALSVID-UQ

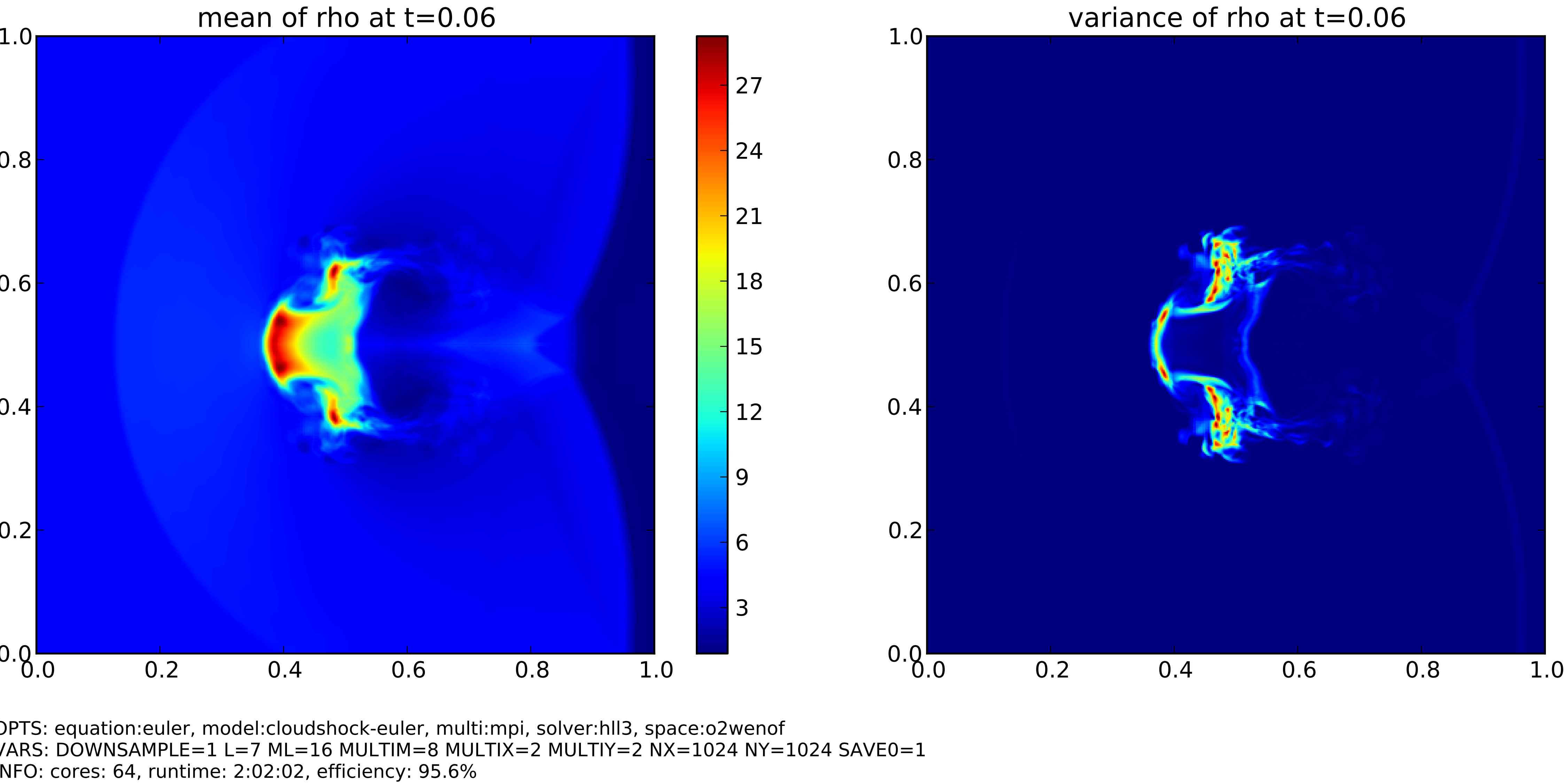

Figure 1: Cloud shock at $t = 0$ and $t = 0.06$ using MLMC-FVM

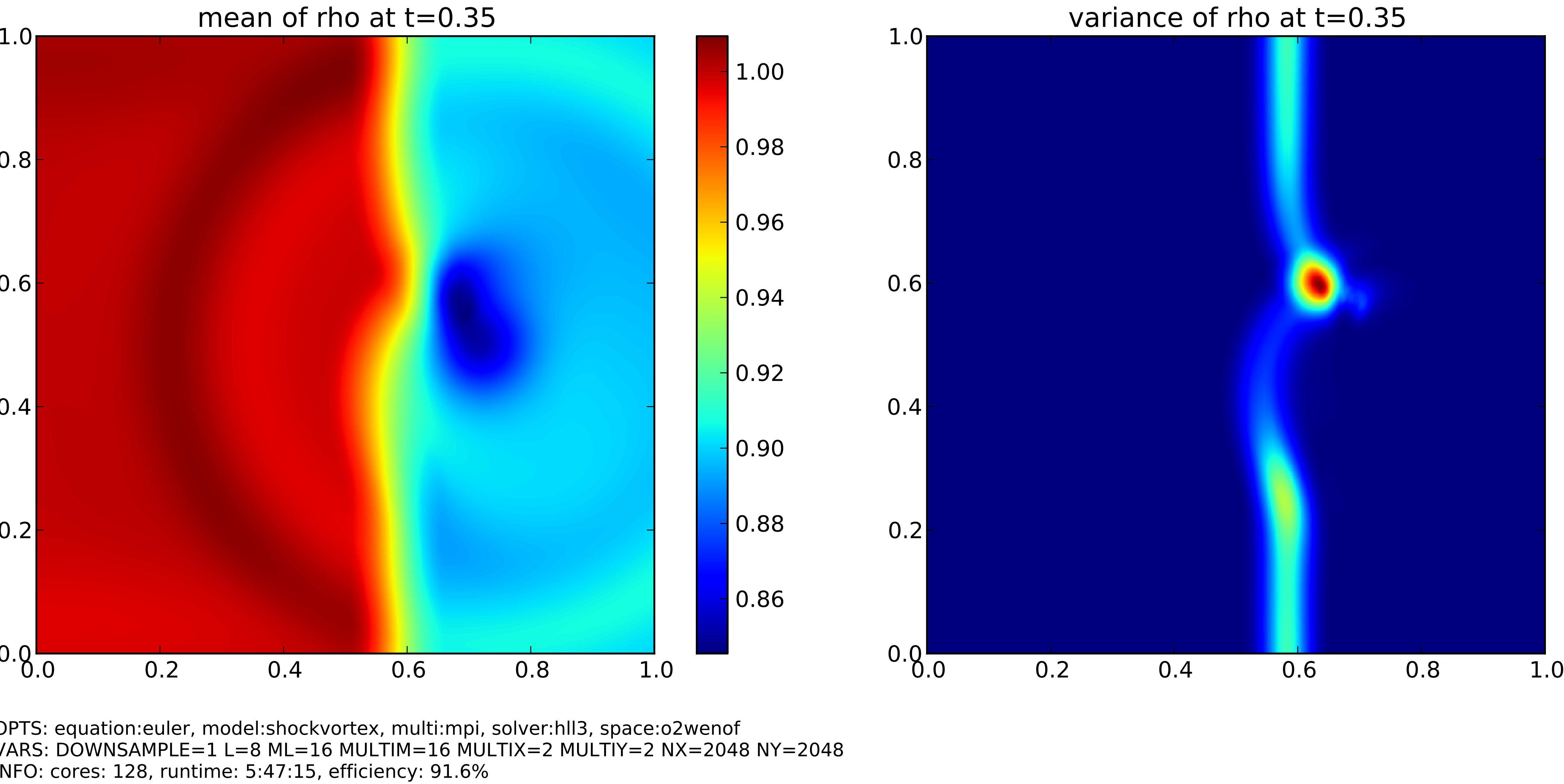

Figure 2: Shock-vortex interaction solution at time $t = 0.35$ using MLMC

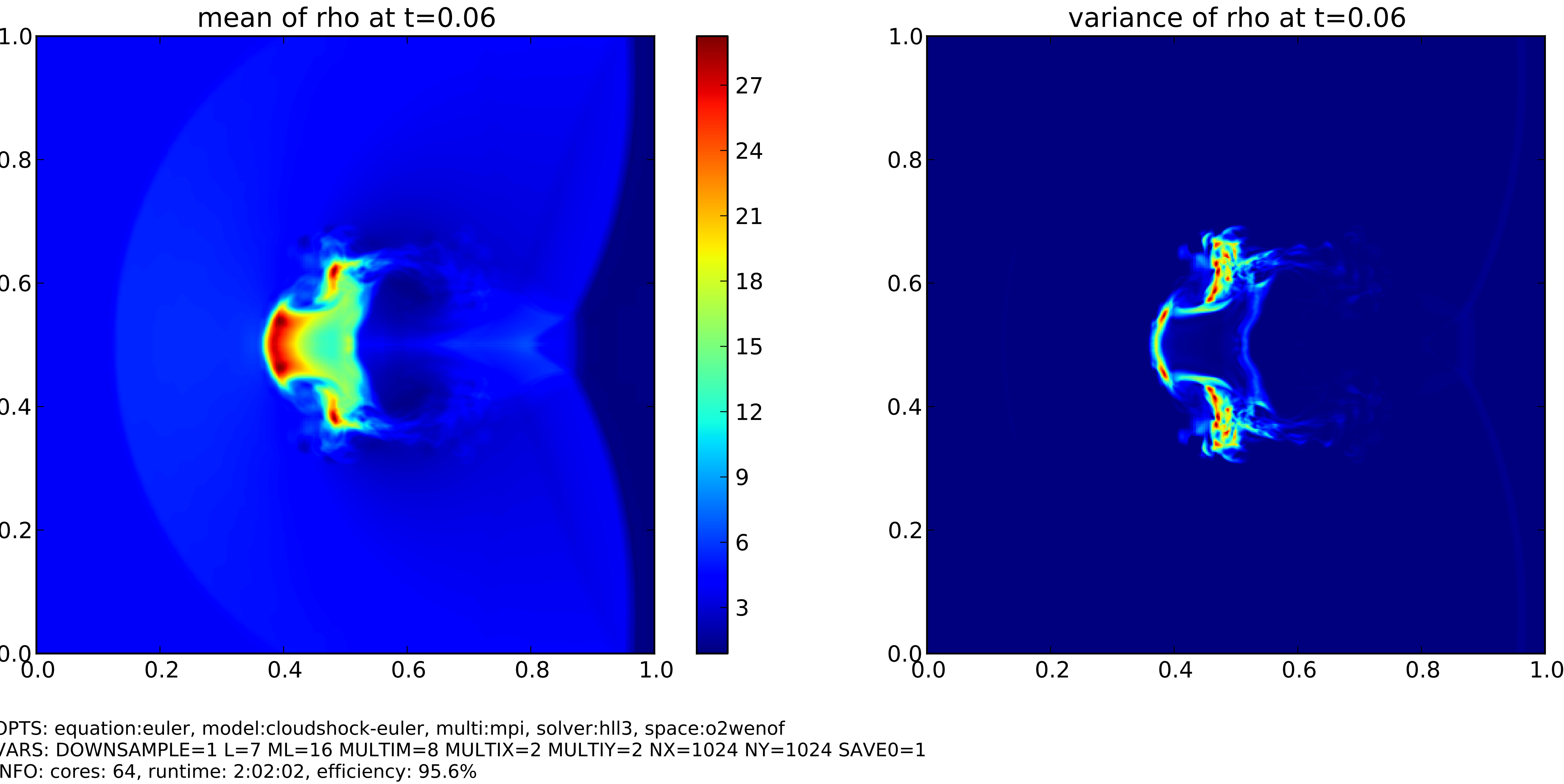

Figure 3: Cloud shock at $t = 0$ and $t = 0.06$ using MLMC-FVM

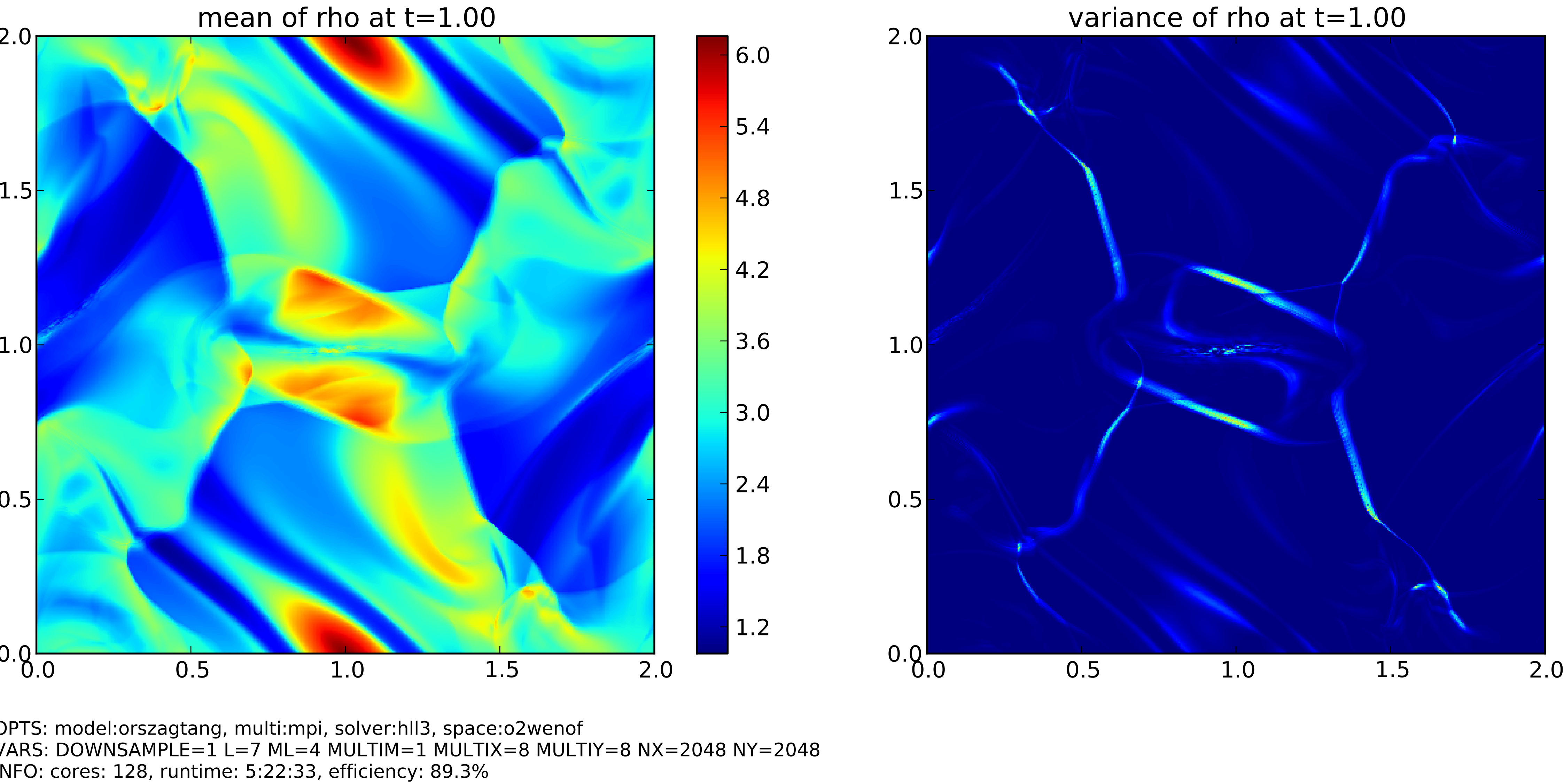

Figure 4: Uncertain Orszag-Tang vortex solution at $t = 1.0$ using MLMC-FVM. Variance is very large near discontinuities of the path-wise solutions.

$$u_{tt} = \nabla \cdot (A \nabla u) \ , \qquad A = A(x, \omega),$$

$$w := A\nabla u, \quad v := u_t \quad \Longrightarrow w_t = A\nabla u_t = A\nabla v$$

$\Longrightarrow$ Linear Hyperbolic SCLs ($d(d+1)/2 + d + 1$ eqns.) with *random initial data*:

$$v_t = \nabla \cdot w$$

$$w_t = A\nabla v$$

$$A_t = 0$$

$$u|_{t=0} = g_1(x), \quad v|_{t=0} = \dot{u}|_{t=0} = g_2(x)$$

$$w|_{t=0} = A\nabla u_0 = A\nabla g_1(x) \quad A|_{t=0} = A(x, \omega) \ .$$

Conclusions

- Monte-Carlo, MLMC for Galerkin FEM and FVM: framework, convergence analysis.
  Error bounds in probability (L^1), mean square (L^2) and $\mathbb{P}$-a.s.

- For low order discretizations in physical space, MLMC optimal (also vs. gpc methods).

- general theory for gpc discretizations of linear PDEs in engineering interest

- theory for nonlinear PDEs under development (hyperbolic SCL with random initial data, random flux)

- Sparse Tensor Galerkin FEM for k-point correlations:
  regularity in anisotropic spaces; sparse tensor product spaces,
  log-linear complexity of k-point correlation computations.

- gpc: trade stochasticity and MC for high-dimensionality + deterministic FEM

- a-priori and a-posteriori error estimates, adaptivity: for elliptic sPDEs
  $\rightarrow$ framework of Cohen, Dahmen, DeVore in tensor product Besov spaces
  (P.A. Nitsche: Constr. Approx. 2006, Stevenson and Sc.: MathComp 2008)

$$\mathcal{M}^k(u) \in B_q^{\alpha}(L_q(D)) \otimes_q \ldots \otimes_q B_q^{\alpha}(L_q(D))$$

  for arbitrarily large α with

$$q = [\alpha/2 + 1/2]^{-1} < 1 \quad \text{indep. of} \quad k.$$

- ‘small’, bounded random perturbations: nonlinear (Fréchet-differentiable) problems:
  First Order Second Moment (FoSM) calculus
  - linearize around “nominal” solution,
  - get 2nd order statistics of random solution from gradient and Hessian at “nominal” solution,
  - Sparse Tensor Discretization of FoSM problems
    (H. Harbrecht, R. Schneider and Sch. Numer. Math. (2008))
    (A. Chernov and CS (2011))
- Multilevel solvers in physical domain *wavelets, any hierarchic basis [BPX, spectral, hp]*;
  for PDEs frames are sufficient... :
  H. Harbrecht, R. Schneider and Ch. Sch. (Numer. Math. (2008)),
  p-FEM, Spectral FEM: A. Chernov and CS: AppNum. (2009) .
- `www.sam.math.ethz.ch/~reports`
- Acta Numerica 2011 (C.S. & C.J. Gittelson)