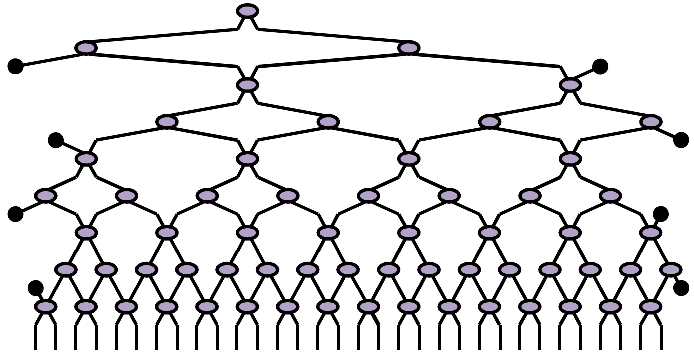
IPAM/UCLA

Wed 28th Jan 2009

Recent progress in Entanglement Renormalization:



Guifre Vidal

In collaboration with:

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Robert Koenig (Caltech)
Ben Reichardt (IQC Waterloo)

THE UNIVERSITY OF QUEENSLAND AUSTRALIA

Australian Government
Australian Research Council

Outline

Foundations

Entanglement Renormalization

MERA (Multi-scale Entanglement Renormalization Ansatz)

Applications

Quantum Criticality

- MERA ⟺ CFT

Topological Order

- Exact MERA for quantum double and string-net models

Frustrated Antiferromagnets

- Kagome lattice
- Triangular lattice

Foundations

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MERA

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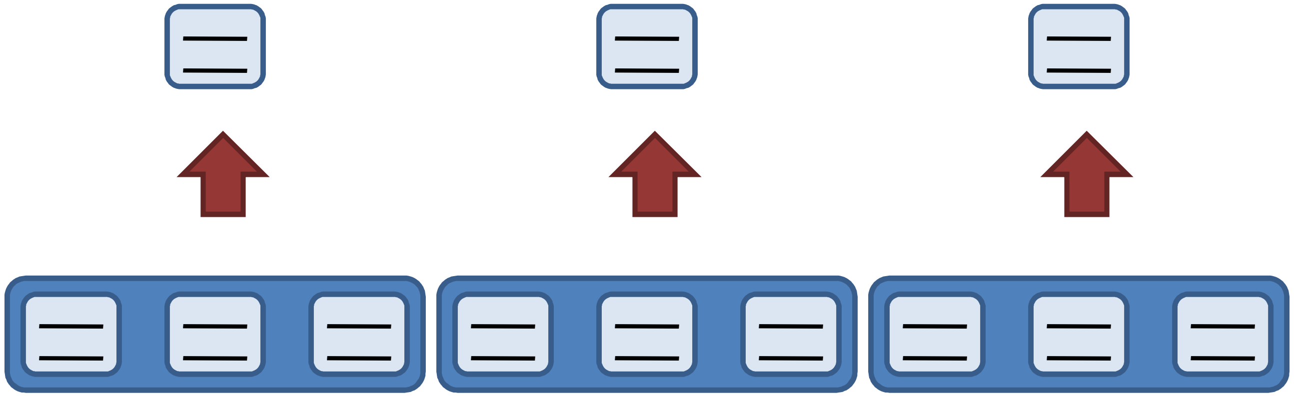
Introduction

- real space RG transformation



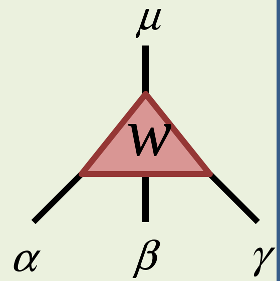
- Kadanoff (1960's): spin blocking

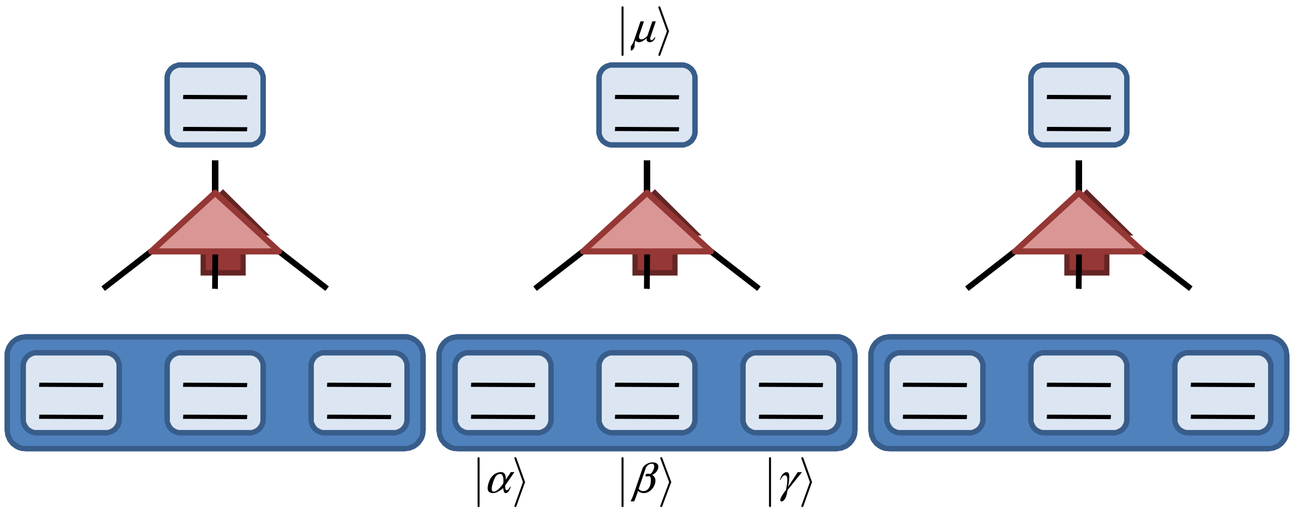
Introduction

• real space RG transformation

Change of basis

$$|\mu\rangle = \sum_{\alpha\beta\gamma} w^{\mu}_{\alpha\beta\gamma} |\alpha\rangle|\beta\rangle|\gamma\rangle$$



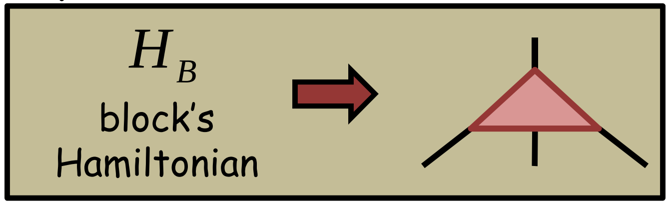


• Kadanoff (1960's): spin blocking

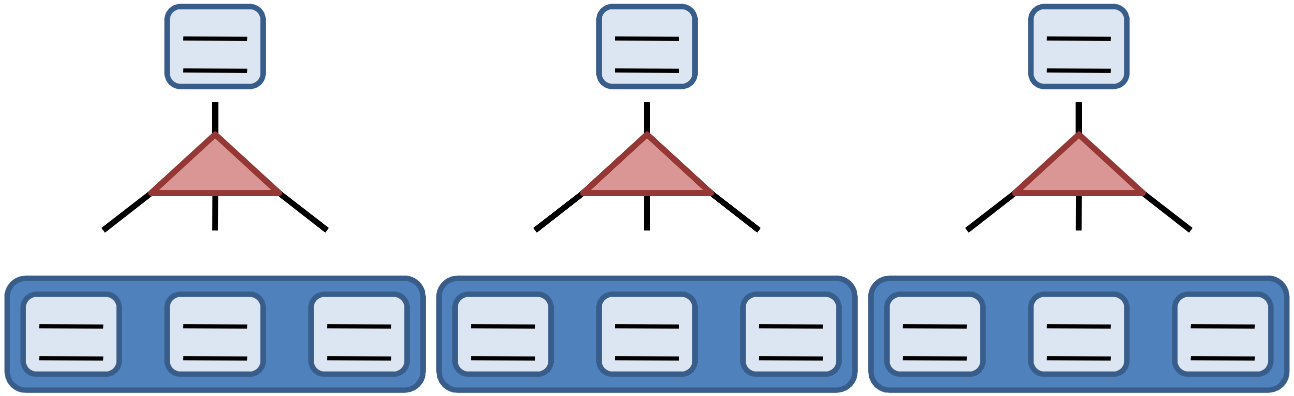
• Wilson (1970's): keep low energy subspace
Kondo impurity problem: Numerical Renormalization Group (NRG)

Introduction

• real space RG transformation



(Wilson)



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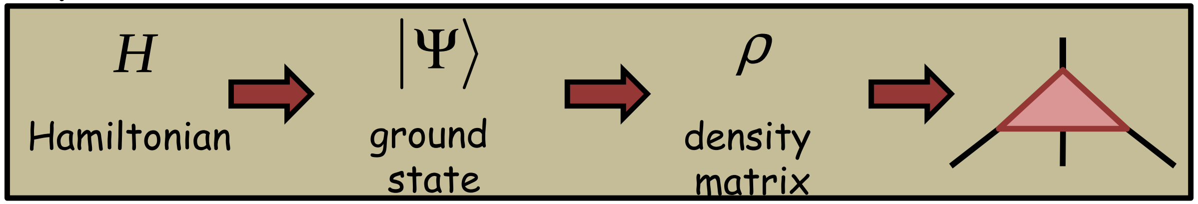
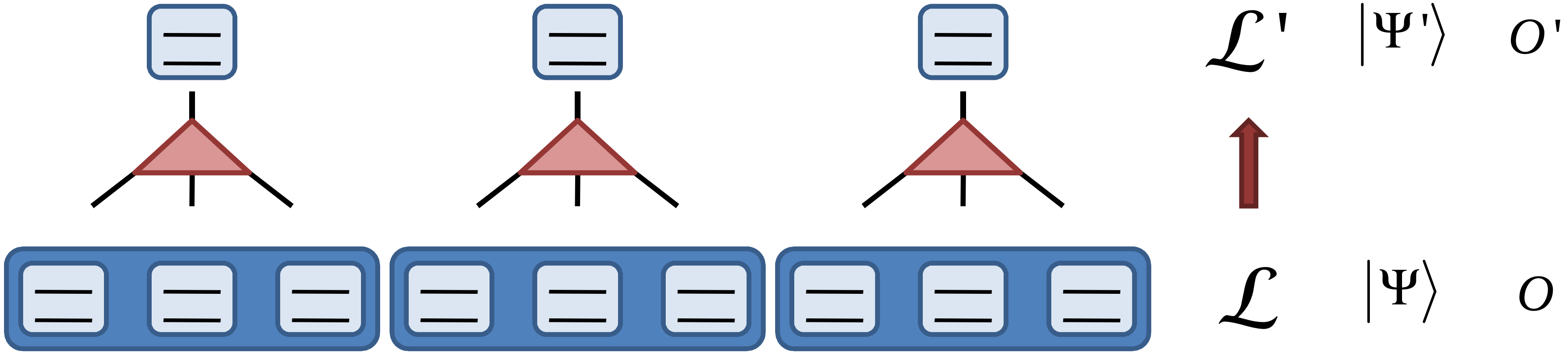


(White)



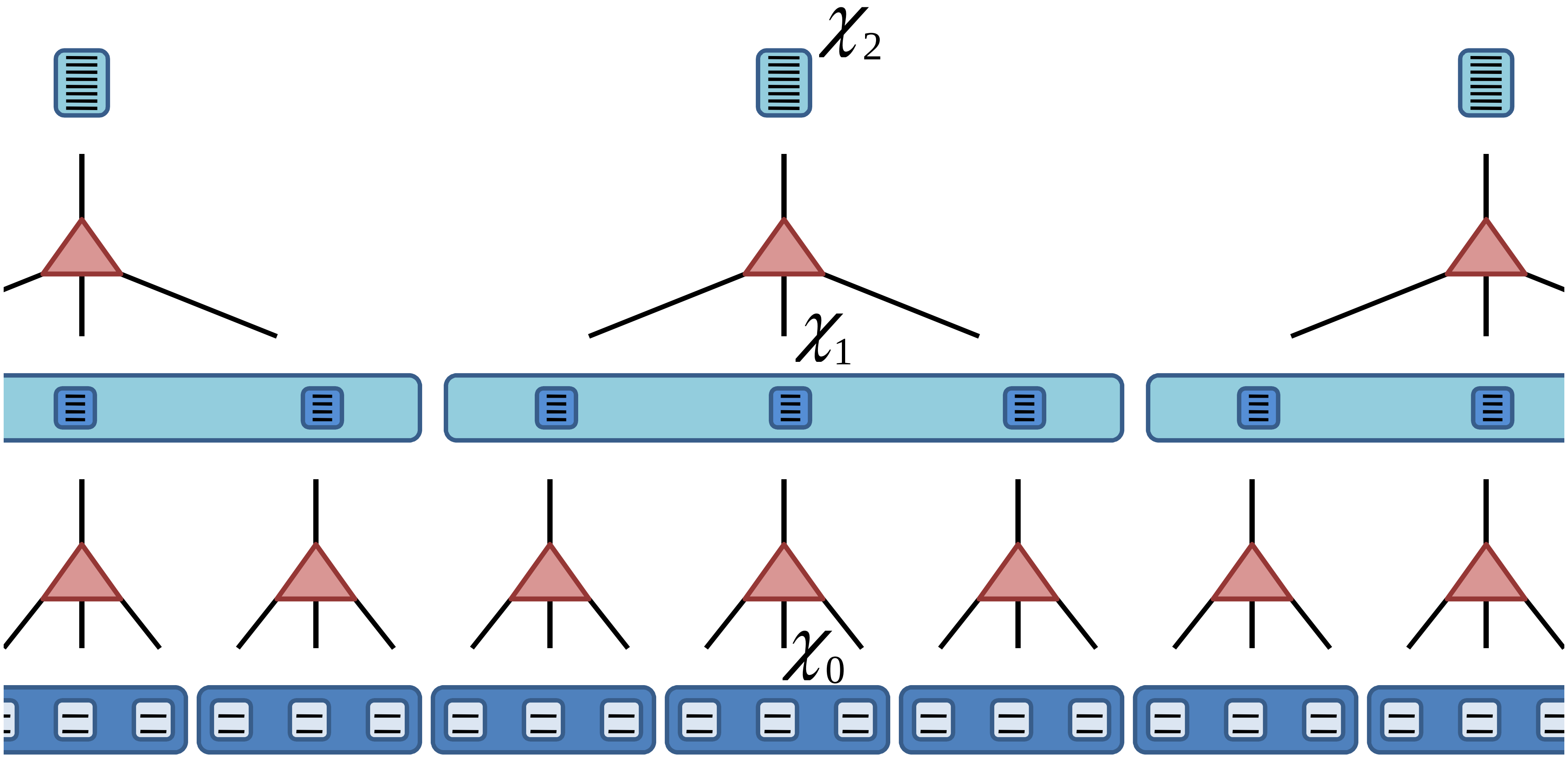
$$\langle\Psi|O|\Psi\rangle = \langle\Psi'|O'|\Psi'\rangle$$

- Kadanoff (1960's): spin blocking
- Wilson (1970's): keep low energy subspace
  Kondo impurity problem: Numerical Renormalization Group (NRG)
- White (1990's): keep local support of ground state reduce density matrix
  Density Matrix Renormalization Group (DMRG)

Introduction

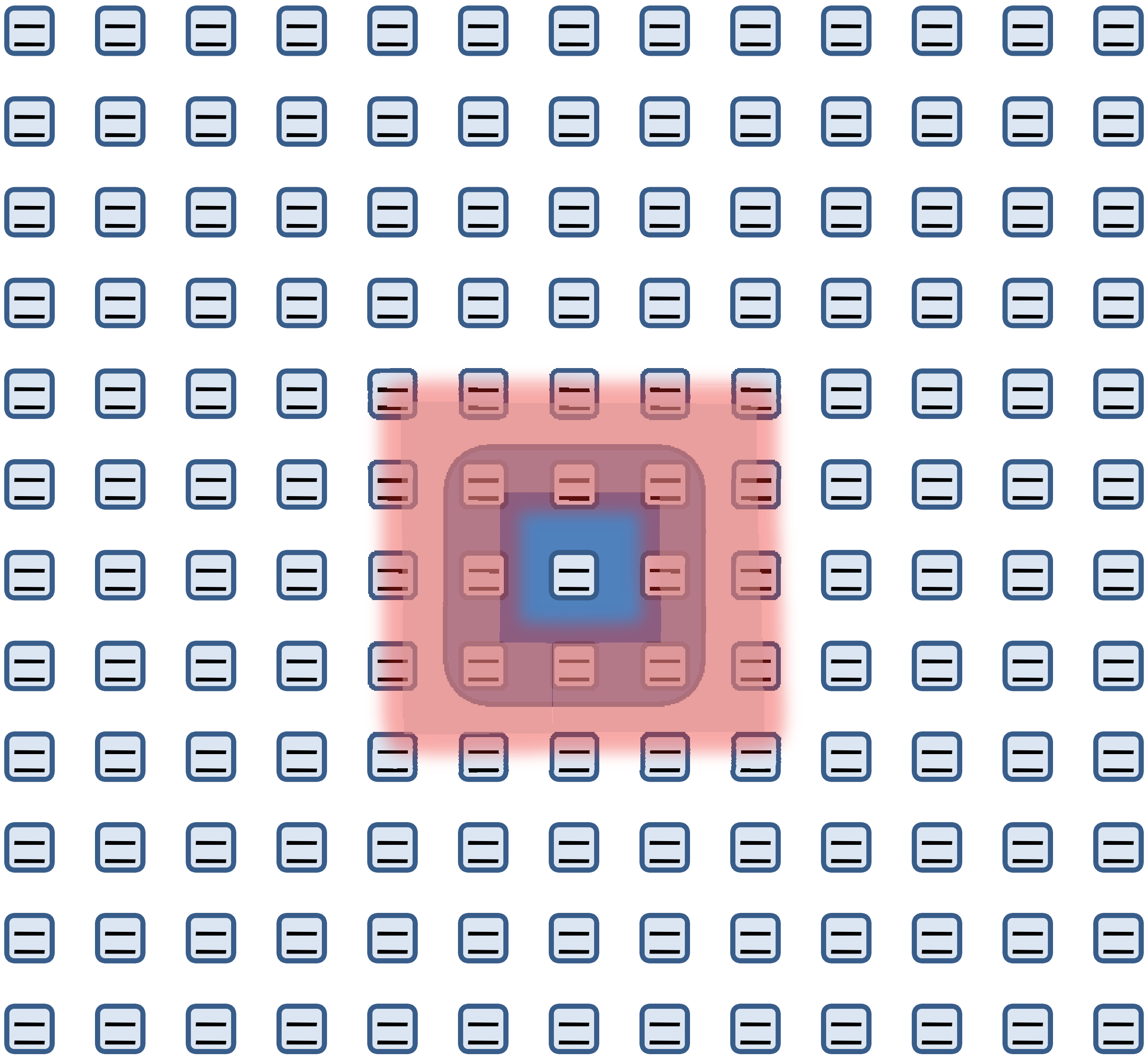
- entanglement in 1D critical

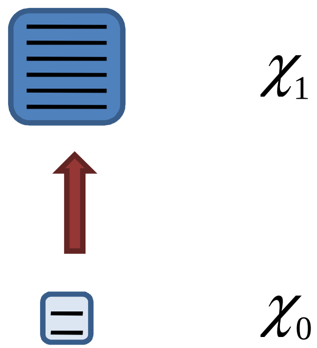
$$S(\rho_L) \sim k \log L \quad \Rightarrow \quad \boxed{\chi_\tau \sim e^\tau}$$



Introduction

• entanglement in 2D: boundary law $S(\rho_{L\times L}) \sim L$

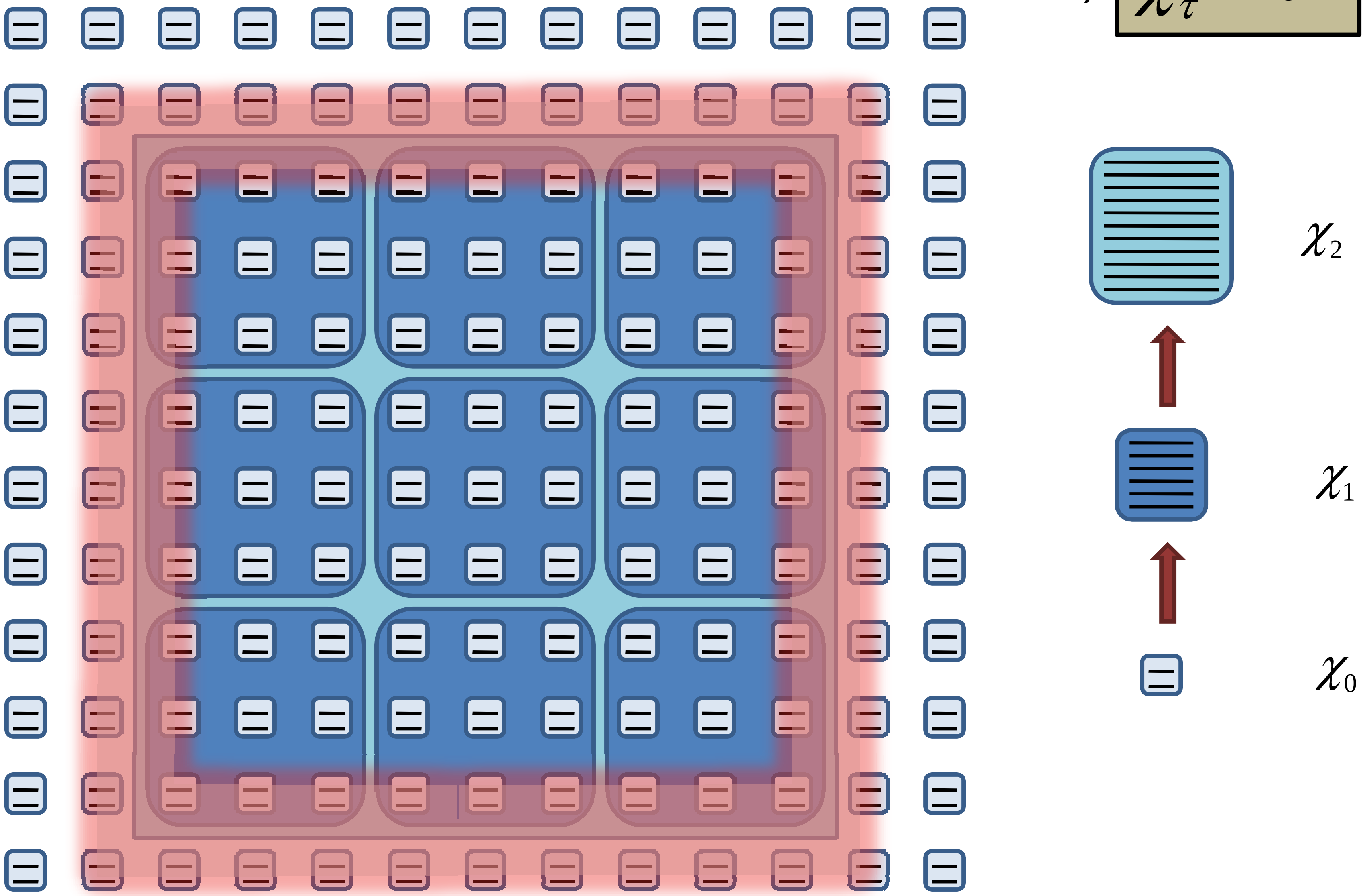




Introduction

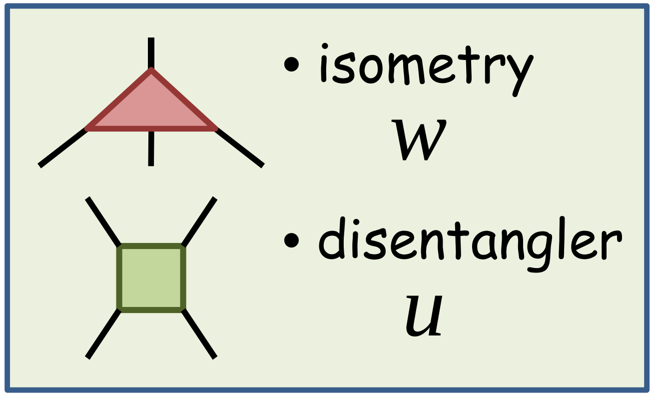
• entanglement in 2D: boundary law $S(\rho_{L\times L}) \sim L$

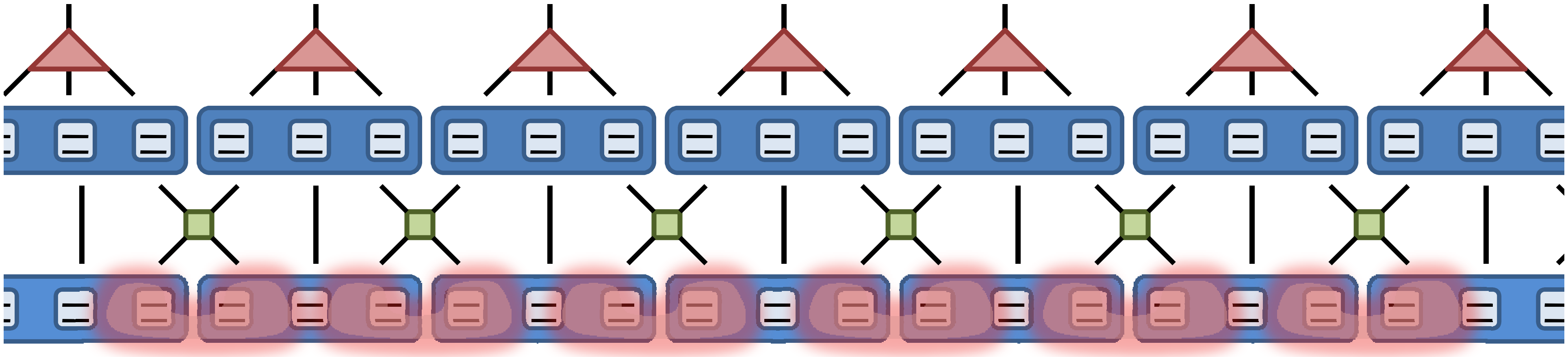
$$\Rightarrow \chi_\tau = e^{e^\tau}$$



Entanglement Renormalization

Vidal, Phys. Rev. Lett. 99, 220405 (2007)



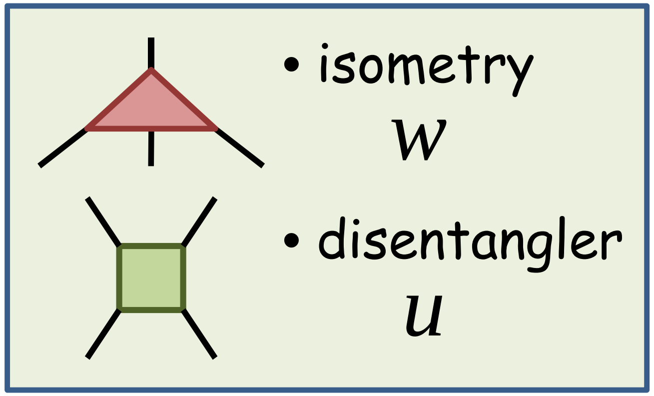


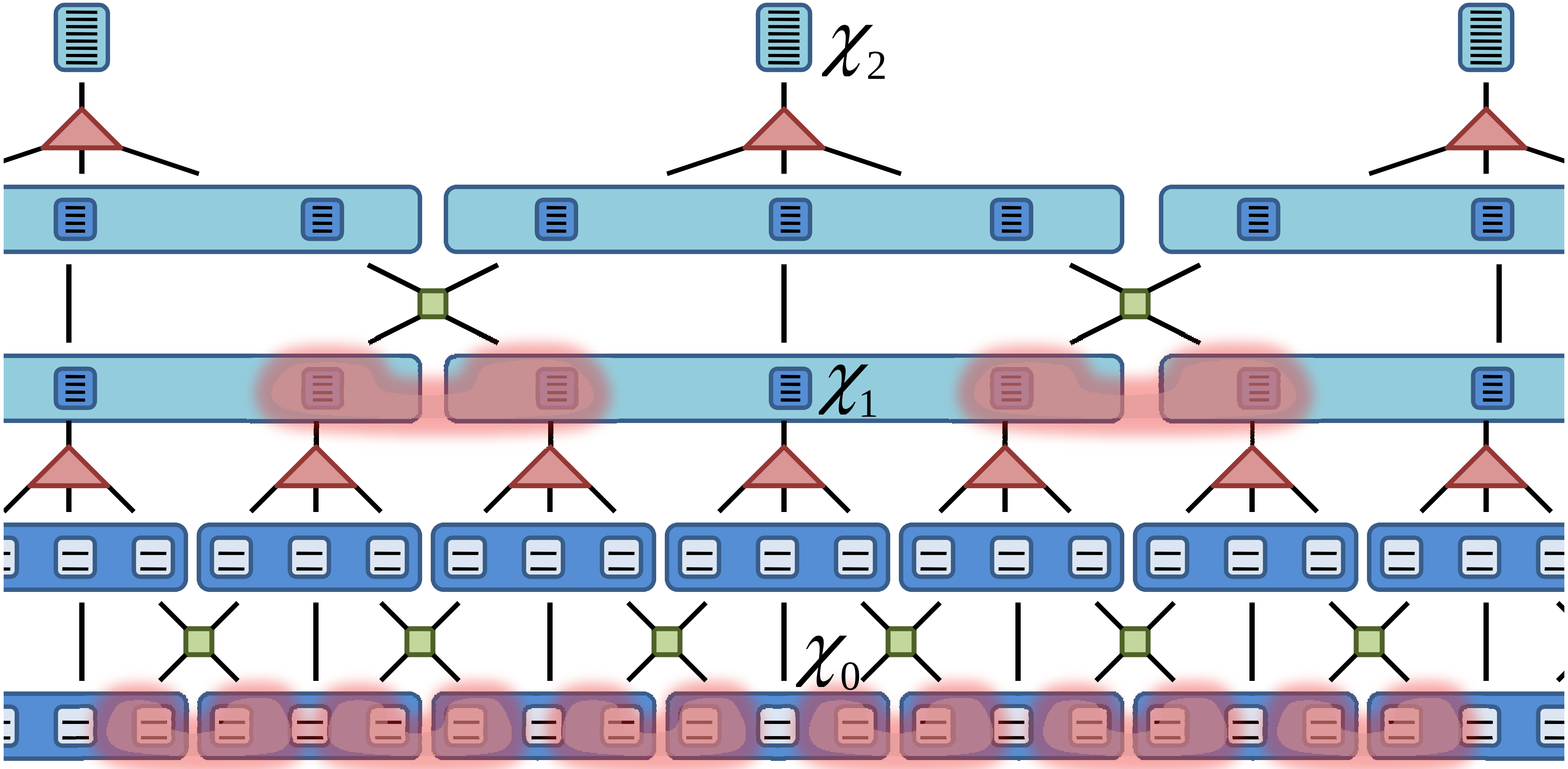
Entanglement Renormalization

Vidal, Phys. Rev. Lett. 99, 220405 (2007)

$$\chi_\tau \sim e^\tau \;\Rightarrow\; \chi_\tau \sim const.$$

(1D critical)

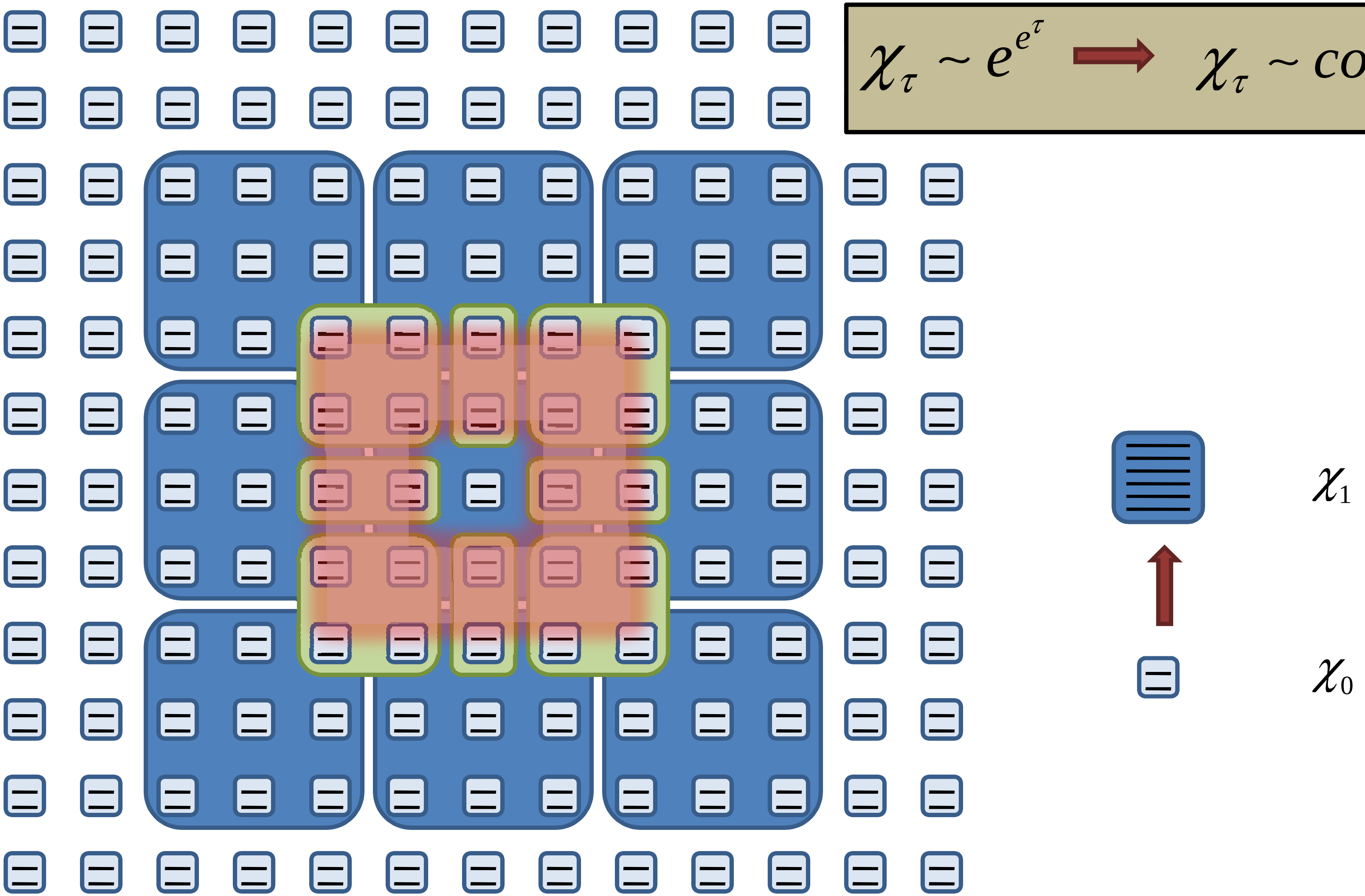




Evenbly, Vidal, arXiv:0811.0879

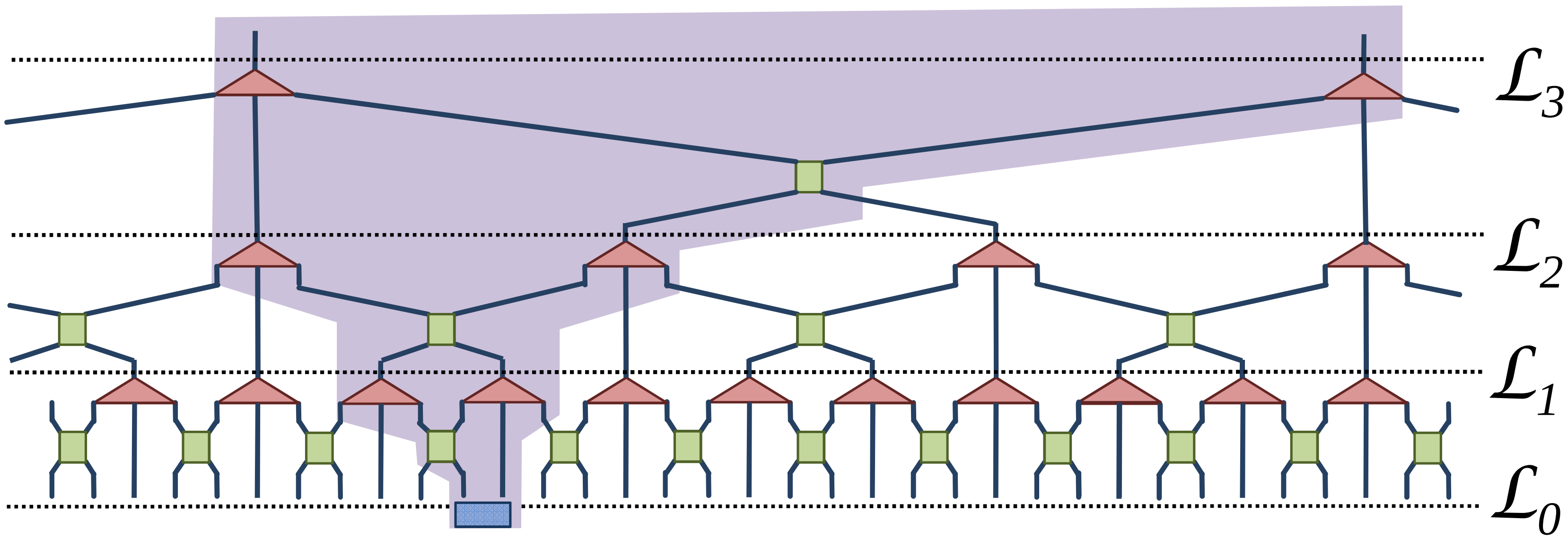
- entanglement in 2D: boundary law $S(\rho_{L\times L}) \sim L$

$$\chi_\tau \sim e^{e^\tau} \Rightarrow \chi_\tau \sim const.$$

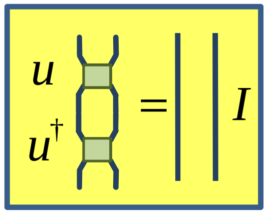


Entanglement Renormalization

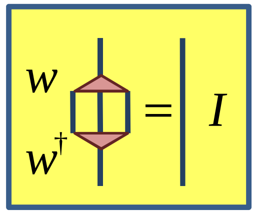
Vidal, Phys. Rev. Lett. 99, 220405 (2007)



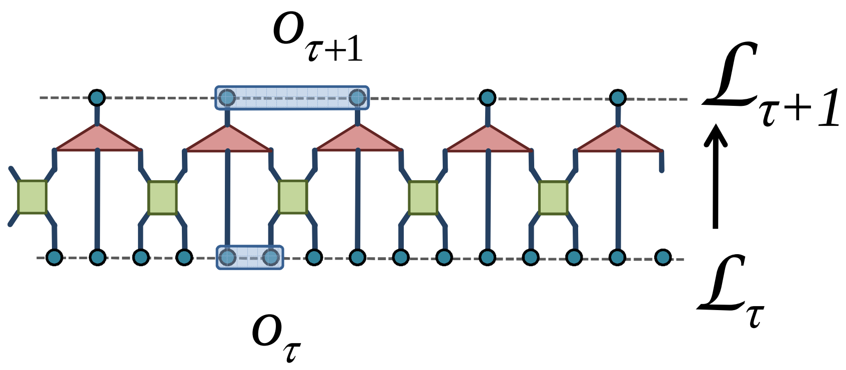
disentangler



isometry



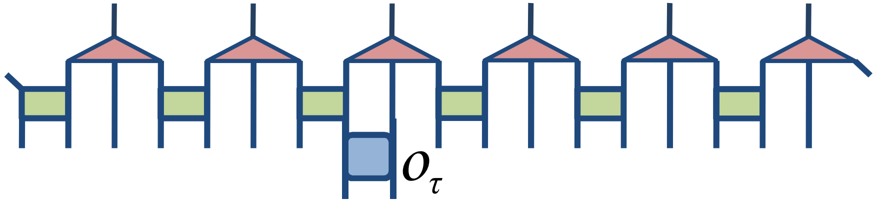
- transformation of local operators ?



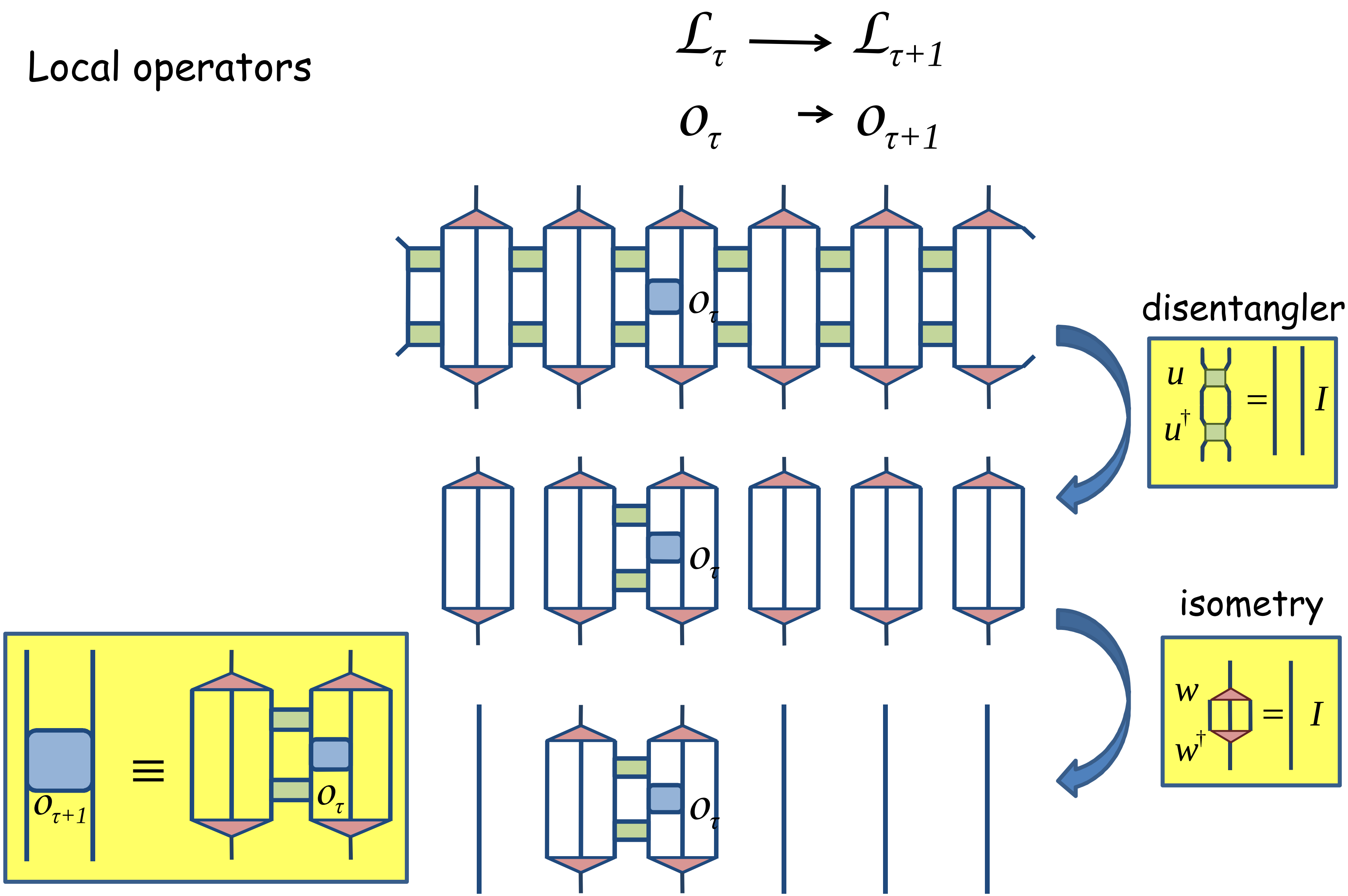
Local operators

$$\mathcal{L}_\tau \longrightarrow \mathcal{L}_{\tau+1}$$

$$O_\tau \rightarrow O_{\tau+1}$$

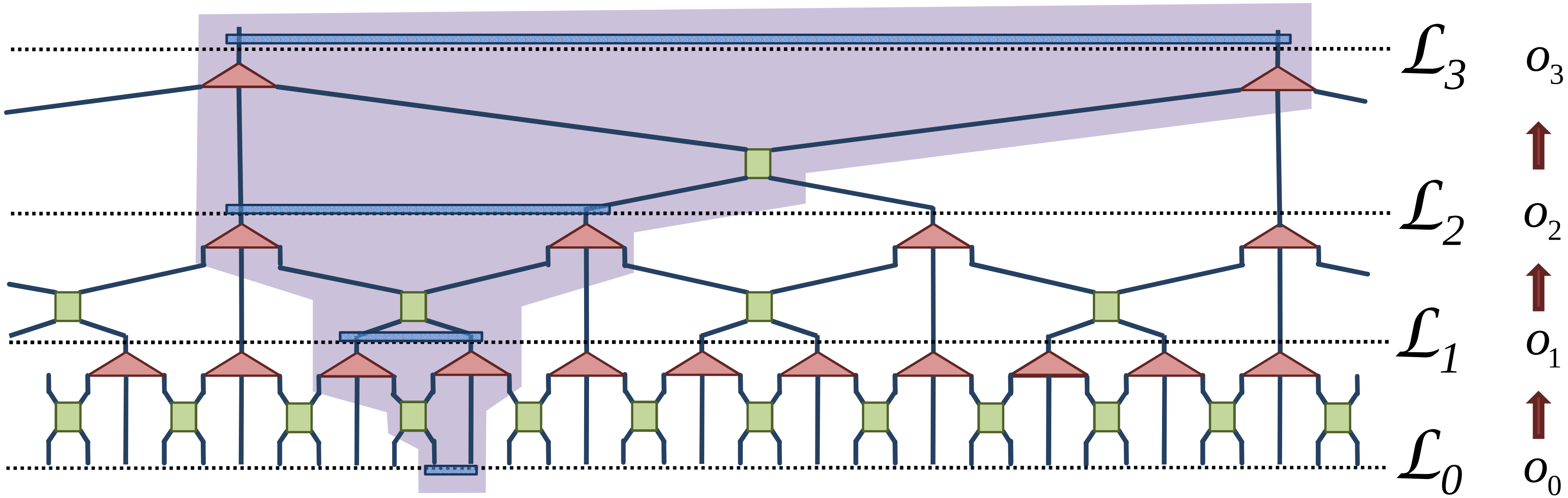


Local operators



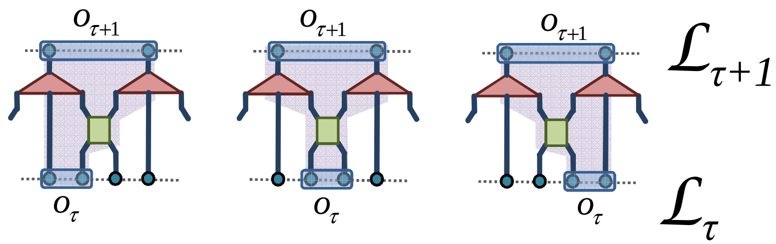
Ascending superoperator

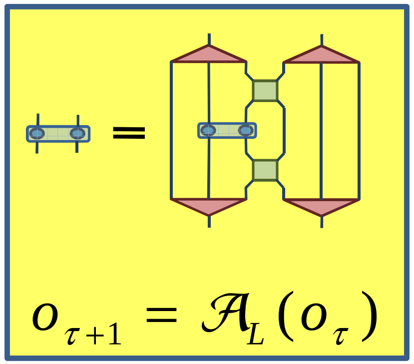
- RG transformation

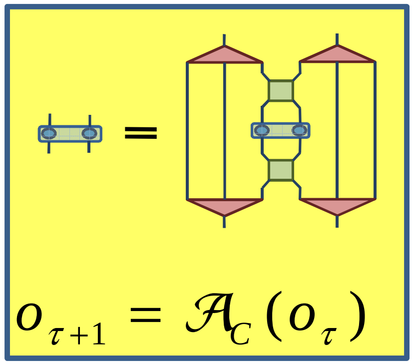


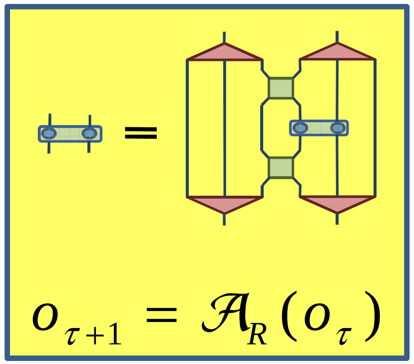
- ascending superoperator $o_{\tau+1} = \mathcal{A}(o_\tau)$

o_0 ➡ o_1 ➡ o_2 ➡ . . .









Foundations

Entanglement Renormalization

MERA

(Multi-scale Entanglement Renormalization Ansatz)

Applications

Quantum Criticality

- MERA ⟺ CFT

Topological Order

- Exact MERA for quantum double and string-net models

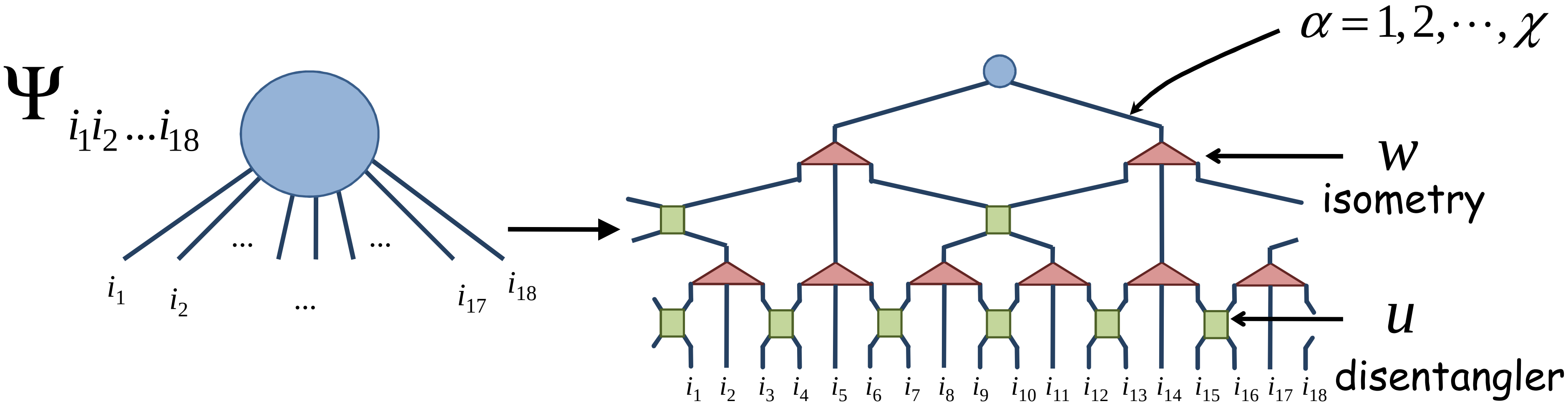
Frustrated Antiferromagnets

- Kagome lattice
- Triangular lattice

MERA (Multi-scale entanglement renormalization ansatz)

Vidal, Phys. Rev. Lett. 101, 110501 (2008)

$$|\Psi\rangle = \sum_{i_1 i_2 \ldots i_{18}} \Psi_{i_1 i_2 \ldots i_{18}} |i_1 \otimes i_2 \otimes \ldots \otimes i_{18}\rangle$$



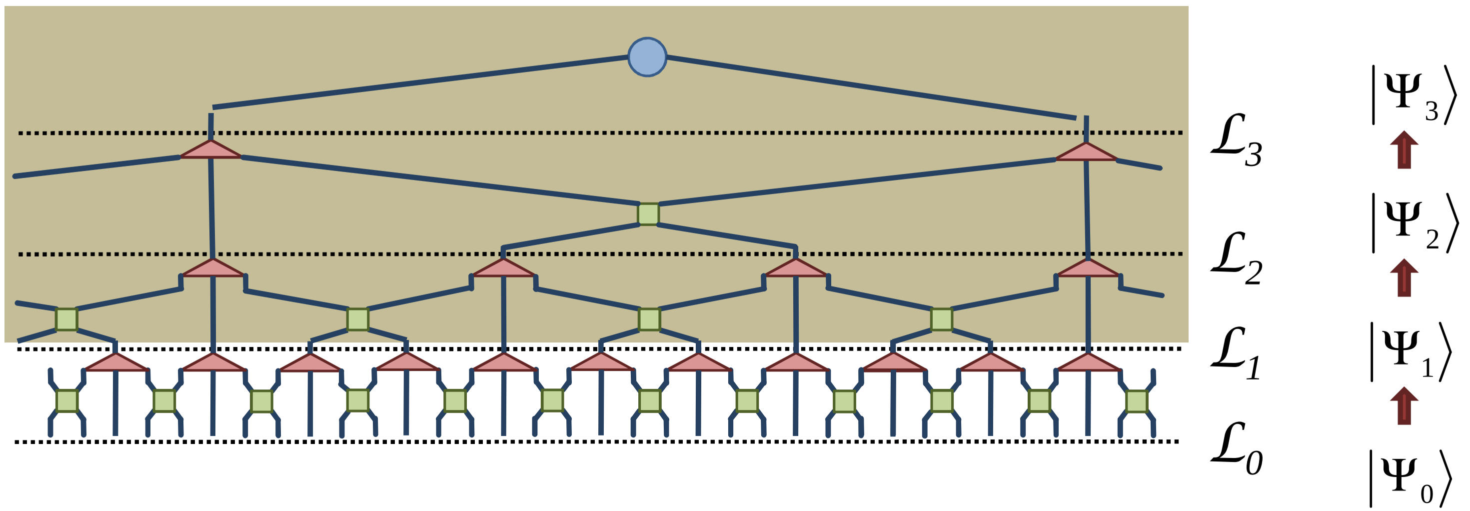
MERA (Multi-scale entanglement renormalization ansatz)

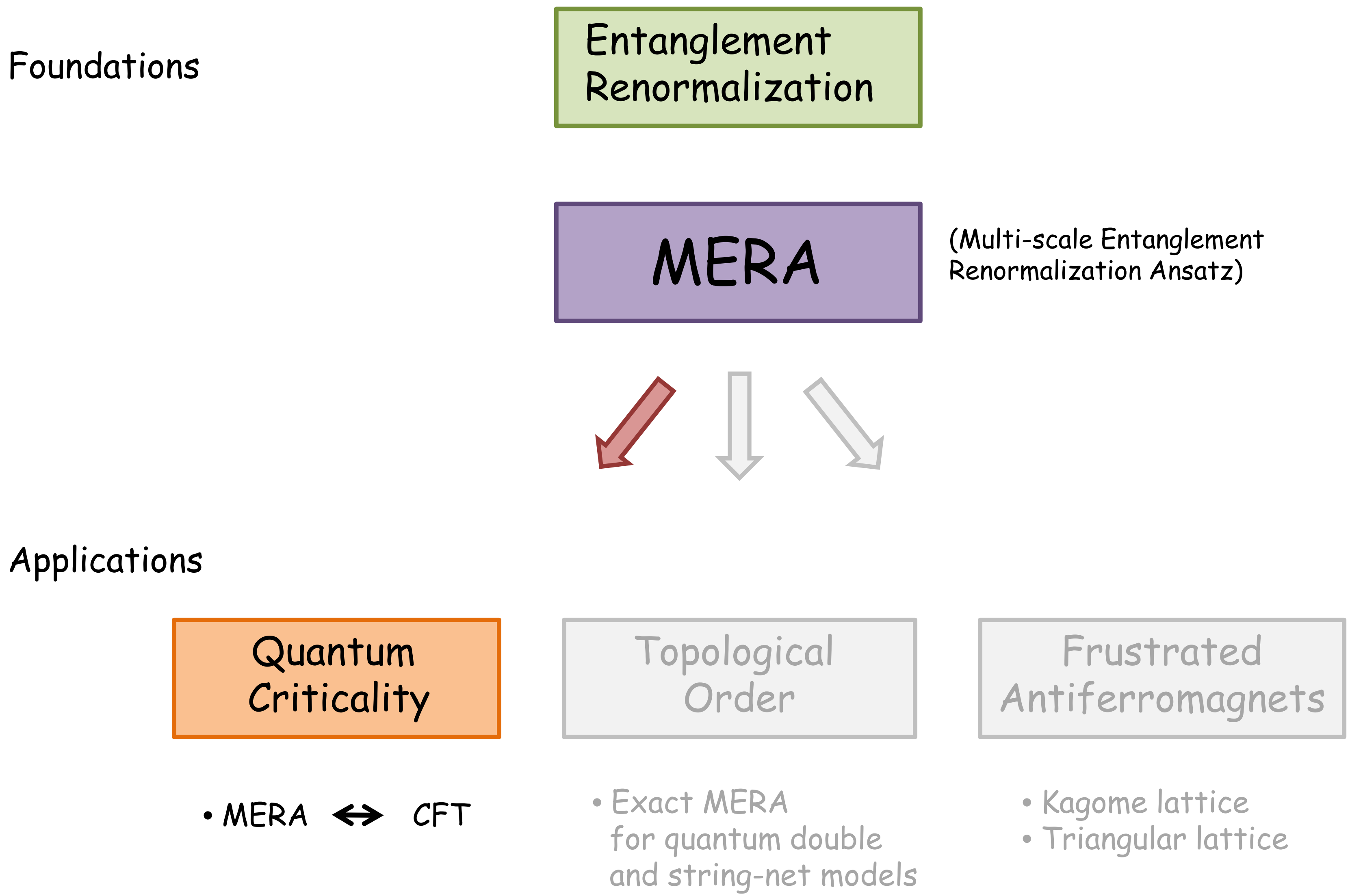
- Optimization algorithms (Evenbly, Vidal, arXiv:0707.1454v3)

H local Hamiltonian

$$\min_{u,w} \langle \Psi | H | \Psi \rangle \quad \Rightarrow \quad (u, w)$$

- RG transformation for ground states

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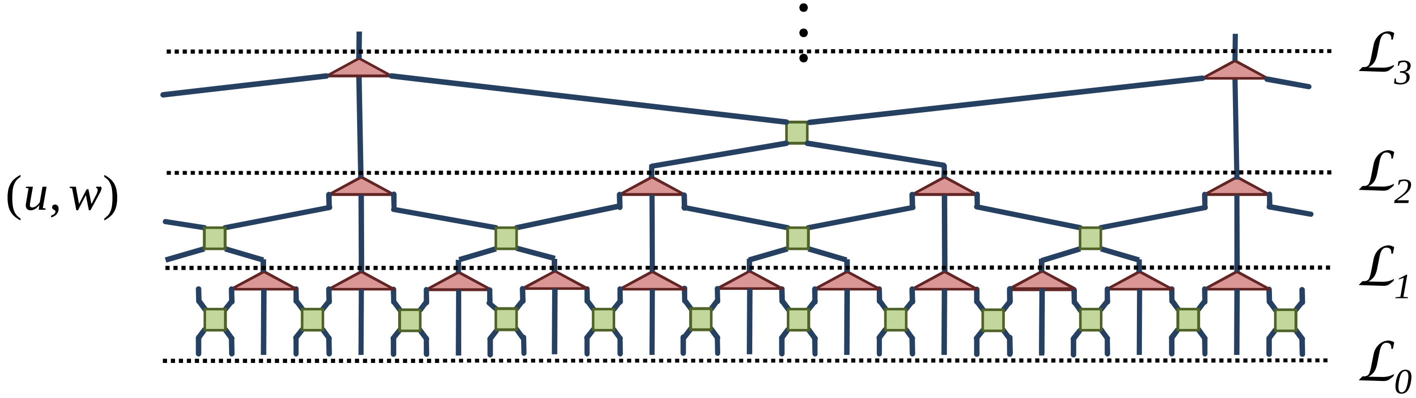
Quantum Criticality

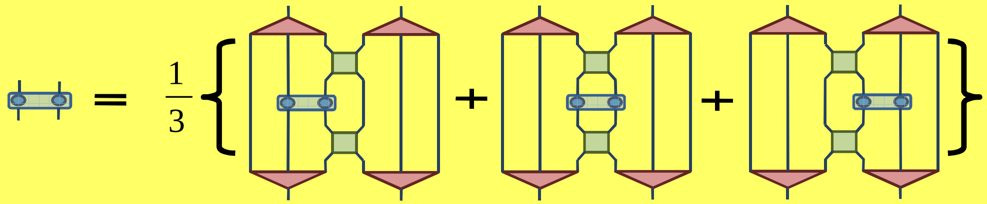
Pfeifer, Evenbly, Vidal, arXiv:0810.0580

- scale invariant MERA:



- scaling superoperator:

$$o_{\tau+1} = \mathcal{S}(o_\tau)$$



Quantum Criticality

• scaling superoperator $\mathcal{S}$ is a fixed point RG map

What are the scaling operators/dimensions of the theory?

• scaling operators ϕ_α

• scaling dimensions Δ_α

$$\Delta_\alpha \equiv -\log_3 \lambda_\alpha$$

$$\phi_\alpha \Rightarrow 3^{-\Delta_\alpha}\phi_\alpha \Rightarrow 3^{-2\Delta_\alpha}\phi_\alpha \Rightarrow \cdots$$

• From spectral decomposition of the scaling superoperator:

Giovannetti, Montangero, Fazio, Phys. Rev. Lett. 101, 180503 (2008)
Pfeifer, Evenbly, Vidal, arXiv:0810.0580

$$\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha$$

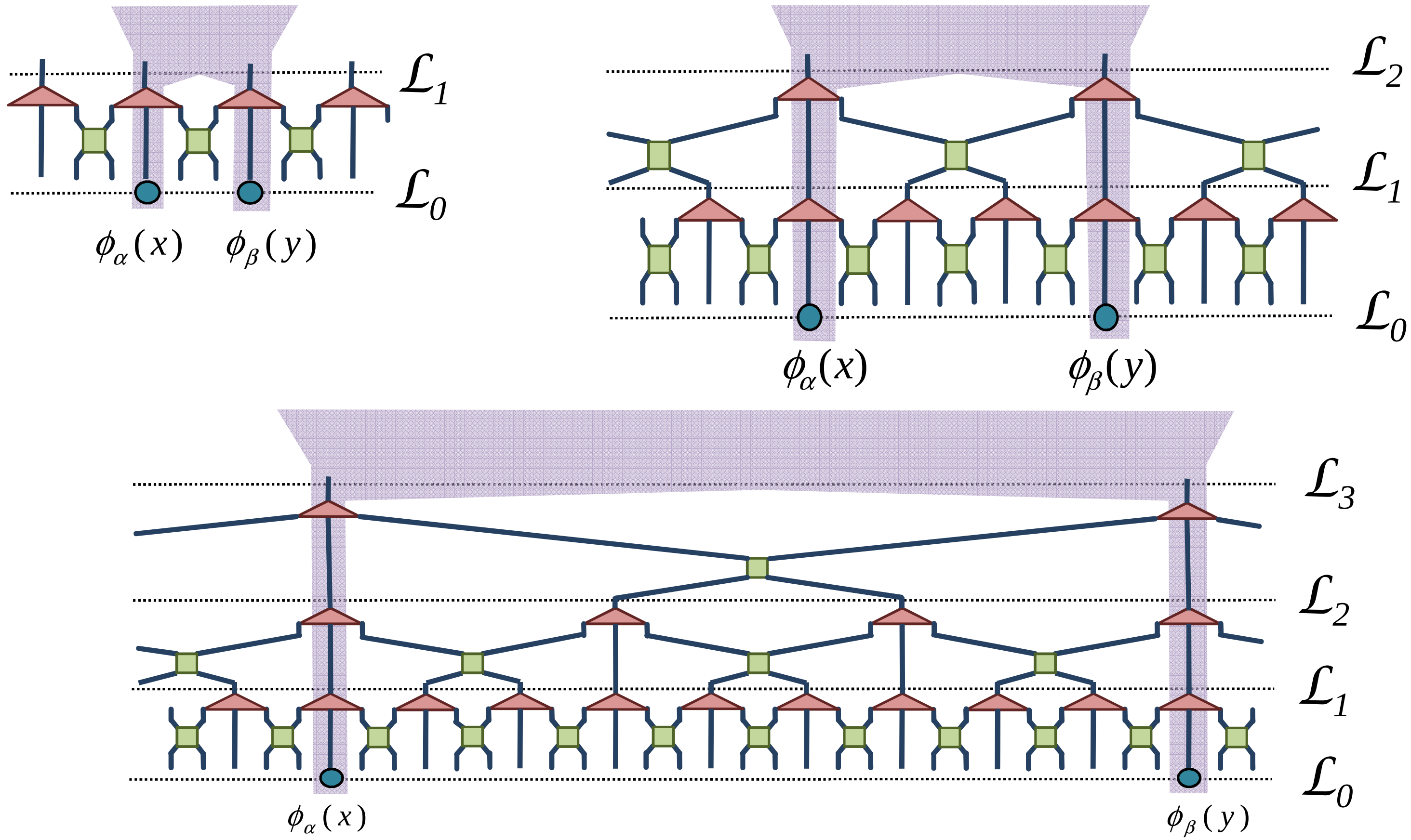
Quantum Criticality

Pfeifer, Evenbly, Vidal, arXiv:0810.0580

- local observables: $\langle \phi_\alpha(x) \rangle = \delta_{\alpha \mathbb{I}}$

all critical exponents

- two-point correlator: $\langle \phi_\alpha(x)\phi_\alpha(y) \rangle = (\lambda_\alpha)^{2\log_3|x-y|} \langle \phi_\alpha(1)\phi_\alpha(0) \rangle = \dfrac{\langle \phi_\alpha(1)\phi_\alpha(0) \rangle}{|x-y|^{2\Delta_\alpha}}$



Quantum Criticality

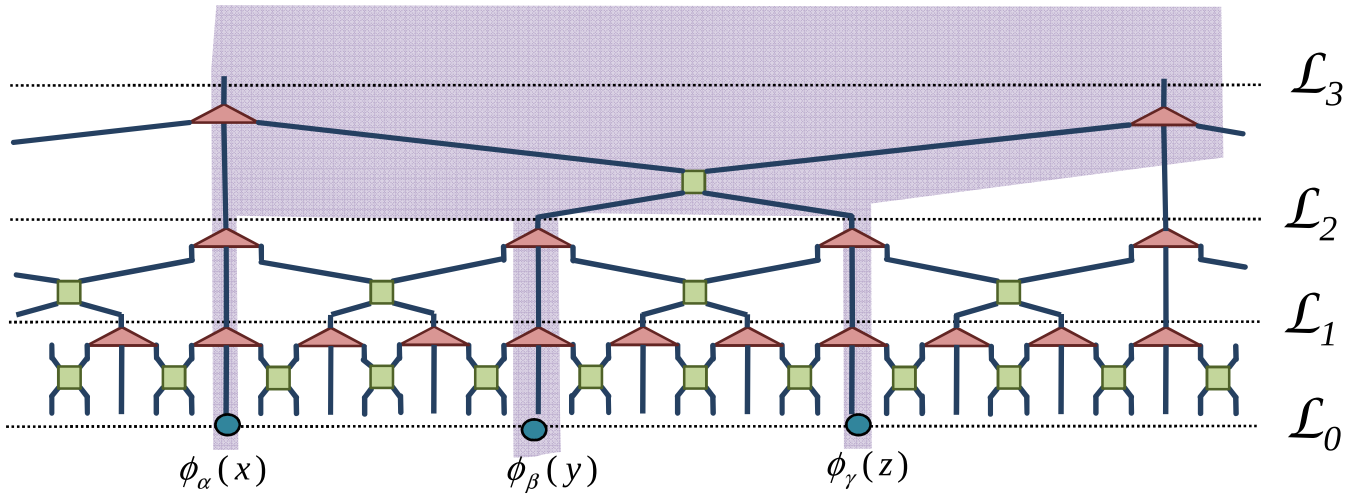
Pfeifer, Evenbly, Vidal, arXiv:0810.0580

- two-point correlator

$$\langle \phi_\alpha(x)\phi_\beta(y)\rangle = \frac{C_{\alpha\beta}}{|x-y|^{\Delta_\alpha+\Delta_\beta}}$$

$$C_{\alpha\beta} = \langle \phi_\alpha(0)\phi_\beta(1)\rangle$$
$$= tr\left(\rho^{(2)}(\phi_\alpha \otimes \phi_\beta)\right)$$

- three-point correlator



$$\langle \phi_\alpha(x)\phi_\beta(y)\phi_\gamma(z)\rangle = \frac{C_{\alpha\beta\gamma}}{|x-y|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}|y-z|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}|z-x|^{\Delta_\gamma+\Delta_\alpha-\Delta_\beta}}$$

$$C_{\alpha\beta\gamma} = 2^{\Delta_\alpha-\Delta_\beta+\Delta_\gamma} tr\left(\rho^{(3)}(\phi_\alpha \otimes \phi_\beta \otimes \phi_\gamma)\right)$$

Quantum Criticality

Pfeifer, Evenbly, Vidal, arXiv:0810.0580

scale invariant MERA (lattice)		CFT (continuum)
scaling operators ϕ_α^{MERA}	⟷	quasi-primary fields ϕ_α^{CFT}
scaling dimensions Δ_α^{MERA}	⟷	scaling dimensions $\Delta_\alpha^{CFT} = h_\alpha + \bar{h}_\alpha$
three-point correlator $C_{\alpha\beta\gamma}^{MERA}$	⟷	operator product expansion (OPE) $C_{\alpha\beta\gamma}^{CFT}$
entanglement entropy $c = 3\left(S(\rho^{(2)}) - S(\rho^{(1)})\right)$	⟷	central charge c

$$S(\rho^{(L)}) = \frac{c}{3}\log(L) + k$$

Pfeifer, Evenbly, Vidal, arXiv:0810.0580

Quantum Criticality

Given a critical hamiltonian on the lattice

H_{crit}

↓

optimize MERA (u, w)

↓

compute scaling super-operator $\mathcal{S}$

↓

$\{\phi_\alpha, \Delta_\alpha\}$ $\{\rho^{(1)}, \rho^{(2)}, \rho^{(3)}\}$

↓

$C_{\alpha\beta\gamma}$ c

$$C_{\alpha\beta\gamma} = 2^{\Delta_\alpha - \Delta_\beta + \Delta_\gamma} tr\left(\rho^{(3)}(\phi_\alpha^{(1)} \otimes \phi_\beta^{(1)} \otimes \phi_\gamma^{(1)})\right)$$

$$c = 3\left(S(\rho^{(2)}) - S(\rho^{(1)})\right)$$

example 1D quantum Ising:

$$H_{\text{Ising}} = \sum_i \left[\sigma_x^{[i]}\sigma_x^{[i+1]} + \sigma_z^{[i]}\right]$$

$\Delta_{\mathbb{I}} = 0$

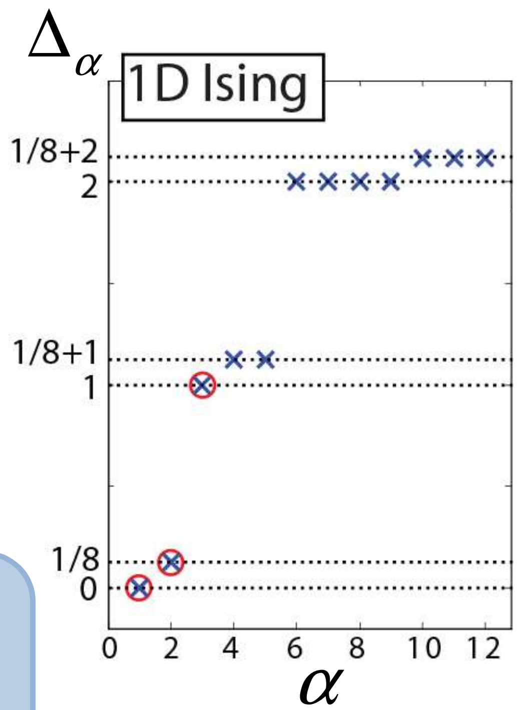
$\Delta_\sigma = 0.124997$ (0.002%)

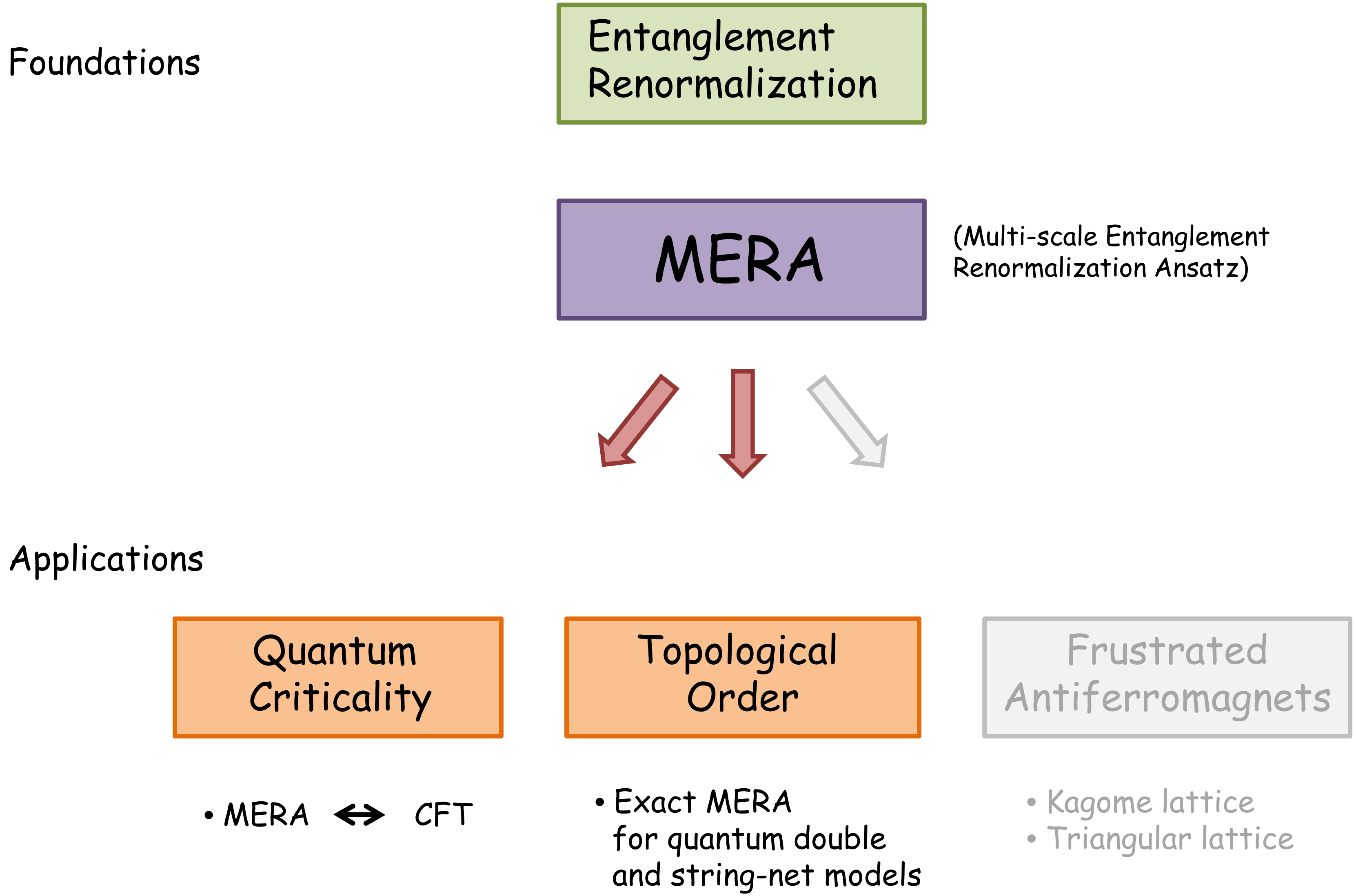
$\Delta_\varepsilon = 1.0001$ (0.01%)

$C_{\alpha\beta\mathbb{I}} = \delta_{\alpha\beta}$ $C_{\sigma\sigma\varepsilon} = 1/2$

$C_{\sigma\varepsilon\varepsilon} = C_{\sigma\sigma\sigma} = C_{\varepsilon\varepsilon\varepsilon} = 0$ $(\pm 5\times 10^{-4})$

$c = 0.5008$ (0.15%)

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Topological Order

Exact MERA representation for

- quantum double models

Aguado, Vidal, Phys. Rev. Lett. 100, 070404 (2008)

- Levin-Wen's string net models

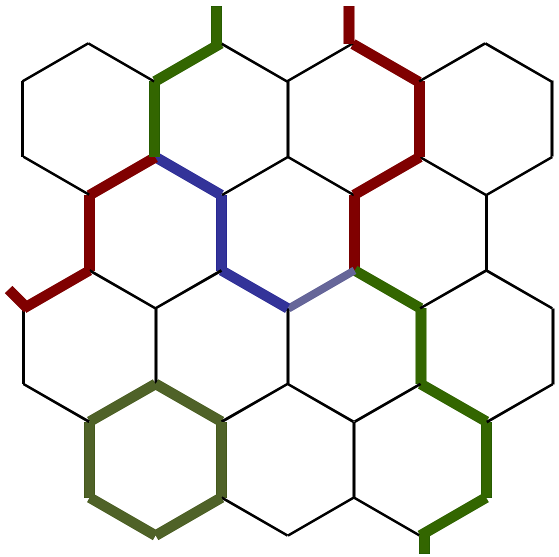
Koenig, Reichardt, Vidal, arXiv:0806.4583

Topological Order

Koenig, Reichardt, Vidal, arXiv:0806.4583

• Exact MERA representation for Levin-Wen's string net models

Levin, Wen, Phys. Rev. B71 (2005) 045110



• string types $i = \{1, 2, \cdots, N\}$

• fusion rules $\delta_{i\,j\,k}$

• 6-j symbols $F^{i\,j\,m}_{k\,l\,n}$

• fixed-point Hamiltonian

$$H = -\sum_{\curlywedge} A_{\curlywedge} - \sum_{\hexagon} B_{\hexagon}$$

$\delta_{i\,j\,k}$ $F^{i\,j\,m}_{k\,l\,n}$

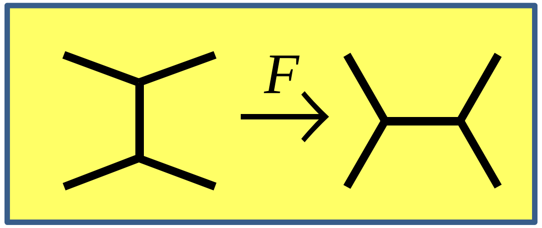
'physical' subspace

Topological Order

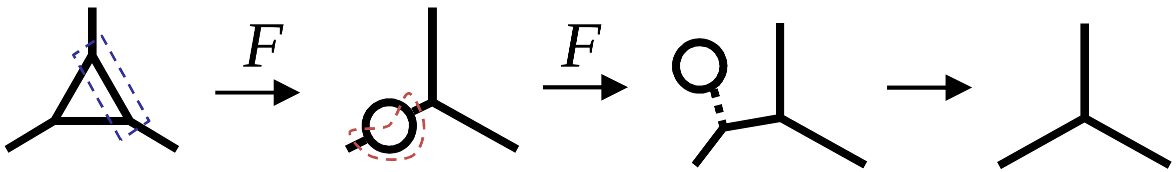
Koenig, Reichardt, Vidal, arXiv:0806.4583

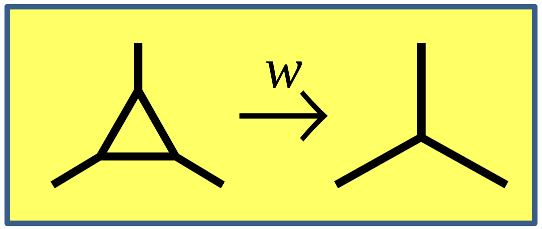
- Exact MERA representation for Levin-Wen's string net models

The tensor $F^{i\,j\,m}_{k\,l\,n}$ can be used to define an "F-move"



Disentangler $u \quad (=F)$





Isometry w

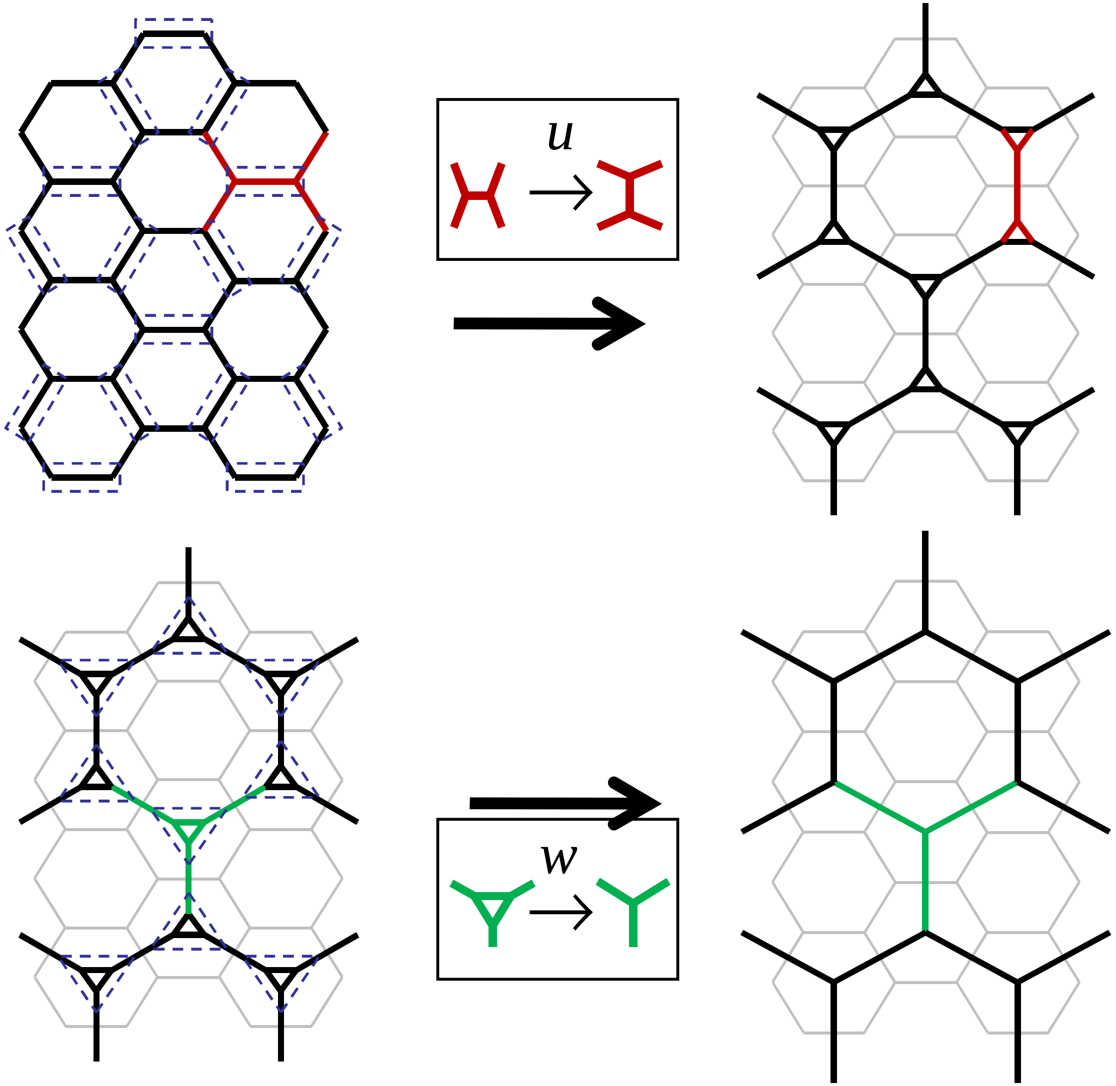
Topological Order

Koenig, Reichardt, Vidal, arXiv:0806.4583

- Exact MERA representation for Levin-Wen's string net models

$$|\Psi_{GS}\rangle$$

$$H = -\sum_{\hexagon} B_{\hexagon}$$



sequence used for classical partition functions by Levin, Nave, Phys. Rev. Lett. 99, 120601 (2007)

$$|\Psi'_{GS}\rangle$$

$$H' = -\sum_{\hexagon} B_{\hexagon}$$

$\Rightarrow$ Fixed-point of RG flow

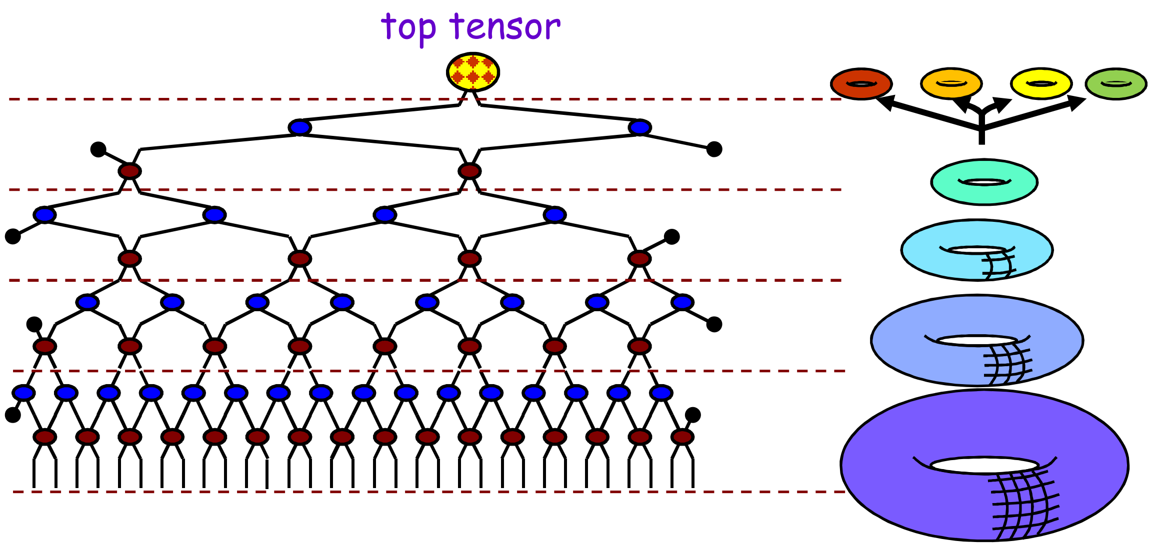
Topological Order

- MERA for ground states with topological order

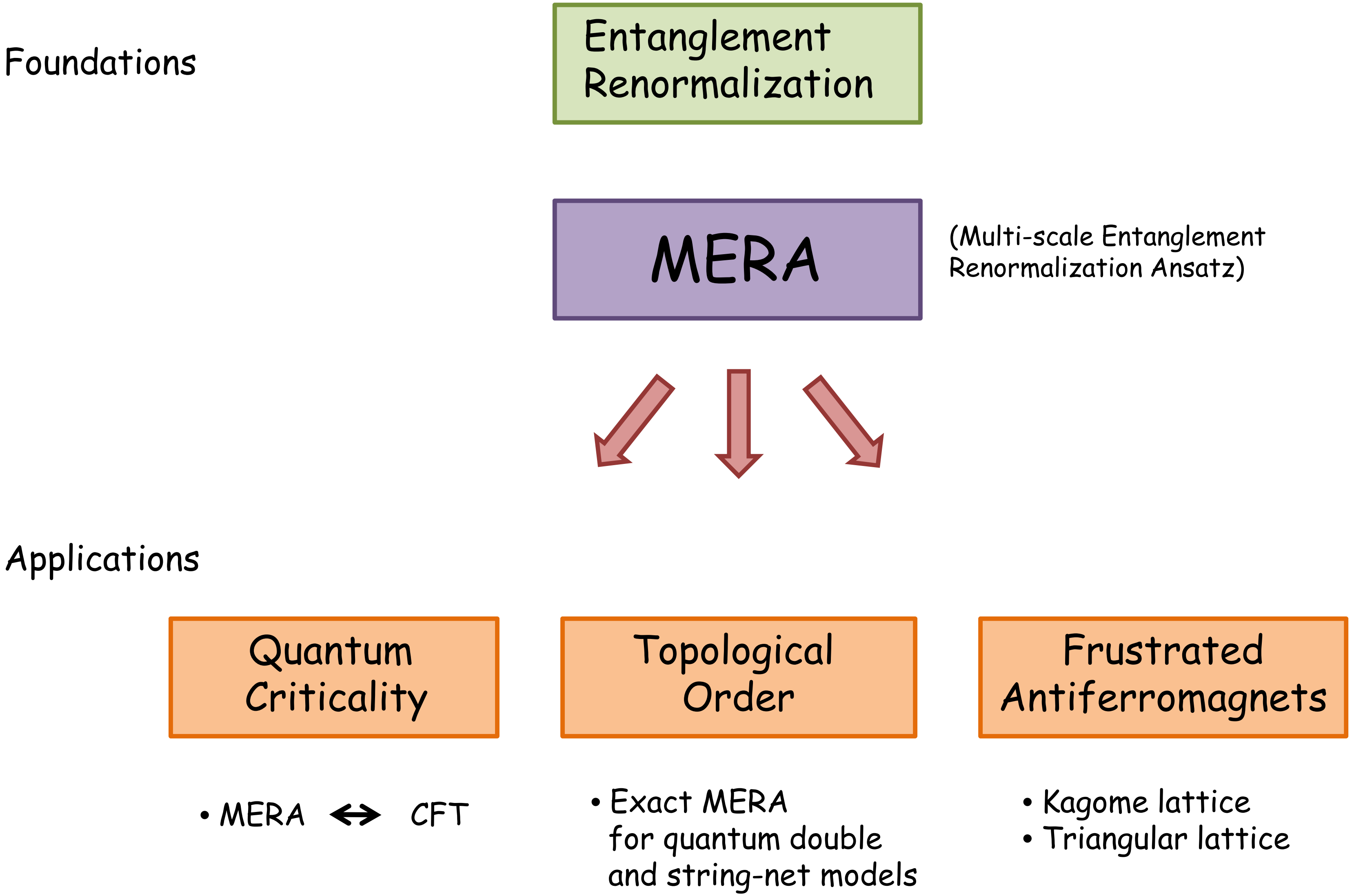
degenerate ground states

$$|\Psi_{gr}^{(1)}\rangle \quad |\Psi_{gr}^{(2)}\rangle \quad |\Psi_{gr}^{(3)}\rangle \quad |\Psi_{gr}^{(4)}\rangle$$

Topological information is stored in the top tensor of the MERA



$$|\Psi_{gr}^{(1)}\rangle \quad |\Psi_{gr}^{(2)}\rangle \quad |\Psi_{gr}^{(3)}\rangle \quad |\Psi_{gr}^{(4)}\rangle$$



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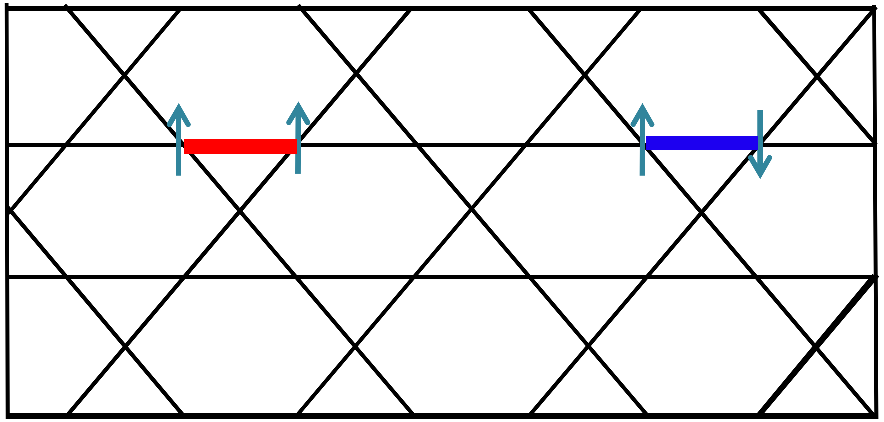
Frustrated Antiferromagnets

Evenbly, Vidal, in preparation

Glen Evenbly

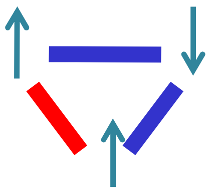
- AF Heisenberg Model on Kagome Lattice



Hamiltonian: $$H = \sum_{\langle i,j \rangle} S_i \cdot S_j$$

- Favours antiferromagnetic alignment

- Geometrically frustrated



Frustrated Antiferromagnets

- AF Heisenberg Model on Kagome Lattice

What is the ground state? Open question...

- Valence Bond Crystal

Marston, Zeng (1991), Syromyatnikov, Maleyev (2002) Nikolic, Senthil (2003), Budnik, Auerbach (2004)

Rajiv R. P. Singh, David A. Huse, Phys. Rev. B **76**, 180407 (2007):
series expansion

- Spin Liquid

Hermele, Senthil, Fisher (2005)

Y. Ran, M. Hermele, P. A. Lee, X. Wen, Phys. Rev. Lett. **98**, 117205 (2007)
Gutzwiller ansatz

H. C. Jiang, Z. Y. Weng, D. N. Sheng, Phys. Rev. Lett. **101**, 117203 (2008)
DMRG

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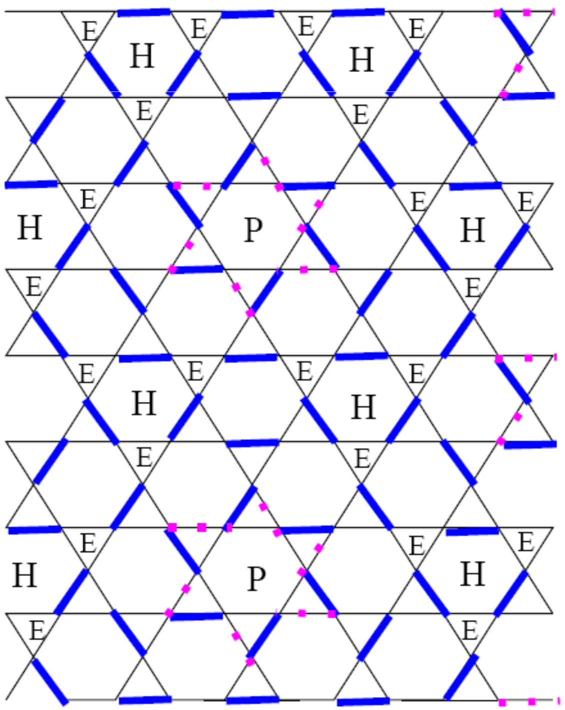

Order	VBC (∞ system)	36-site VBC
0	-0.375	-0.375
1	-0.375	-0.375
2	-0.42187	-0.42187
3	-0.42578	-0.42578
4	-0.43155	-0.43400
5	-0.43208	-0.43624
∞	-0.433±0.001	

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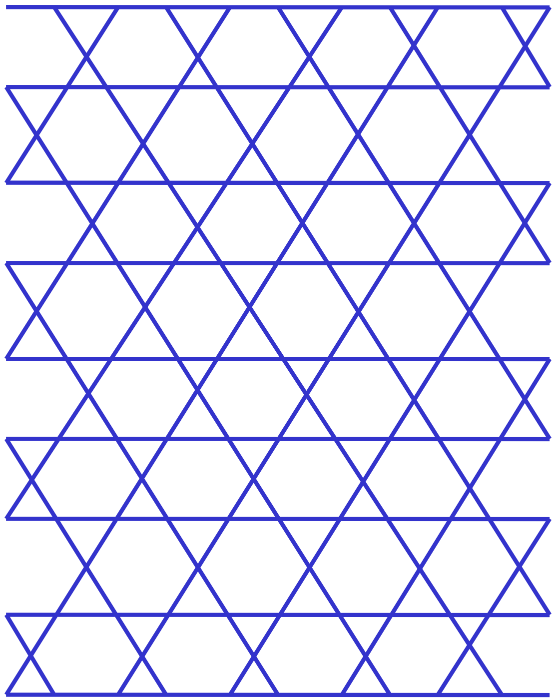


DMRG (Jiang)

N	E_0/N
48	-0.43591
108	-0.43111

Gutzwiller (Ran)

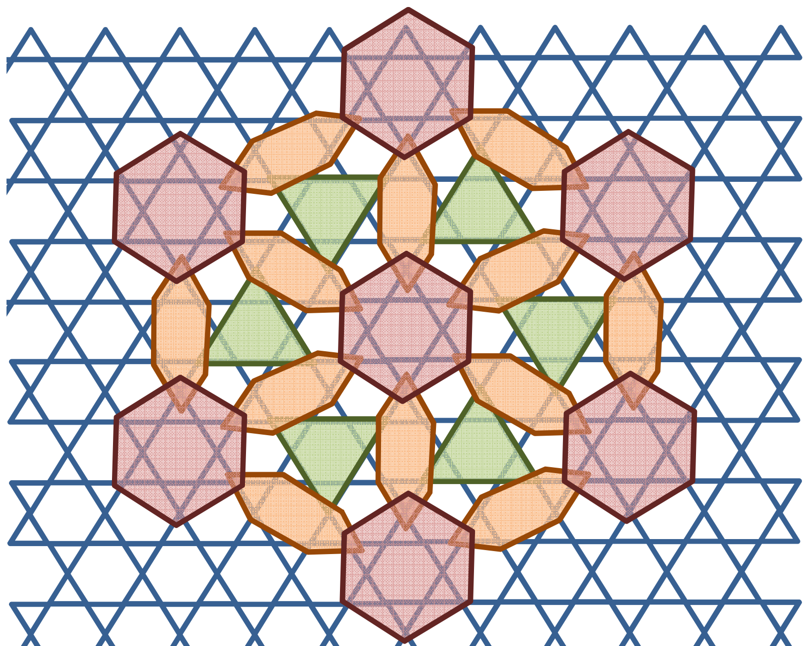
N	E_0/N
192	-0.42866(2)
432	-0.42863(2)

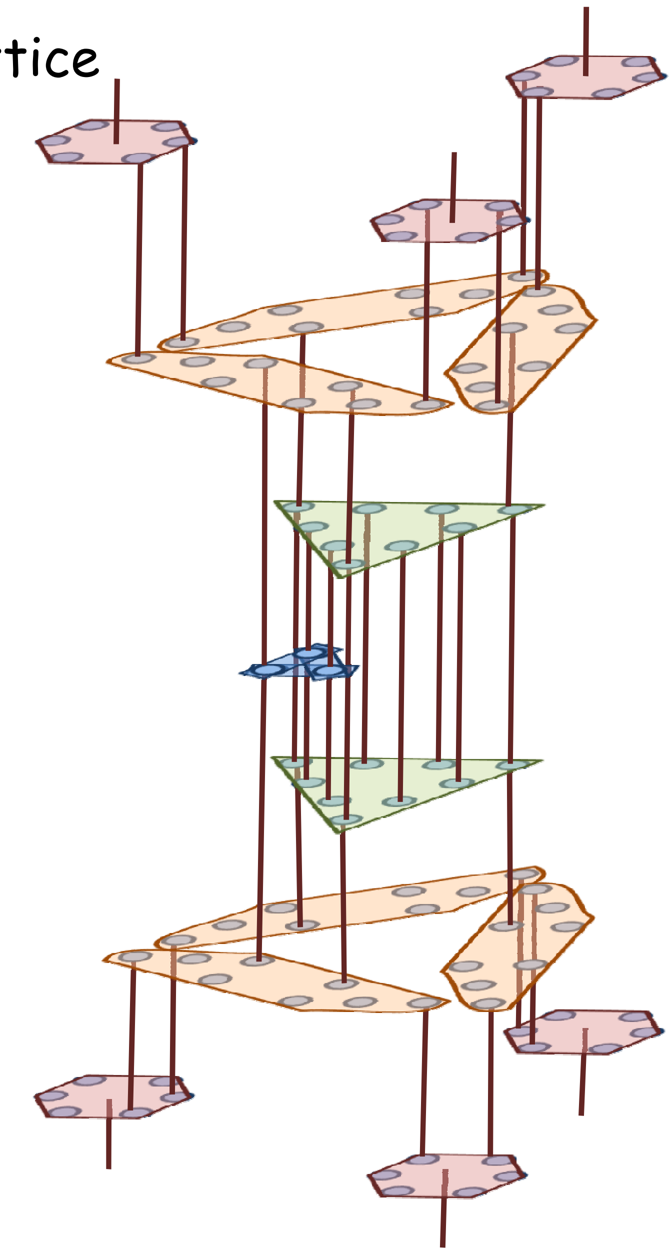
Frustrated Antiferromagnets

- AF Heisenberg Model on Kagome Lattice

MERA ?

Movie!!!





Frustrated Antiferromagnets

- AF Heisenberg Model on Kagome Lattice

Series Expansion (Singh)

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Entanglement Renormalization

N	E_0/N
∞	-0.4315

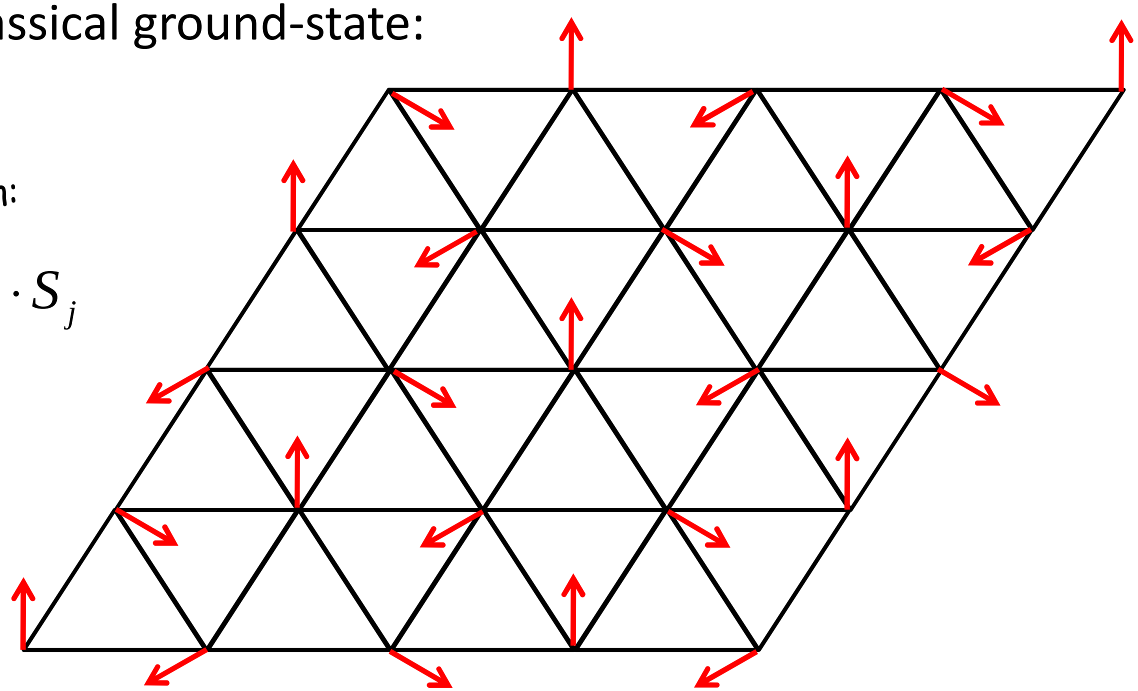
36-site Exact Diag (Leung)

N	E_0/N
36	-0.43837

Frustrated Antiferromagnets

- AF Heisenberg Model on Triangular Lattice

Classical ground-state:

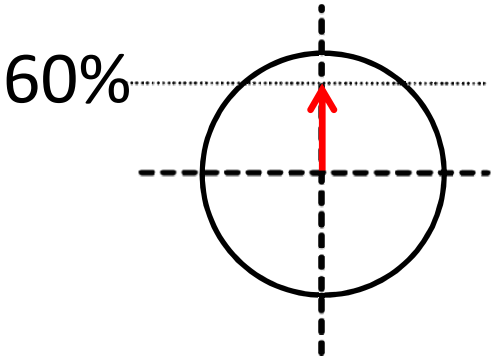


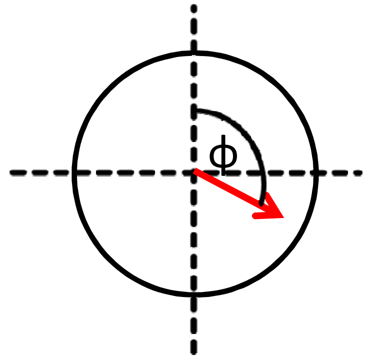
Hamiltonian:

$$H = \sum_{\langle i,j \rangle} S_i \cdot S_j$$

MERA ?

Magnetization:





$\phi = 120° \pm 0.1$

Frustrated Antiferromagnets

- AF Heisenberg Model on Triangular Lattice

(from W. Zheng, J. Fjaerestad, R. Singh, R. McKenzie, R. Coldea, Phys. Rev. B **74**, 224420 (2006))

Method	**E_0/N**	**Mag**
DMRG	-0.5442	
GFQMC	-0.5458(1)	41%
VQMC, RVB	-0.5357	0%
Series Expansion	-0.5502(4)	39%
VQMC, BCS+Neel	-0.532(1)	72%
SWT + 1/S	-0.5466	50%
Entanglement Renormalization	-0.5451	60%

- ER gives ***exact upper bound*** for ground energy of the infinite system (not based upon finite size scaling)

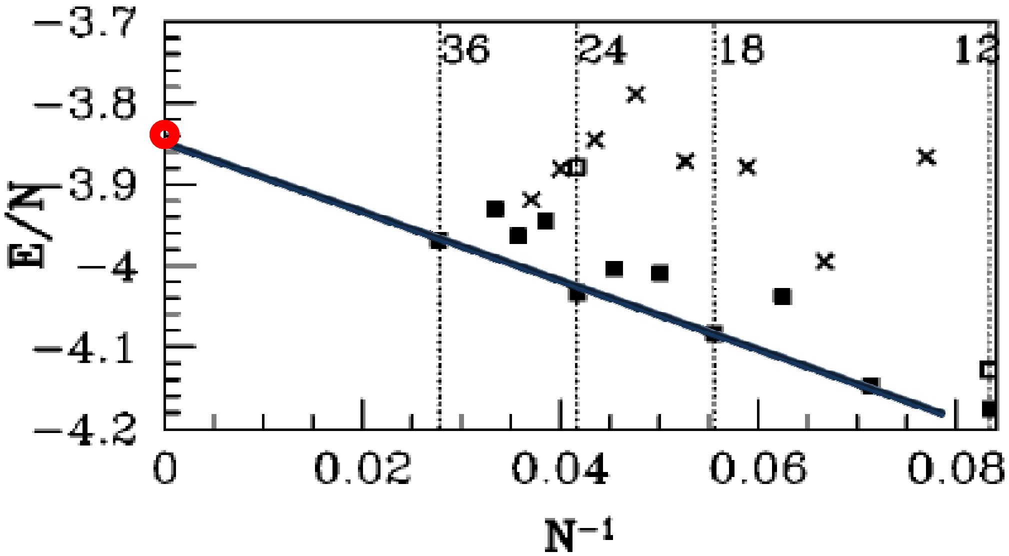
Frustrated Antiferromagnets

- Multiple spin exchange (MSE) on Triangular Lattice

$$H = J_2 \sum P_{ij} + J_4 \sum \left(P_{ijkl} + P_{lkji} \right)$$

- G. Misguich, C. Lhuillier, B. Bernu, and C. Waldtmann, Phys. Rev. B **60**, 1064 (1999)
  - exact diagonalization (36 sites)

Spin Liquid phase (J2 = -2, J4 = 1)



Entanglement Renormalization

N	E_0/N
∞	-3.84

Summary

Foundations

Entanglement Renormalization

MERA (Multi-scale Entanglement Renormalization Ansatz)

Applications

Quantum Criticality	Topological Order	Frustrated Antiferromagnets
• MERA ⟺ CFT	• Exact MERA for quantum double and string-net models	• Kagome lattice • Triangular lattice

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MERA

(Multi-scale Entanglement Renormalization Ansatz)

Applications

Quantum Criticality

- MERA ⟺ CFT

Topological Order

- Exact MERA for quantum double and string-net models

Frustrated Antiferromagnets

- Kagome lattice
- Triangular lattice

Fermions ?